Computer Power
or
The Liberation of Applied Probability

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Abstract

A number of arguments are presented in favor of a greater effort in computational probability. Examples are given to illustrate the use of numerical methods in the teaching and research of probability models.

In this discussion, I shall describe some of the profound changes, which are taking place in Applied Probability through the wide availability of the computer. Many of my remarks are applicable to all of applied mathematics, but given the interests of this audience and my own area of competence, I shall limit myself to examples taken from the study of probability models.

It is difficult to keep the development of modern computing facilities in a proper historical perspective. Without exaggeration, one can say that up to World War II, computations of great complexity were essentially impossible. Only very special problems of paramount scientific or military significance were subjected to extensive computations. These were accomplished only at a great cost in money and, perhaps more importantly, in excruciatingly tiresome effort by individuals whose talents could have served greater ends. This is neither the time, nor the place to survey the history of the computer; I shall only refer you to the excellent book "The Computer from Pascal to von Neumann" by H. H. Goldstine [4]. It is still a source of wonder and awe that in the span of approximately twenty five years, a tool of such power and versatility has been fully developed and made widely available. Today we perform at very high speed and low cost, computations beyond the wildest dreams of scientists of a few decades ago. The changes being wrought in scientific investigation by the computer, are far greater and perhaps more fundamental, than the impact of the introduction of the printing press on the learning of past centuries. This is not yet sufficiently appreciated, which is not really surprising. The first generation of students of the computer era, has only recently left the universities. For a visionary discussion of the broad cultural impact of the computer, I refer to the pleasantly written book "Man and the Computer" by John Kemeny [6].

In order to derive the greatest benefit from the computer in scientific work, it should be integrated quite early in the development of an investigation. The traditional approach in applied mathematics is to postpone all computation to the end; more commonly still it is left to the initiative of others and so, does not get done.

At the risk of walking a well-trodden path, I should like to retrace for you the classical evolution of a mathematical problem of practical significance. This, in order to stress which steps in this evolution are most profoundly affected by the computer. I shall also discuss in passing, some related issues such as the value of tables and social attitudes which affect computational mathematics.

If necessary, kindly overlook these digressions. In discussing computational probability, my prime objective is to indicate the resulting and considerable gain in the practical importance of mathematical models to the applied worker, as well as the rich source of new problems to challenge the theoretical probabilist.

In the first or premathematical stage the qualitative formulation of a problem in applied mathematics originates in a related field of science. This stage usually does not require computation.

The second stage deals with the development of the formal properties of the problem; the latter becomes embedded in a mathematical structure and the relevant questions are stated in precise mathematical terms. This step usually results from a dialogue between those interested in the quantitative aspects of an application and mathematicians interested in reality-related problems. This dialogue may last for many decades and for certain problems of physics has continued for centuries. Many of us have been involved with instances of this process and we are all aware of the somewhat different criteria and values held by the mathematician and the applied worker in such discussions. For the applied researcher, the mathematical model stands or falls on the degree of agreement between its theoretically predicted values and the empirical results of careful observation. He is also interested in specific questions of significance to his speciality; the model is of use only if it gives satisfactory answers to these questions, and this often regardless of the inherent mathematical elegance of the structure. The mathematician is more charmed by the intellectual beauty of the problem for its own sake. By the nature of his training, he searches for generality and elegance, sometimes at the expense of the specific questions raised by the practioner.

Much of the strength of modern mathematics springs from its concern with generality and elegance. In my opinion however, this concern has also led to a truncation of the solution concept in much of mathematics. An elegant existence - and - uniqueness theorem may be admired, while a good algorithm for the computation of that unique solution is disdained as "a routine application of the theory".

In the theory of queues, my area of research for the past years, this truncation is particularly apparent. Practioners tell me that most papers require a large expenditure of time and effort to read, only to discover in the end that few, if any, specific questions are answered by the intricate structural and analytic material in the paper. After a few frustrating experiences, many practioners turn exclusively to simulation studies or use only the most trivial explicit results of the theory. With some justification, they may even become hostile to all theoretical work and question the support given to this type of mathematical research.

At the risk of yet another digression, it is worthwhile to compare the evolution of mathematical programming to that of the theory of probability models. A comparison between those two fields is appropriate; they blossomed during the same period in time and occupy substantially similar positions in relation to applications and to pure mathematics. Mathematical programming has a much narrower base of formal mathematical structure, yet its social impact far exceeds that of probability theory. In my estimation, this is due to the ardent concern for good algorithms for the solution of linear and nonlinear programs, on the part of the same talented mathematicians to whom we owe the theoretical developments.

Taking an example from statistics, a distinguished colleague drew my attention to some highly regarded optimality results in large sample theory, which are mathematically correct, but only for sample sizes of the order of 10^{20}!! Such sample sizes are hardly meaningful, but it took substantial theoretical insight to see that such forbidding limitations were inherent in this work.

I believe it is in the best tradition of science for society to require that theoretical results be backed up by an assessment of their concrete range of applicability. One important way to do this is through numerical analysis. Having the power of the computer available to us, we can as a community no longer afford not to do so.

Passing to another aspect of research, if the mathematician hopes to be at all successful in solving a problem, he must be able to relate it quite closely to the realm of presently known mathematics. For example, a problem leading to a nasty partial differential equation of higher order with nonlinearities will probably not receive much mathematical attention. Even today we do not have a good theoretical framework for the analysis of such problems.

This shows the other side of the coin. Without a sound understanding of the structure involved, computation is usually meaningless. However relevant and urgent a problem might be, the quality of computer solutions only reflects the degree of understanding of the designer of the algorithm.

The third stage is reached when the problem is well formulated and its investigation is likely to be fruitful. A list of precise questions can now be drawn up; some of these arise from the area of application, some originate with the mathematician and others still spring from the answers to earlier questions. At this stage, classical applied mathematics begins its quest for explicit solutions, which express the quantities of interest in terms of known functions. This endeavor has evolved the parts of mathematics, colloquially referred to as "hard analysis". Explicit solutions can be derived only in rare cases however. Even then, the resulting expressions may not be very elucidating, nor lend themselves well to numerical analysis. In the absence of simple explicit results, the mathematician looks for qualitative theorems, which are primarily of two types. The first type clarifies the nature of the solution, rather than its explicit form. Examples abound in the qualitative theory of differential equations. Equilibrium conditions for queues are results of this type in probability theory. The second type encompasses limit and approximation theorems. These are much more useful to numerical work, as they facilitate computations and are useful in testing the validity of the model at least in the range of the approximation or limit results.

The central limit theorem of probability is an outstanding result of this type. Among its many merits, it accounts for the robustness of most useful statistical procedures. The diffusion approximations of recent vintage are other important results of this type. However even for such theorems of central importance, far too little numerical work has been done to date. Valid limit theorems may be practically useless or misleading, when very slow convergence is involved. While this is well-known in classical analysis, the corresponding pitfalls in probability theory are not so clearly marked. Let me interrupt this general discussion with an illustrative example.

Consider a sequence of independent Bernoulli trials in which the probability of success at the n-th trial is $p_n = \frac{1}{n}$, and denote the number of successes in n trials by X_n. An application of the central limit theorem yields that

$$\lim_{n \to \infty} P\{X_n \leq \log n + x\sqrt{\log n}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du. \tag{1}$$

This result is so easy to apply that one is tempted to use it as an approximation of the distribution $F_n(x) = P\{X_n \leq x\}$, rather than to compute the latter. I used both methods for n=1000, and offer the following numerical results for your consideration.

Table 1

$F_{1000}(x)$, for $x = 0, \ldots, 15$

x	exact	normal approximation
1	0.00100	0.000 003
2	0.00848	0.000 094
3	0.03567	0.001 471
4	0.09980	0.013 460
5	0.21059	0.073 289
6	0.36047	0.244 857
7	0.52615	0.527 980
8	0.68028	0.797 057
9	0.80365	0.944 320
10	0.89003	0.990 690
11	0.94365	0.999 077
12	0.97348	0.999 947
13	0.98850	0.999 998
14	0.99538	1.000 000
15	0.99828	1.000 000

The approximation is clearly atrociously poor and remains so, even for n=10000. In this simple case, it is easy to see theoretically why the convergence is so slow, but many limit theorems are being recommended to the practioner without any concern for their quality as approximations. In an informal survey, many students assured me that the central limit theorem "holds" as soon as one has thirty or more summands. In the case of non-identically distributed random variables, they may be in for unpleasant surprises.

Without denying the genuine value of limit and approximation theorems, we should be much more prepared to use the structural features of probability models to perform exact calculations. As long as we remain committed to the scientific method over scholastic speculation, we must validate our models by comparing their numerical results to empirical quantities. Rejecting a valid model, on the basis of a poor approximation theorem is a scientific tragedy.

A word about numerical tables is in order here. One of the barriers to scientific computation is the considerable difficulty of getting tables published. The economic reasons for this are sound. Extensive tables are costly to reproduce and take up journal space, which can be put to better use. The number of users of any given table is typically small. The obvious alternative to this is to make well-documented computer programs in a general purpose language available to the scientific community. Most scientists today have ready access to a computer at their home institutions. They may, in most cases, compute a table for the range of values of interest to them at a fraction of the cost of general purpose tables.

Costly tables should be computed once and could be available, through a clearing house, on punched cards or magnetic tapes. An example of this approach is the table of coefficients for use in Faá di Bruno's formula which was computed by Professor E. Klimko of Purdue University. This table of more than a quarter million entries exists on a magnetic tape in a format usable on all Control Data computers of the 6000 series.

As of today, there exist too few vehicles to convey such material to the scientific community. This is, in my opinion, a regretable situation, since I have seen many computer programs of far greater import than many marginal papers which are given journal space.

The fourth stage in the development of a problem of applied mathematics is again outside of the realm of mathematics. It is the stage of validation and testing against empirical data. After this stage, either a scientific discovery is made or the problem returns to stage one and stimulates further investigation to explain the negative results of the proposed model. In comparison with the amount of model building and structural theory, too few studies of validation are undertaken. I am convinced that this is due in large part to the reticence of many workers to do numerical work. Some of the more valid criticisms of science, which find wide currency today, might be avoided by pushing more scientific investigations to this final stage. This will certainly involve a much greater effort in computational mathematics.

Today few of us are involved with more than one of these four stages. The most fruitful use of the computer requires its integration into the latter three stages. A fairly broad range of interests and competence is therefore needed of those who become involved in computational mathematics.

After this general discussion, I now turn specifically to the area of probability models. Through illustrative examples, I shall argue for the following three theses:

1. The computer is an invaluable tool to the teaching of probability;

2. Well-planned computation enhances the understanding as well as the practical value of probability models;

3. Far from being a threat to theoretical investigation, computer investigations raise large numbers of theoretically important questions. Every algorithm has natural limitations which call for approximation theorems. Every large computation poses serious problems of efficiency, which can only be resolved through a profound understanding of the theoretical structure, both of the problem and of the computer.

I now turn to these theses in order.

The Computer in the Teaching of Probability

Our non-measure-theoretic texts on probability or stochastic processes place nearly exclusive emphasis on special distributions and on problems with easily tractable answers. The authors and lecturers properly do not want to burden the student with problems, requiring tedious analytic calculations with unwieldy answers that offer little insight. As a result however, many students underestimate the value of structure and overestimate that of "explicit" simple answers. As an illustration, we consider problems involving two classical games, table tennis and billiards.

Table tennis

Let the successive points in a game between players I and II be modeled as Bernoulli trials. Denote by p the probability that I wins any given point. The probability that player I wins a set is found by setting n=21 in the formula

$$P = \sum_{r=0}^{n-2} \binom{n+r-1}{n-1} p^n q^r + \binom{2n-2}{n-1} p^n q^{n-1} \left(\frac{p}{p^2 + q^2} \right). \tag{2}$$

This is readily shown by a combinatorial argument, which students with little mathematical maturity find rather difficult. If one generalizes the problem by saying that in order to win, a player must score at least n points, and must lead his opponents by at least $k \geq 2$ points, then a result from the gambler's ruin problem is needed to show the probability that player I wins a set is given by

$$(3) \qquad P = \sum_{r=0}^{n-k} \binom{n+r-1}{n-1} p^n q^r + \frac{p^k}{p^k + q^k} \sum_{j=1}^{k-1} \binom{2n-j-1}{n-1} p^{n-j} q^{j-1} (p^j + q^j),$$

This pushes the problem almost beyond the scope of an introductory probability course. It is possible however to present a computational solution, which illustrates all the probabilistic arguments involved, yet is accessible to high school students. Denote by $A(i,j)$ the probability that I wins, given that the score has reached i for I and j for II, then for a general n and k=2, the $A(i,j)$ satisfy the equations

$$(4) \qquad A(i,j) = pA(i+1,j) + qA(i,j+1),$$

for $i \geq 0$, $j \geq 0$, with boundary conditions

$$(5) \qquad \begin{aligned} &A(n,j) = 1, && 0 \leq j \leq n-2, \\ &A(i,n) = 0, && 0 \leq i \leq n-2, \\ &A(n,n-1) = A(n+\nu,\ n+\nu-1), && \nu \geq 0, \\ &A(n-1,n) = A(n+\nu-1,\ n+\nu), && \nu \geq 0, \\ &A(n-1,n-1) = A(n+\nu,\ n+\nu), && \nu \geq 0. \end{aligned}$$

We can readily show that

$$(6) \qquad A(n,n) = \frac{p^2}{p^2 + q^2},$$

$$A(n-1,n) = \frac{p^3}{p^2 + q^2},$$

$$A(n,n-1) = \frac{p^3 + pq}{p^2 + q^2}.$$

Setting $B(i,j) = A(n-i, n-j)$, we obtain a simple recurrence relation with initial conditions i.e.

$$(7) \qquad \begin{aligned} B(0,0) &= p^2/(p^2 + q^2), \\ B(1,0) &= p^3/(p^2 + q^2), \\ B(0,1) &= (p^3 + pq)/(p^2 + q^2), \\ B(0,j) &= 1, && 2 \le j \le n, \\ B(i,0) &= 0, && 2 \le i \le n, \\ B(i,j) &= pB(i-1,j) + qB(i,j-1), \end{aligned}$$

for $1 \le i \le n$, $1 \le j \le n$. For general k, only the initial values of $B(i,0)$ and $B(0,j)$ are different.

Programming these recurrence relations for the computer is trivial. The probability that I wins a set is given by $B(n,n)$. Several conditional probabilities, whose explicit formulas are uninspiring, may be readily computed from the array $B(i,j)$. If one further assumes that the probability of I winning a point is p_1, if I serves and is p_2, if II serves, then the explicit probability, which generalizes (2) is messy and involves quite tedious calculations. In contrast, generalizing the recurrence relations (7) and the computer program to handle this case, involves only trivial changes in the formulas (7).

The following is a short table of the probability P for n=21, k=2, given for those interested in the lore of table tennis, this delightful game of recent global significance.

Table 2

The probability of winning a set in table tennis

p	P
0.40	0.091
0.41	0.116
0.42	0.144
0.43	0.177
0.44	0.213
0.45	0.254
0.46	0.298
0.47	0.346
0.48	0.396
0.49	0.447
0.50	0.500

Billiards

Probability problems related to billiards are beyond most undergraduates, largely because of the complicated expressions involved. They are not commonly found in probability texts. The game may be formalized as follows. Successive trials are modeled as independent Bernoulli trials. If player I is at play, he makes a point with probability p_1; the corresponding probability for player II is p_2. Player I wins if he scores n_1 points before player II scores n_2 points. Player II wins in the alternate case. The game starts with a turn for player I and he continues playing for as long as his trials are met with success. At the first failure of player I, his opponent gets a turn which lasts until his first failure. The players take turns alternatingly until one of them reaches his allotted score. Many probabilities related to billiards were calculated explicitly by O. Bottema and S.C. Van Veen and published in two papers in 1943 [3]. These papers are in Dutch and are not easily accessible.

Among the many problems, related to this game consider <u>the handicap problem</u>. Let $p_1 > p_2$, and let n_1 be given. Find n_2 so that the game is approximately even. More precisely, let $P_1(n_1,n_2)$ be the probability that I wins if he starts and $P_2(n_1,n_2)$, the probability that I wins if II starts the game. For given ϵ, p_1,p_2 and n_1, we wish to find the values n_2 for which $P_1(n_1,n_2)$ or $P_2(n_1,n_2)$ or both, are in the interval $(0.5-\epsilon,\ 0.5+\epsilon)$.

Assuming that the formulas in [3] have been derived in class, the computation of n_2 is still not easy, since the explicit expressions are not well suited for numerical computation. There is however, a computational solution, which can be done by a freshman class; it completely obviates the messy analytic calculations. It suffices to note that the probabilities $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ satisfy the recurrence relations, $(p_1+q_1 = p_2+q_2 = 1)$,

$$\begin{aligned} P_1(n_1,n_2) &= p_1P_1(n_1-1,n_2) + q_1[1-P_2(n_1,n_2)], \\ P_2(n_1,n_2) &= p_2P_2(n_1,n_2-1) + q_2[1-P_1(n_1,n_2)], \end{aligned} \tag{8}$$

for $n_1 \geq 1$, $n_2 \geq 1$, with boundary conditions

$$\begin{aligned} P_1(n_1,0) &= P_2(n_1,0) = 0, \\ P_1(0,n_2) &= P_2(0,n_2) = 1, \end{aligned} \tag{9}$$

for $n_1 \geq 1$, $n_2 \geq 1$. The equations (8) may be written as

$$\begin{aligned} P_1(n_1,n_2) &= \frac{p_1}{1-q_1q_2} P_1(n_1-1,n_2) - \frac{p_2q_1}{1-q_1q_2} P_2(n_1,n_2-1) + \frac{q_1p_2}{1-q_1q_2}, \\ P_2(n_1,n_2) &= -\frac{p_1q_2}{1-q_1q_2} P_1(n_1-1,n_2) + \frac{p_2}{1-q_1q_2} P_2(n_1,n_2-1) + \frac{q_2p_1}{1-q_1q_2}. \end{aligned} \tag{10}$$

By using a scheme analogous to Pascal's triangle, the probabilities $P_1(n_1,n_2)$

and $P_2(n_1,n_2)$ may be computed recursively. Whenever the first index is equal to the given value n_1, it suffices to examine for which n_2, the numerical values of $P_1(n_1,n_2)$ and $P_2(n_1,n_2)$ lie in the stated interval about 0.5. For further details, see [11]. This problem is pedagogically interesting, since it prepares the students for ideas arising in renewal and inventory theory.

I conclude the discussion of the educational value of computational probability, by noting that in the preparation of a forthcoming text book in this field, Professor Klimko and I found few problems, currently in books on probability, which are sufficiently complex to demonstrate the great power of the modern computer. In contrast, a formidable combinatorial problem such as the joint distribution of the Goren point count in Bridge (for all four hands!) is quite accessible to the intermediate student and will be included in the text.

The Computer and Probability Models

Probability models in queues, dams, inventories, reliability, epidemics and learning offer a rich and largely untapted sourse of computational problems. To treat any one of these in detail is beyond the limitations of this discussion. For some detailed examples, we refer to the papers [1, 5, 7, 8, 9, 10]. The algorithms which are developed there, use the special structural properties of the queueing models to the fullest extent. They offer an alternative to the intricacies of the nonlinear integral equations and to the curse of Laplace transforms, which make much of the theory of these models inaccessable to the practical worker.

In [9] an algorithm is developed for a single server queue. It can easily give numerical results on the transient behavior for such queues in discrete time over long time intervals. Queue lengths up to one hundred can be handled, using computer times which are between 1/10 and 1/5 of those needed for simulation studies. The numerical accuracy is far greater than that of simulation methods.

By use of the special properties of the equations for the stationary phase, the limit behavior of queues up to a maximum length of eight hundred was studied numerically in [7III].

While our early efforts dealt only with models from queueing theory, it is clear that a similar approach can be fruitfully applied to machine repair models and to small scale epidemics.

On the other hand, the limitations of the algorithms are also clear. Many server queues, for instance, present as many computational as analytic difficulties. It must be remembered however that even for the single server there are only a few explicit analytic results available. Several attempts by other workers [8] have shown that the classical results on the transient behavior even for the M|M|1 queue (which is of limited utility), are not well-suited for numerical evaluation.

Just as there may be several approaches to the analytic solution of the classical models, there are usually also a number of alternate ways of organizing their numerical solution. These may differ substantially in the amounts of computer time and memory storage involved. The computer study of large stochastic models requires good probabilistic insight and also a degree of skill, which can only be acquired through experience. For obvious economic reasons, careful planning of the algorithm is indicated. Truly large scale stochastic models involve much more than routine computer programming. In many cases they may require a team effort by probabilists and computer specialists.

In many instances, well-planned algorithms also offer an intellectually welcome alternative to the panacea of simulation. This is particularly welcome in the study of unstable queues, which are difficult to simulate accurately and for which the time dependent behavior is of paramount importance.

Even for stable queues, which exhibit considerable fluctuations the numerical study of the transient behavior is important. In the paper [7III], an example is given of such a queue. The most noteworthy aspect of this example is that all the simple formulas for the stationary phase, which are commonly taught to the practioners of applied queueing theory, yield results which are highly misleading when used in decisions of design and control. A similar remark applied to priority queues. Many economic decisions based on the rare explicit, asymptotic and average results may have adverse effects on the behavior of the queue. These effects cannot be inferred from the complicated analytic expressions arising in the theory of priority queues, and require much further numerical study.

The numerical analysis often requires assumptions which yield computational tractability, such as the use of a discrete time parameter. The merits of such assumptions are discussed in [9]. Their introduction is comparable to the Markovian and other assumptions which are routinely stated in analytical investigations. There is still considerable reticence in accepting assumptions dictated by computational needs. These require a reexamination of the nature of applied mathematics, such as suggested by R. Bellman in [2].

Unexpected and exciting results abound in this work and one example in point is the following. In examining the higher moments of the busy period distribution for queues of the classical $M|G|1$ type Professor E. Klimko and I, [7II] were led to the computation of higher order derivatives of composite functions of the form $f_1[f_2[f_3[f_4(x)]]]$. Beyond the second or third derivative, even the most hardy calculator becomes bogged down in the messy expressions which arise. We recalled the classical formula of Fáa di Bruno, published in 1857, to wit:

(11) $$\frac{d^n}{dx^n} f[g(x)]_{x=0} = \sum_{r=1}^{n} f^{(r)}[g(0)]Y_{n,r},$$

where

(12) $$Y_{n,r} = \sum_{\substack{j_1+\ldots+j_n=r, \\ j_1+2j_2+\ldots+nj=n, \\ j_1 \geq 0,\ldots,j_n \geq 0.}} \frac{n!}{j_1!\ldots j_n!} \left[\frac{g^{(1)}(0)}{1!}\right]^{j_1} \ldots \left[\frac{g^{(n)}(0)}{n!}\right]^{j_n}.$$

The number of elementary terms in the right hand side of (11) grows slowly for n small, but for n=40, reaches 37,338 and is equal for n=50, to 204,226. By use of a tightly written computer program, it was possible to compute all the sets of indices j_ν appearing on the right hand side of (12) and to differentiate three and fourfold functional iterates up to order forty with central processing times of 140 seconds approximately. Through comparisons with the few cases where the moments can be calculated explicitly, we were also able to demonstrate the astounding numerical accuracy of our algorithm.

Some Theoretical Questions arising from Computational Probability

From the abundant collection of such questions, I select some for purposes of illustration. Many are of interest to numerical analysis generally. We do not have e.g. good algorithms for the evaluation of convolution products, at this time. Some algorithms for special functions and some based on transform inversion are available, but their practical utility is limited.

Other problems arise from the need to approximate expressions, whose direct evaluation is prohibitively expensive. For instance, let $F(\cdot)$ be a probability distribution and $A_n(\cdot)$ a sequence of nondecreasing mass functions such that $\sum_{n=0}^{\infty} A_n(\cdot)$ is a probability distribution. Expressions of the general form (13), as well as their matrix analogues, arise frequently in probability problems:

$$(13) \qquad \sum_{n=0}^{\infty} \int_0^x F^{(n)}(x-u)\,dA_n(u).$$

$F^{(n)}(\cdot)$ is the n-fold convolution of the distribution $F(\cdot)$ with itself. Such expressions are difficult to compute for large x and bounds or good approximations for large x would be very desirable, but do not appear to be available.

Finally, many difficult problems are related to what Bellman calls "the curse of dimensionality". Stochastic models, say m-server queues, lead immediately to multidimensional stochastic processes. These exceed, even for small values of m, the storage capacity of modern computers and lead also to prohibitive computer times. The numerical solution of such models, if it ever comes about, will result from a thorough use of approximation results, combined with many of the more esoteric software techniques available to the modern computer expert.

Concluding Remarks

The numerical investigation of probability models is an essential, but underdeveloped part of their solution. For complex problems, the difficulties of the numerical analysis are comparable to those of an analytic discussion. A thorough understanding of the structural properties is essential to a well-planned algorithm.

For many of the classical models, the use of the computer will have a liberating effect. The analytic complexity of the current models is likely to confine and stunt the future growth of our field of endeavor, but computational work promises to give a viable alternative and to stimulate further research.

To do work in computational mathematics is not an abdication in the face of analytic complexity, but a commitment to a more demanding definition of what consititutes the solution to a mathematical problem. When done properly, it conforms to the highest standards of scientific research.

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