Moments, Product Moments and Percentage Points of the Order Statistics from the Lognormal Distribution for Samples of Size Twenty and Less

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1. Introduction and Summary

This paper deals with the computation of the moments and percentage points of the order statistics of the lognormal distribution. This distribution is applicable in a variety of situations, as for example in reliability and life test problems when the time to failure arises as the multiplicative effects of many random phenomena [2,3]. The lognormal family of distributions has been proposed as a reasonable family for describing the length of time to repair a piece of equipment, partly because of its mathematical tractability and partly because the failure rate increases at first and then eventually decreases to zero. The average concentration of an air pollutant such as sulfur dioxide, carbon monoxide, nitrous oxides, etc., is approximately lognormally distributed [12]. In a recent paper by Singpurwalla [17], an attempt is made to show how certain empirical relationships observed in an analysis of air pollution data can be interpreted using extreme-value theory applied to the maximum term of a random series from the lognormal law. Lognormal models have also been used in diverse sets of medical and environmental problems which include incubation periods during an epidemic of streptococcal sore throat and systolic blood measurements among middle-aged industrial employees [10]. For other applications in econometrics reference is made to Aitchison and Brown [2].

Order statistics in themselves have received wide attention in recent years as their usefulness and properties are investigated and

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systematically collected [6,16]. In particular, the order statistics from the lognormal distribution arise in the study of Munro and Wixley [14]. Estimation in lognormal linear models has been considered by Lambert [11] and Bradu and Mundlak [5]. The means, variances and other moments of the various order statistics have direct bearing in all problems where the lognormal distribution is an appropriate model. It should be pointed out that the moments of the lognormal order statistics cannot be computed by a transformation of the corresponding quantities for the normal distribution [7,18].

Section 2 of this paper derives the formulae for the moments, product moments and percentage points of the order statistics from a standard lognormal distribution. Section 3 discusses the method of computation for the moments, percentage points and product moments of the order statistics, which are given in Tables 1, 2 and 3 respectively. An overall error percentage has been computed in the analysis as a partial check on the accuracy of the computations of the moments. In addition, the moments and product moments of the order statistics from a normal distribution have been computed using the method herein described. These results were then compared with existing tables to ascertain the accuracy of computation. Finally, in Section 4 applications of the included tables are indicated for least squares estimation of location and scale parameters.

2. Moments of the Order Statistics

Let X have a standard lognormal distribution; i.e., let the density of the random variable X be given by

$$(2.1) \qquad f(x) = \frac{1}{x\sqrt{2\pi}} e^{-(\ln x)^2/2}, \qquad 0 < x < \infty.$$

Let F(x) denote the cumulative distribution function

$$(2.2) \qquad F(x) = \int_0^x f(t)dt \ .$$

Suppose that we have n independent random variables $X_1, X_2, \ldots, X_n$ each distributed according to the lognormal density function (2.1). Let $X_{(k)}$ denote the k^{th} order statistic from the sample $X_1, \ldots, X_n$; i.e.,

$$(2.3) \qquad X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)} \ .$$

The j^{th} moment about the origin, $\mu_j'(k,n)$, of the k^{th} order statistic from a sample of size n from the standard lognormal distribution is given by

$$(2.4) \qquad \mu_j'(k,n) = k\binom{n}{k} \int_0^\infty x^j [F(x)]^{k-1}[1-F(x)]^{n-k} f(x)dx \ .$$

Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal cumulative distribution function and probability density respectively. Then $F(x) = \Phi(\ln x)$, $x > 0$, and

$$(2.5) \qquad \mu_j'(k,n) = k\binom{n}{k} \int_{-\infty}^\infty e^{jx} [\Phi(x)]^{k-1}[1-\Phi(x)]^{n-k} \phi(x)dx \ .$$

Let

$$(2.6) \qquad g(x) = e^{jx\sqrt{2}}[\Phi(x\sqrt{2})]^{k-1}[1-\Phi(x\sqrt{2})]^{n-k}, \qquad -\infty < x < \infty.$$

Then equation (2.5) can be written as

$$(2.7) \qquad \mu_j'(k,n) = \pi^{-1/2} k \binom{n}{k} \int_{-\infty}^{\infty} g(x)\, e^{-x^2}\, dx ,$$

which may be numerically evaluated using Hermite quadratures. Since the sum of the expectations of the moments of the ordered random variables is the same as the corresponding sum of the unordered random variables, it follows that

$$(2.8) \qquad \sum_{k=1}^{n} \mu_j'(k,n) = \sum_{k=1}^{n} E(X_k^j)$$

$$= ne^{j^2/2} .$$

The corresponding central moments of the order statistics, i.e., the moments about the mean, will be denoted by $\mu_j(k,n)$ and can be obtained from the relation

$$(2.9) \qquad \mu_j(k,n) = \sum_{r=0}^{j} (-1)^{j-r} \binom{j}{r} \mu_r'(k,n) (\mu_1'(k,n))^{j-r} .$$

The product moments of the order statistics can be similarly expressed. Letting $1 \leq i < j \leq n$, then $E(X_{(i)}X_{(j)})$ can be written as

$$(2.10) \qquad E(X_{(i)}X_{(j)}) = C \int_{-\infty}^{\infty} q(x) e^{-x^2}\, dx ,$$

where

$$(2.11) \qquad C = C(i,j,n) = i(j-i) \binom{n}{i,j-i,n-j} ,$$

and

$$(2.12) \qquad q(x) = \int_{-\infty}^{x} e^{x+y+x^2} [\Phi(y)]^{i-1} [\Phi(x) - \Phi(y)]^{j-i-1} [1-\Phi(x)]^{n-j} \phi(x)\phi(y)dy \ .$$

To evaluate (2.10) one first computes (2.12) for a fixed x by using, for example, Gauss-Laguerre quadrature. Then the integral in (2.10) can be computed with the Gauss-Hermite quadrature. Since the sum of squares of the ordered random variables is the same as the corresponding sum of the unordered random variables, it follows that

$$(2.13) \qquad \sum_{i=1}^{n} \sum_{j=i+1}^{n} E(X_{(i)}X_{(j)}) = n(n-1)[E(X_1)]^2/2$$

$$= n(n-1)e/2.$$

The covariance of $X_{(i)}$ and $X_{(j)}$ is given by

$$(2.14) \qquad \mathrm{Cov}(X_{(i)},X_{(j)}) = E(X_{(i)}X_{(j)}) - E(X_{(i)})E(X_{(j)}) \ ,$$

where $E(X_{(i)}X_{(j)})$ is given by (2.10) and the first moments are computed from (2.7).

The 100α percentage points of the k^{th} order statistic in a sample of size n can be obtained from the incomplete beta integral, $I_x(p,q)$. Let ξ_α be defined by the equation $P[X_{(k)} \leq \xi_\alpha] = \alpha$. Then it follows that

$$(2.15) \qquad \alpha = P[X_{(k)} \leq \xi_\alpha]$$

$$= k\binom{n}{k} \int_{0}^{F(\xi_\alpha)} u^{k-1}(1-u)^{n-k} du$$

$$\equiv I_{F(\xi_\alpha)}(k,n-k+1) \ .$$

Thus for a given α, $F(\xi_\alpha)$ can be determined [4,9,15] and $F(\xi_\alpha) = \Phi(\ell n\, \xi_\alpha)$ yields

$$(2.16) \qquad \xi_\alpha = \exp[\Phi^{-1}(F(\xi_\alpha))],$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal cdf.

3. Method of Computation and Error Percentages

All computations were performed on the IBM 370/165. The integral appearing in equation (2.7) was evaluated using the Gaussian-Hermite quadrature formula with 64 nodes. The subroutine is available in the IBM Fortan Subroutine Library (QH64) in a double precision format for both the weights and nodes. The function $g(x)$ appearing in (2.7) was evaluated at the nodes by calling a subroutine NORM, which computes $\Phi(x)$ and $\phi(x)$, for a given value of x. This subroutine is adapted from Abramovitz and Stegum [1]. A comparison of the output of this subroutine with the normal probability functions tabulated in [19] showed NORM to be accurate to at least fourteen decimal places.

In light of the relation given in (2.8), an overall error percentage for the central moments can be defined, for each j and n, as follows:

$$(3.1) \qquad O_j(n) \equiv 100[\mu_j'(\cdot,n) - ne^{j^2/2}]/ne^{j^2/2} \ ,$$

where

$$(3.2) \qquad \mu_j'(\cdot,n) = \sum_{k=1}^{n} \mu_j'(k,n) \ .$$

It is necessary for these overall percentages to be small if the computations are accurately executed; however, the smallness of these measures is not sufficient to guarantee accuracy of computation, although it would certainly preclude certain types of errors. In our computations, the percentages $0_j(n)$ satisfied $-2.7(10^{-5}) \leq 0_j(n) \leq -4.4(10^{-4})$, $j=1,\ldots,4$; $n=1,\ldots,20$.

As a further check on the method of computation, $g(x)$ as defined in (2.6) was modified so that equation (2.7) now yielded the moments of order statistics from a normal population. The computation of the first two moments was made employing the subroutines and method described above. These results were then compared with the corresponding results of Teichroew [18]. Similar accuracy can be expected in the lognormal computations. For n less than 13, the results were in agreement through at least five significant digits; for n less than 17 through at least four significant digits; and at none of the cases investigated was agreement ever less than three significant digits. For n=20, the percentage difference between our calculations and those in [18], ranged from 10^{-4} to .18. The high and low values were achieved for k=1 and k=10 respectively for both of the moments which were checked.

The percentage points in Table 2 were computed by rewriting (2.15) in the form

$$(3.3) \qquad \alpha = \sum_{\ell=k}^{n} a_\ell (-1)^{\ell-k} F^{\ell}(\xi_\alpha) ,$$

where

$$(3.4) \qquad a_\ell = \binom{n}{\ell}\binom{\ell-1}{k-1} , \qquad 1 \leq k \leq \ell \leq n .$$

For a given n,k and α, the quantity $F(\xi_\alpha)$ was determined using a Newton-Raphson algorithm with a 10^{-6} convergence criterion. An initial estimate of the solution $F(\xi_\alpha)$ was taken to be

$$(3.5) \qquad \hat{F}(\xi_\alpha) = 1 - g_n(1-\alpha) + (k-1)[g_n(\alpha) - 1 + g_n(1-\alpha)]/(n-1),$$

where $g_n(x) = x^{1/n}$, $x > 0$. For a fixed n, this is simply a linearized estimate of the exact values obtained at the extremes k=1 and k=n, and appeared to be quite adequate for rapid convergence. The percentage point ξ_α was then obtained from equation (2.16). The inverse normal distribution function employed in these calculations, is described in Milton and Hotchkiss [13]. It should be noted that the natural logarithm of the percentage points for the lognormal order statistics are the percentage points of the same order statistics from a normal population. Percentage points of order statistics from the normal distribution have been tabulated by Gupta [8] for α = .50, .75, .90, .95 and .99, and for all the order statistics from samples of size 10 or less, and for selected order statistics from samples of size 11 to 20. Thus, Table 2 can be used to extend the tabulations given in [8] for the normal distribution. As a further check on the accuracy of ξ_α, the tabulated values of ξ_α were used to compute $I_{F(\xi_\alpha)}(k,n-k+1)$ for n=5(5)20 and all values of k for α = .50, .75, .90, .95 and .99. Here the incomplete beta function was evaluated by using a different method, namely, the continued fraction expansion. In all cases it was found that $I_{F(\xi_\alpha)}(k,y-k+1) - \alpha| \leq 8.10^{-7}$.

The product moments in Table 3 were computed from equations (2.10) to (2.12) using the subroutines NORM, QH64 and QL32. This subroutine QL32 employs the Gauss-Laguerre quadrature formula with 32 nodes, and is available in the IBM Fortran Subroutine Library in the same precision as QH64. As in the case of the central moments, an overall error percentage for the product moments can be defined using (2.13), for each value of n, as follows:

$$(3.6) \qquad Q(n) \equiv 100[\sum_{i \neq j}\sum E(X_{(i)}X_{(j)}) - n(n-1)e]/n(n-1)e.$$

In our computations, the percentages Q(n) satisfied $0 < Q(n) < 1.7(10^{-8})$.

As before, q(x) as defined in (2.12) was modified so that the computation schemes yielded the product moments of the order statistics from a normal population. These results were then compared with the corresponding values in Teichroew [18], and similar accuracy should prevail in the lognormal computations. For n less than 8, the results were in agreement through at least five significant digits; for n=10 and the various cases computed for n=15, the agreement was never less than three significant digits; and only in one of the investigated cases for n=20, did agreement drop to two significant digits.

4. Least Squares Estimation of Location and Scale Parameters

Let X have a standard lognormal distribution and suppose that a linearly transformed process of X is generating the observations; i.e., $Y = \sigma X + \mu$, $0 < \sigma < \infty$, $|\mu| < \infty$, is the random variable being observed. The expected value of Y is $\sigma\, e^{1/2} + \mu$ and the standard deviation is $\sigma\, e^{1/2}(e-1)^{1/2}$. Equivalently, the r.v. Y has a three parameter lognormal distribution $\Lambda(\mu, \ell n\sigma, 1)$ (see the notations in Aitchison in Brown [2]) where μ is the threshhold parameter and $\ell n\sigma$ and 1 are, respectively, the mean and variance of the transformed variable $\ell n\ (Y-\mu)$. Note that these estimators yield the best linear unbiased estimators of the mean and standard deviation of Y since these are estimable functions of μ and σ. Now we discuss the best linear unbiased estimators of μ and σ.

Since

$$Y_{(i)} = \sigma\, X_{(i)} + \mu, \qquad i = 1,\ldots,n,$$

then

$$EY_{(i)} = \sigma\alpha_i + \mu, \ \mathrm{Cov}(Y_{(i)}, Y_{(j)}) = \sigma^2\beta_{ij},$$

where the α_i and β_{ij} are the expected value of the i^{th} order statistic and covariance of the i^{th} and j^{th} order statistics from a sample of size n drawn from a standard lognormal population, i.e. $E(X_{(i)}) = \alpha_i$, $\mathrm{Cov}(X_{(i)}, X_{(j)}) = \beta_{ij}$. These α_i and β_{ij} are essentially given in Table 1 and Table 3 for samples of size 20 and less. The Gauss-Markov least-squares theorem may therefore be applied to give unbiased estimators of μ and σ, say μ^* and σ^*, which have minimum variance in the class of linear unbiased estimators [6, pp. 102]. These estimators are given by

(4.1) $$\mu^* = -\underline{\alpha}'\,\Gamma\,\underline{Y}$$

and

(4.2) $$\sigma^* = \underline{1}'\,\Gamma\,\underline{Y},$$

where $\underline{Y}$, $\underline{\alpha}$ are, respectively, the column vectors of the $Y_{(i)}$, α_i; $\underline{1}$ is a column of n 1's, and Γ is the skew-symmetric matrix

(4.3) $$\Gamma = \frac{\Omega(\underline{1}\,\underline{\alpha}' - \underline{\alpha}\,\underline{1}')\Omega}{\Delta} .$$

The matrix $\Omega = B^{-1}$ where B is the covariance matrix of the $X_{(i)}$, and Δ is the determinant of the matrix $A'\Omega A$, where $A = (\underline{1}, \underline{\alpha})$. The variances and covariance of these estimators are given by

(4.4) $$\mathrm{Var}(\mu^*) = \frac{\underline{\alpha}'\Omega\underline{\alpha}\sigma^2}{\Delta} ,$$

$$(4.5) \qquad \mathrm{Var}(\sigma^*) = \frac{\underline{1}'\Omega\underline{1}\sigma^2}{\Delta} ,$$

and

$$(4.6) \qquad \mathrm{Cov}(\mu^*,\sigma^*) = \frac{-\underline{1}'\Omega\underline{\alpha}\sigma^2}{\Delta} .$$

As an example now, suppose five observations are available on Y, and it is desired to estimate μ and σ. From Table 1 and Table 3, we have

$$(4.7) \qquad \underline{\alpha}' = (.38568, .71024, 1.15434, 1.92348, 4.06986)$$

and

$$(4.8) \qquad B = \begin{pmatrix} .06807 & .06714 & .07257 & .08487 & .11822 \\ & .17368 & .18860 & .22079 & .30715 \\ & & .44537 & .52178 & .72431 \\ & & & 1.46575 & 2.02352 \\ & & & & 12.54316 \end{pmatrix}$$

The estimators μ^* and σ^* are then given by

$$(4.9) \qquad \mu^* = \sum_{i=1}^{5} a_i Y_{(i)}, \qquad \sigma^* = \sum_{i=1}^{5} b_i Y_{(i)},$$

where

$$(4.10)\quad \begin{cases} a_1 = 1.3924 & b_1 = -1.1098 \\ a_2 = -.1410 & b_2 = .4763 \\ a_3 = -.1356 & b_3 = .3511 \\ a_4 = -.0889 & b_4 = .2165 \\ a_5 = -.0269 & b_5 = .0658\ . \end{cases}$$

Equations (4.4)-(4.6) yield

$$(4.11)\qquad \sigma^{-2}\,\mathrm{Var}(\mu^*) = .1232,\ \sigma^{-2}\,\mathrm{Var}(\sigma^*) = .4140,\ \sigma^{-2}\,\mathrm{Cov}(\mu^*,\sigma^*) = -.1516.$$

Similar estimates may be derived assuming a censored sample consisting of $n-r_1-r_2$ observations, where r_1 observations are missing in the beginning and r_2 observations are missing at the end.

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Table 1. The first four moments about the origin, the second, third and fourth central moments, and the standard deviation of the k^{th} order statistic from a sample of size n from the standard lognormal distribution; n = 1(1)20, k = 1(1)n.

n	k	$\mu_1'(k,n)$	$\mu_2'(k,n)$	$\mu_3'(k,n)$	$\mu_4'(k,n)$	$\mu_2(k,n)$	$\mu_3(k,n)$	$\mu_4(k,n)$	Std. Dev.
1	1	1.64872	7.38905	90.01691	2980.94511	4.67077	62.43285	2485.63996	2.16120
2	1	0.79056	1.16229	3.05112	13.94412	0.53730	1.28271	7.48242	0.73301
2	2	2.50688	13.61580	176.98271	5947.94610	7.33136	106.09186	4568.17265	2.70765
3	1	0.55992	0.51132	0.72653	1.55836	0.19781	0.21872	0.59813	0.44476
3	2	1.25185	2.46425	7.70029	38.71564	0.89711	2.36932	15.96019	0.94716
3	3	3.13439	19.19158	261.62392	8902.56133	9.36717	142.74921	6464.15222	3.06058
4	1	0.45059	0.30908	0.30762	0.43055	0.10605	0.07279	0.12895	0.32565
4	2	0.88788	1.11802	1.98325	4.94179	0.32969	0.40512	1.32209	0.57419
4	3	1.61582	3.81047	13.41734	72.48949	1.19958	3.38369	25.01120	1.09525
4	4	3.64058	24.31862	344.35944	11845.91860	11.06479	175.26099	8238.14134	3.32638
5	1	0.38568	0.21682	0.16942	0.17834	0.06807	0.03329	0.04411	0.26090
5	2	0.71024	0.67813	0.86043	1.43938	0.17368	0.13208	0.28399	0.41675
5	3	1.15434	1.77786	3.66746	10.19542	0.44537	0.58700	2.14878	0.66736
5	4	1.92348	5.16554	19.91726	114.01887	1.46575	4.34275	34.37984	1.21068
5	5	4.06986	29.10689	425.46998	14778.89354	12.54316	204.91138	9922.12737	3.54163
6	1	0.34213	0.16552	0.10808	0.09235	0.04847	0.01828	0.01959	0.22016
6	2	0.60342	0.47328	0.47612	0.60831	0.10917	0.05879	0.09534	0.33040
6	3	0.92389	1.08783	1.62906	3.10150	0.23426	0.19117	0.46670	0.48400
6	4	1.38479	2.46790	5.70587	17.28934	0.55026	0.76436	3.04687	0.74180
6	5	2.19283	6.51436	27.02296	162.38363	1.70585	5.25681	43.93724	1.30608
6	6	4.44526	33.62540	505.15939	17702.19552	13.86505	232.41814	11535.21688	3.72358
7	1	0.31059	0.13337	0.07563	0.05495	0.03690	0.01129	0.01027	0.19208
7	2	0.53137	0.35848	0.30277	0.31672	0.07612	0.03137	0.04134	0.27591
7	3	0.78354	0.76029	0.90951	1.33729	0.14635	0.08446	0.15658	0.38255
7	4	1.11101	1.52455	2.58845	5.45379	0.29019	0.24985	0.67064	0.53869
7	5	1.59012	3.17541	8.04394	26.16600	0.64694	0.93723	3.99696	0.80433
7	6	2.43392	7.84993	34.61457	216.87069	1.92598	6.13313	53.61085	1.38780
7	7	4.78049	37.92131	583.58353	20616.41633	15.06827	258.23396	13090.08528	3.88179
8	1	0.28652	0.11151	0.05637	0.03592	0.02941	0.00756	0.00603	0.17151
8	2	0.47910	0.28636	0.21049	0.18818	0.05682	0.01885	0.02112	0.23836
8	3	0.68819	0.57486	0.57960	0.70232	0.10126	0.04462	0.06745	0.31821
8	4	0.94247	1.06933	1.45936	2.39557	0.18108	0.11023	0.22599	0.42553
8	5	1.27956	1.97976	3.71754	8.51200	0.34249	0.30785	0.89120	0.58523
8	6	1.77645	3.89280	10.63977	36.75841	0.73703	1.10582	4.98642	0.85850
8	7	2.65307	9.16898	42.60616	276.90811	2.13018	6.97716	63.35642	1.45951
8	8	5.08440	42.02879	660.86601	23522.06036	16.17765	282.66745	14595.73585	4.02214

(Table 1 continued)

n	k	$\mu'_1(k,n)$	$\mu'_2(k,n)$	$\mu'_3(k,n)$	$\mu'_4(k,n)$	$\mu_2(k,n)$	$\mu_3(k,n)$	$\mu_4(k,n)$	Std. Dev.
9	1	0.26743	0.09577	0.04396	0.02511	0.02425	0.00537	0.00384	0.15573
9	2	0.43922	0.23739	0.15564	0.12240	0.04448	0.01230	0.01209	0.21090
9	3	0.61870	0.45774	0.40246	0.41843	0.07495	0.02651	0.03415	0.27377
9	4	0.82715	0.80909	0.93389	1.27011	0.12491	0.05801	0.09731	0.35343
9	5	1.08663	1.39463	2.11619	3.80240	0.21388	0.13594	0.30212	0.46247
9	6	1.43390	2.44786	4.99863	12.27969	0.39179	0.36506	1.12512	0.62593
9	7	1.94773	4.61527	13.46034	48.99777	0.82164	1.27040	6.00653	0.90644
9	8	2.85460	10.47003	50.93354	342.02535	2.32129	7.79308	73.14531	1.52358
9	9	5.36313	45.97363	737.10757	26419.56473	17.21051	305.94092	16058.88010	4.14855
10	1	0.25186	0.08395	0.03547	0.01846	0.02051	0.00399	0.00261	0.14323
10	2	0.40764	0.20223	0.12037	0.08500	0.03606	0.00853	0.00752	0.18990
10	3	0.56555	0.37803	0.29671	0.27200	0.05819	0.01710	0.01935	0.24122
10	4	0.74273	0.64374	0.64921	0.76009	0.09210	0.03428	0.04911	0.30347
10	5	0.95379	1.05712	1.36091	2.03514	0.14741	0.07146	0.13036	0.38394
10	6	1.21947	1.73215	2.87147	5.56965	0.24505	0.16152	0.38386	0.49503
10	7	1.57686	2.92501	6.41673	16.75305	0.43852	0.42143	1.37000	0.66221
10	8	2.10667	5.33967	16.47903	62.81693	0.90162	1.43127	7.05108	0.94954
10	9	3.04158	11.75262	59.54717	411.82746	2.50139	8.58420	82.95823	1.58158
10	10	5.62108	49.77597	812.39206	29309.31332	18.17947	328.22118	17484.70496	4.26374
11	1	0.23885	0.07475	0.02938	0.01410	0.01770	0.00307	0.00185	0.13304
11	2	0.38191	0.17589	0.09631	0.06201	0.03004	0.00620	0.00499	0.17331
11	3	0.52340	0.32075	0.22860	0.18843	0.04680	0.01173	0.01191	0.21633
11	4	0.67794	0.53079	0.47835	0.49486	0.07119	0.02199	0.02769	0.26681
11	5	0.85611	0.84140	0.94821	1.22426	0.10849	0.04213	0.06576	0.32937
11	6	1.07100	1.31598	1.85616	3.00820	0.16892	0.08491	0.16613	0.41100
11	7	1.34318	2.07895	3.71756	7.70418	0.27482	0.18689	0.47035	0.52423
11	8	1.71039	3.40846	7.95912	21.92382	0.48304	0.47696	1.62401	0.69501
11	9	2.25527	6.06388	19.67400	78.15185	0.97761	1.58871	8.11544	0.98874
11	10	3.21632	13.01679	68.40787	485.97759	2.67208	9.35320	92.78171	1.63465
11	11	5.86155	53.45188	886.79047	32191.64689	19.09410	349.63741	18877.33318	4.36968
12	1	0.22779	0.06741	0.02486	0.01111	0.01552	0.00243	0.00137	0.12458
12	2	0.36048	0.15550	0.07915	0.04702	0.02555	0.00467	0.00347	0.15986
12	3	0.48905	0.27785	0.18214	0.13700	0.03868	0.00843	0.00780	0.19667
12	4	0.62646	0.44945	0.36796	0.34273	0.05700	0.01499	0.01694	0.23874
12	5	0.78090	0.69347	0.69913	0.79912	0.08366	0.02694	0.03701	0.28924
12	6	0.96139	1.04851	1.29692	1.81946	0.12424	0.05001	0.08389	0.35247
12	7	1.18062	1.58344	2.41541	4.19695	0.18958	0.09832	0.20428	0.43541
12	8	1.45930	2.43289	4.64766	10.20935	0.30334	0.21202	0.56090	0.55076
12	9	1.83593	3.89625	9.61485	27.78106	0.52561	0.53165	1.88574	0.72499
12	10	2.39506	6.78642	23.02705	94.94211	1.05013	1.74295	9.19609	1.02476
12	11	3.38057	14.26286	77.48403	564.18469	2.83459	10.10230	102.60615	1.68362
12	12	6.08709	57.01452	960.36379	35066.87073	19.96180	370.29226	20240.11600	4.46786

(Table 1 continued)

n	k	$\mu_1'(k,n)$	$\mu_2'(k,n)$	$\mu_3'(k,n)$	$\mu_4'(k,n)$	$\mu_2(k,n)$	$\mu_3(k,n)$	$\mu_4(k,n)$	Std. Dev.
13	1	0.21825	0.06142	0.02139	0.00897	0.01379	0.00197	0.00104	0.11742
13	2	0.34231	0.13929	0.06644	0.03675	0.02211	0.00363	0.00251	0.14871
13	3	0.46044	0.24467	0.14902	0.10345	0.03266	0.00629	0.00537	0.18073
13	4	0.58444	0.38846	0.29256	0.24884	0.04690	0.01071	0.01104	0.21656
13	5	0.72100	0.58666	0.53762	0.55397	0.06681	0.01829	0.02258	0.25848
13	6	0.87674	0.86437	0.95755	1.19135	0.09569	0.03192	0.04719	0.30935
13	7	1.06015	1.26334	1.69285	2.55225	0.13943	0.05789	0.10334	0.37340
13	8	1.28388	1.85781	3.03475	5.60669	0.20947	0.11168	0.24448	0.45768
13	9	1.56894	2.79232	5.65573	13.08601	0.33076	0.23689	0.65498	0.57511
13	10	1.95460	4.38689	11.37446	34.31219	0.56644	0.58554	2.15406	0.75262
13	11	2.52719	7.50628	26.52283	113.13109	1.11957	1.89424	10.29029	1.05810
13	12	3.53573	15.49133	86.74971	646.19443	2.98993	10.83332	112.42456	1.72914
13	13	6.29971	60.47479	1033.16496	37935.26042	20.78846	390.26893	21575.82810	4.55944
14	1	0.20991	0.05644	0.01867	0.00740	0.01238	0.00163	0.00082	0.11127
14	2	0.32666	0.12612	0.05676	0.02946	0.01941	0.00288	0.00188	0.13931
14	3	0.43618	0.21832	0.12455	0.08052	0.02807	0.00483	0.00385	0.16754
14	4	0.54940	0.34127	0.23876	0.18751	0.03943	0.00794	0.00755	0.19858
14	5	0.67203	0.50644	0.42704	0.40216	0.05482	0.01302	0.01466	0.23414
14	6	0.80916	0.73104	0.73665	0.82723	0.07630	0.02164	0.02875	0.27623
14	7	0.96685	1.04214	1.25207	1.67684	0.10734	0.03692	0.05816	0.32763
14	8	1.15345	1.48455	2.13363	3.42766	0.15410	0.06578	0.12394	0.39255
14	9	1.38170	2.13776	3.71058	7.24097	0.22867	0.12495	0.28650	0.47819
14	10	1.67296	3.15596	6.73637	16.33325	0.35718	0.26150	0.75217	0.59764
14	11	2.06725	4.87926	13.22969	41.50377	0.60573	0.63866	2.42806	0.77829
14	12	2.65263	8.22274	30.14823	132.66581	1.18628	2.04275	11.39589	1.08917
14	13	3.68292	16.70276	96.18329	731.78254	3.13890	11.54784	122.23175	1.77169
14	14	6.50100	63.84187	1105.24047	40797.06641	21.57886	409.63609	22886.80236	4.64531
15	1	0.20255	0.05224	0.01649	0.00620	0.01122	0.00137	0.00065	0.10592
15	2	0.31302	0.11522	0.04920	0.02411	0.01723	0.00234	0.00144	0.13128
15	3	0.41530	0.19695	0.10593	0.06425	0.02447	0.00381	0.00285	0.15644
15	4	0.51968	0.30381	0.19902	0.14559	0.03374	0.00607	0.00537	0.18369
15	5	0.63114	0.44430	0.34805	0.30279	0.04597	0.00961	0.00999	0.21440
15	6	0.75380	0.63072	0.58503	0.60091	0.06250	0.01536	0.01863	0.25000
15	7	0.89219	0.88152	0.96409	1.16671	0.08552	0.02500	0.03541	0.29244
15	8	1.05217	1.22569	1.58119	2.25985	0.11863	0.04192	0.06986	0.34443
15	9	1.24207	1.71105	2.61701	4.44949	0.16830	0.07366	0.14553	0.41025
15	10	1.47478	2.42223	4.43964	9.10196	0.24725	0.13810	0.33018	0.49724
15	11	1.77205	3.52283	7.88474	19.94890	0.38268	0.28586	0.85206	0.61862
15	12	2.17460	5.37250	15.17331	49.34191	0.64362	0.69104	2.70702	0.80226
15	13	2.77214	8.93531	33.89196	153.49679	1.25054	2.18869	12.51113	1.11827
15	14	3.82303	17.89776	105.76657	820.74958	3.28216	12.24717	132.02388	1.81167
15	15	6.69228	67.12359	1176.63147	43652.51761	22.33693	428.45122	24175.02581	4.72620

(Table 1 continued)

n	k	$\mu_1'(k,n)$	$\mu_2'(k,n)$	$\mu_3'(k,n)$	$\mu_4'(k,n)$	$\mu_2(k,n)$	$\mu_3(k,n)$	$\mu_4(k,n)$	Std. Dev.
16	1	0.19598	0.04866	0.01471	0.00528	0.01025	0.00116	0.00053	0.10123
16	2	0.30101	0.10606	0.04316	0.02008	0.01546	0.00193	0.00114	0.12432
16	3	0.39712	0.17930	0.09142	0.05234	0.02160	0.00306	0.00217	0.14696
16	4	0.49410	0.27343	0.16880	0.11586	0.02930	0.00475	0.00395	0.17116
16	5	0.59642	0.39494	0.28966	0.23479	0.03923	0.00732	0.00708	0.19806
16	6	0.70753	0.55290	0.47652	0.45239	0.05231	0.01131	0.01268	0.22871
16	7	0.83094	0.76042	0.76587	0.84844	0.06996	0.01774	0.02291	0.26450
16	8	0.97095	1.03723	1.21894	1.57590	0.09449	0.02835	0.04254	0.30740
16	9	1.13339	1.41415	1.94344	2.94381	0.12957	0.04694	0.08217	0.35996
16	10	1.32660	1.94197	3.14089	5.62057	0.18210	0.08152	0.16802	0.42673
16	11	1.56369	2.71039	5.21889	11.19080	0.26526	0.15112	0.37539	0.51503
16	12	1.86675	3.89212	9.09649	23.92985	0.40736	0.30999	0.95427	0.63825
16	13	2.27721	5.86596	17.19892	57.81259	0.68026	0.74268	2.99037	0.82478
16	14	2.88636	9.64362	37.74419	175.57776	1.31257	2.33222	13.63458	1.14567
16	15	3.95685	19.07692	115.48405	912.91698	3.42029	12.93249	141.79805	1.84940
16	16	6.87465	70.32670	1247.37463	46501.82432	23.06594	446.76307	25442.20998	4.80270
17	1	0.19008	0.04556	0.01324	0.00455	0.00942	0.00100	0.00044	0.09708
17	2	0.29033	0.09827	0.03827	0.01697	0.01398	0.00162	0.00091	0.11824
17	3	0.38111	0.16450	0.07988	0.04339	0.01926	0.00251	0.00169	0.13877
17	4	0.47181	0.24836	0.14528	0.09411	0.02575	0.00380	0.00299	0.16047
17	5	0.56651	0.35490	0.24526	0.18654	0.03397	0.00571	0.00519	0.18430
17	6	0.66818	0.49103	0.39622	0.35059	0.04456	0.00857	0.00897	0.21110
17	7	0.77966	0.66634	0.62373	0.63902	0.05847	0.01305	0.01557	0.24180
17	8	0.90419	0.89482	0.96892	1.14763	0.07727	0.02011	0.02748	0.27797
17	9	1.04605	1.19745	1.50022	2.05771	0.10322	0.03168	0.05014	0.32128
17	10	1.21103	1.60677	2.33741	3.73145	0.14019	0.05203	0.09497	0.37442
17	11	1.40751	2.17661	3.70332	6.94296	0.19554	0.08929	0.19138	0.44220
17	12	1.64888	3.00153	6.04556	13.50780	0.28273	0.16402	0.42200	0.53172
17	13	1.95753	4.26320	10.36771	28.27238	0.43127	0.33392	1.05848	0.65671
17	14	2.37558	6.35912	19.30083	66.90189	0.71575	0.79361	3.27770	0.84602
17	15	2.99581	10.34744	41.69634	198.86544	1.37257	2.47349	14.76497	1.17157
17	16	4.08498	20.24085	125.32242	1008.12385	3.55375	13.60479	151.55213	1.88514
17	17	7.04900	73.45707	1317.50289	49345.18060	23.76867	464.61337	26689.84337	4.87531
18	1	0.18475	0.04285	0.01201	0.00396	0.00872	0.00087	0.00037	0.09336
18	2	0.28076	0.09156	0.03423	0.01452	0.01274	0.00138	0.00074	0.11286
18	3	0.36690	0.15193	0.07054	0.03651	0.01732	0.00208	0.00134	0.13161
18	4	0.45220	0.22736	0.12659	0.07779	0.02287	0.00309	0.00232	0.15123
18	5	0.54046	0.32186	0.21069	0.15125	0.02977	0.00456	0.00390	0.17253
18	6	0.63426	0.44081	0.33513	0.27830	0.03853	0.00667	0.00655	0.19628
18	7	0.73602	0.59146	0.51841	0.49519	0.04974	0.00987	0.01102	0.22301
18	8	0.84824	0.78400	0.78925	0.86503	0.06449	0.01483	0.01862	0.25395
18	9	0.97412	1.03334	1.19351	1.50087	0.08443	0.02242	0.03239	0.29057
18	10	1.11798	1.36155	1.80693	2.61454	0.11166	0.03506	0.05811	0.33416
18	11	1.28546	1.80295	2.76180	4.62497	0.15054	0.05715	0.10815	0.38799
18	12	1.48517	2.41440	4.30247	8.41804	0.20866	0.09688	0.21571	0.45679
18	13	1.73074	3.29510	6.91710	16.05267	0.29966	0.17689	0.46972	0.54741
18	14	2.04476	4.63554	11.69487	32.97226	0.45450	0.35762	1.16447	0.67416
18	15	2.47010	6.85157	21.47396	76.59607	0.75019	0.84383	3.56868	0.86613
18	16	3.10095	11.04661	45.74082	223.31932	1.43072	2.61266	15.90123	1.19613
18	17	4.20799	21.39013	135.27011	1106.22442	3.68296	14.26495	161.28453	1.91910
18	18	7.21612	76.51983	1387.04600	52182.76625	24.44747	482.03827	27919.23164	4.94444

(Table 1 continued)

n	k	$\mu_1'(k,n)$	$\mu_2'(k,n)$	$\mu_3'(k,n)$	$\mu_4'(k,n)$	$\mu_2(k,n)$	$\mu_3(k,n)$	$\mu_4(k,n)$	Std. Dev.
19	1	0.17990	0.04047	0.01096	0.00348	0.00810	0.00076	0.00031	0.09002
19	2	0.27212	0.08573	0.03087	0.01257	0.01168	0.00118	0.00061	0.10808
19	3	0.35417	0.14113	0.06286	0.03112	0.01570	0.00176	0.00108	0.12529
19	4	0.43478	0.20953	0.11148	0.06526	0.02050	0.00256	0.00183	0.14317
19	5	0.51752	0.29419	0.18324	0.12477	0.02636	0.00370	0.00301	0.16236
19	6	0.60468	0.39935	0.28756	0.22540	0.03371	0.00531	0.00490	0.18361
19	7	0.69835	0.53065	0.43821	0.39291	0.04296	0.00763	0.00805	0.20726
19	8	0.80059	0.69570	0.65588	0.67053	0.05475	0.01123	0.01316	0.23400
19	9	0.91375	0.90540	0.97263	1.13247	0.07046	0.01655	0.02188	0.26545
19	10	1.04120	1.17550	1.43894	1.91021	0.09140	0.02467	0.03768	0.30233
19	11	1.18709	1.52900	2.13813	3.24844	0.11982	0.03861	0.06625	0.34615
19	12	1.35701	2.00219	3.21537	5.62609	0.16073	0.06217	0.12178	0.40091
19	13	1.55994	2.65485	4.93662	10.04669	0.22145	0.10432	0.24104	0.47059
19	14	1.80957	3.59060	7.83117	18.82467	0.31607	0.18982	0.51821	0.56220
19	15	2.12876	5.00874	13.07477	38.02497	0.47712	0.38102	1.27222	0.69074
19	16	2.56112	7.34299	23.71374	86.88169	0.78365	0.89340	3.86292	0.88524
19	17	3.20217	11.74104	49.87090	248.90137	1.48716	2.74985	17.04243	1.21949
19	18	4.32632	22.52532	145.31708	1207.08595	3.80827	14.91374	170.99411	1.95148
19	19	7.37666	79.51952	1456.03093	55014.74849	25.10438	499.06929	29131.52861	5.01043
20	1	0.17546	0.03835	0.01006	0.00309	0.00757	0.00068	0.00027	0.08700
20	2	0.26428	0.08062	0.02802	0.01099	0.01077	0.00102	0.00051	0.10379
20	3	0.34269	0.13176	0.05647	0.02683	0.01432	0.00150	0.00089	0.11968
20	4	0.41919	0.19424	0.09909	0.05546	0.01852	0.00214	0.00147	0.13608
20	5	0.49715	0.27072	0.16106	0.10446	0.02356	0.00305	0.00237	0.15348
20	6	0.57863	0.36460	0.24977	0.18570	0.02979	0.00432	0.00376	0.17261
20	7	0.66546	0.48041	0.37574	0.31803	0.03758	0.00603	0.00602	0.19385
20	8	0.75945	0.62397	0.55424	0.53198	0.04720	0.00867	0.00964	0.21726
20	9	0.86231	0.80331	0.80835	0.87836	0.05973	0.01263	0.01537	0.24440
20	10	0.97662	1.03017	1.17340	1.44305	0.07639	0.01811	0.02545	0.27639
20	11	1.10579	1.32083	1.70447	2.37737	0.09807	0.02702	0.04320	0.31316
20	12	1.25361	1.69931	2.49295	3.96113	0.12778	0.04231	0.07441	0.35747
20	13	1.42594	2.20411	3.69699	6.73605	0.17081	0.06693	0.13614	0.41329
20	14	1.63209	2.89756	5.60411	11.82934	0.23385	0.11173	0.26718	0.48358
20	15	1.88563	3.88762	8.78562	21.82266	0.33203	0.20283	0.56715	0.57622
20	16	2.20980	5.38244	14.50449	43.42574	0.49922	0.40403	1.38189	0.70655
20	17	2.64895	7.83313	26.01606	97.74568	0.81619	0.94239	4.15997	0.90343
20	18	3.29979	12.43067	54.08058	275.57591	1.54203	2.88518	18.18785	1.24179
20	19	4.44038	23.64695	155.45447	1310.58707	3.92998	15.55185	180.68001	1.98242
20	20	7.53120	82.46018	1524.48233	57841.28331	25.74116	515.73412	30327.76086	5.07357

Table 2. 100α percentage points of the k^{th} order statistic in a sample of size n from the lognormal distribution.

n	k	α .50	.75	.90	.95	.99
1	1	1.000	1.963	3.602	5.180	10.240
2	1	0.580	1.000	1.613	2.138	3.602
2	2	1.725	3.028	5.115	7.060	13.131
3	1	0.441	0.718	1.094	1.399	2.198
3	2	1.000	1.568	2.355	3.009	4.778
3	3	2.269	3.788	6.161	8.341	15.059
4	1	0.369	0.580	0.855	1.070	1.613
4	2	0.748	1.116	1.594	1.972	2.934
4	3	1.337	2.007	2.912	3.652	5.629
4	4	2.713	4.394	6.981	9.337	16.541
5	1	0.323	0.497	0.716	0.884	1.295
5	2	0.616	0.891	1.236	1.500	2.149
5	3	1.000	1.433	1.984	2.412	3.491
5	4	1.624	2.373	3.369	4.176	6.314
5	5	3.093	4.904	7.663	10.162	17.759
6	1	0.292	0.441	0.624	0.762	1.094
6	2	0.533	0.755	1.026	1.229	1.717
6	3	0.820	1.143	1.539	1.838	2.565
6	4	1.219	1.705	2.313	2.781	3.951
6	5	1.877	2.690	3.762	4.624	6.894
6	6	3.426	5.347	8.253	10.872	18.801
7	1	0.269	0.400	0.559	0.677	0.956
7	2	0.475	0.663	0.888	1.053	1.444
7	3	0.706	0.966	1.276	1.505	2.051
7	4	1.000	1.361	1.799	2.126	2.916
7	5	1.415	1.944	2.600	3.103	4.348
7	6	2.104	2.973	4.108	5.018	7.401
7	7	3.724	5.741	8.773	11.497	19.715
8	1	0.250	0.369	0.510	0.613	0.855
8	2	0.433	0.597	0.789	0.929	1.255
8	3	0.627	0.845	1.101	1.287	1.723
8	4	0.860	1.150	1.492	1.743	2.335
8	5	1.163	1.557	2.029	2.380	3.223
8	6	1.594	2.160	2.858	3.389	4.701
8	7	2.311	3.228	4.420	5.370	7.853
8	8	3.996	6.097	9.242	12.059	20.533

(Table 2 continued)

n	k	α .50	.75	.90	.95	.99
9	1	0.236	0.344	0.471	0.564	0.777
9	2	0.400	0.546	0.714	0.836	1.117
9	3	0.569	0.757	0.975	1.133	1.495
9	4	0.762	1.005	1.386	1.490	1.963
9	5	1.000	1.315	1.685	1.955	2.587
9	6	1.312	1.734	2.237	2.609	3.498
9	7	1.758	2.358	3.092	3.649	5.019
9	8	2.501	3.462	4.704	5.691	8.262
9	9	4.245	6.422	9.668	12.569	21.274
10	1	0.223	0.323	0.440	0.524	0.716
10	2	0.373	0.505	0.656	0.765	1.011
10	3	0.523	0.690	0.881	1.017	1.328
10	4	0.690	0.899	1.139	1.311	1.704
10	5	0.886	1.150	1.453	1.672	2.177
10	6	1.129	1.467	1.861	2.147	2.814
10	7	1.450	1.898	2.428	2.819	3.748
10	8	1.911	2.541	3.307	3.888	5.310
10	9	2.678	3.678	4.965	5.986	8.637
10	10	4.476	6.723	10.061	13.038	21.953
11	1	0.213	0.306	0.414	0.491	0.666
11	2	0.352	0.472	0.609	0.707	0.927
11	3	0.487	0.637	0.807	0.928	1.200
11	4	0.633	0.818	1.027	1.176	1.512
11	5	0.800	1.028	1.287	1.471	1.890
11	6	1.000	1.283	1.607	1.839	2.371
11	7	1.249	1.608	2.023	2.324	3.022
11	8	1.579	2.050	2.604	3.012	3.978
11	9	2.054	2.711	3.507	4.108	5.578
11	10	2.844	3.879	5.208	6.260	8.984
11	11	4.692	7.002	10.426	13.473	22.581
12	1	0.204	0.292	0.392	0.463	0.624
12	2	0.333	0.445	0.571	0.660	0.859
12	3	0.457	0.594	0.748	0.856	1.098
12	4	0.588	0.754	0.940	1.071	1.365
12	5	0.734	0.935	1.160	1.319	1.678
12	6	0.903	1.147	1.422	1.618	2.060
12	7	1.107	1.407	1.748	1.992	2.550
12	8	1.363	1.740	2.174	2.488	3.215
12	9	1.701	2.193	2.769	3.193	4.192
12	10	2.189	2.870	3.694	4.315	5.828
12	11	3.000	4.068	5.435	6.515	9.307
12	12	4.895	7.264	10.766	13.878	23.165

(Table 2 continued)

n	k	α .50	.75	.90	.95	.99
13	1	0.197	0.279	0.373	0.440	0.589
13	2	0.318	0.422	0.538	0.620	0.802
13	3	0.432	0.558	0.698	0.797	1.016
13	4	0.550	0.702	0.869	0.987	1.248
13	5	0.681	0.860	1.061	1.201	1.514
13	6	0.828	1.043	1.282	1.451	1.830
13	7	1.000	1.259	1.549	1.754	2.218
13	8	1.208	1.524	1.881	2.136	2.716
13	9	1.469	1.864	2.316	2.642	3.394
13	10	1.817	2.327	2.924	3.362	4.392
13	11	2.316	3.021	3.870	4.508	6.062
13	12	3.147	4.246	5.649	6.754	9.610
13	13	5.086	7.510	11.086	14.259	23.713
14	1	0.190	0.269	0.357	0.420	0.559
14	2	0.304	0.402	0.510	0.586	0.755
14	3	0.410	0.528	0.657	0.747	0.947
14	4	0.519	0.658	0.811	0.917	1.153
14	5	0.637	0.800	0.980	1.106	1.384
14	6	0.767	0.960	1.172	1.321	1.651
14	7	0.916	1.144	1.397	1.574	1.970
14	8	1.091	1.364	1.668	1.882	2.365
14	9	1.304	1.634	2.006	2.271	2.872
14	10	1.571	1.981	2.450	2.787	3.563
14	11	1.926	2.455	3.071	3.521	4.581
14	12	2.437	3.163	4.036	4.691	6.282
14	13	3.287	4.414	5.850	6.981	9.895
14	14	5.267	7.743	11.387	14.617	24.228
15	1	0.184	0.259	0.343	0.402	0.533
15	2	0.292	0.385	0.486	0.557	0.714
15	3	0.392	0.501	0.622	0.705	0.889
15	4	0.493	0.621	0.762	0.859	1.074
15	5	0.600	0.749	0.914	1.027	1.278
15	6	0.717	0.891	1.083	1.216	1.509
15	7	0.849	1.052	1.277	1.433	1.779
15	8	1.000	1.240	1.504	1.690	2.102
15	9	1.178	1.463	1.780	2.002	2.503
15	10	1.395	1.738	2.124	2.398	3.019
15	11	1.668	2.092	2.576	2.924	3.723
15	12	2.030	2.576	3.210	3.672	4.758
15	13	2.553	3.299	4.193	4.864	6.490
15	14	3.420	4.574	6.041	7.195	10.165
15	15	5.440	7.964	11.673	14.957	24.714

(Table 2 continued)

n	k	α .50	.75	.90	.95	.99
16	1	0.178	0.250	0.330	0.386	0.510
16	2	0.282	0.369	0.465	0.532	0.678
16	3	0.375	0.479	0.591	0.669	0.839
16	4	0.469	0.589	0.720	0.810	1.007
16	5	0.568	0.707	0.858	0.962	1.190
16	6	0.675	0.835	1.009	1.129	1.393
16	7	0.793	0.978	1.179	1.319	1.626
16	8	0.926	1.140	1.375	1.538	1.899
16	9	1.080	1.330	1.606	1.799	2.227
16	10	1.261	1.557	1.886	2.117	2.633
16	11	1.482	1.838	2.237	2.519	3.158
16	12	1.760	2.199	2.697	3.054	3.874
16	13	2.130	2.691	3.342	3.816	4.927
16	14	2.663	3.428	4.343	5.028	6.688
16	15	3.548	4.727	6.224	7.399	10.421
16	16	5.604	8.175	11.945	15.279	25.176
17	1	0.174	0.243	0.319	0.372	0.489
17	2	0.273	0.356	0.447	0.510	0.647
17	3	0.361	0.459	0.564	0.637	0.796
17	4	0.449	0.562	0.684	0.767	0.950
17	5	0.541	0.670	0.810	0.906	1.115
17	6	0.639	0.787	0.946	1.057	1.297
17	7	0.746	0.915	1.098	1.225	1.501
17	8	0.865	1.059	1.270	1.416	1.736
17	9	1.000	1.224	1.469	1.638	2.012
17	10	1.156	1.417	1.703	1.902	2.344
17	11	1.341	1.648	1.987	2.225	2.757
17	12	1.565	1.933	2.344	2.634	3.291
17	13	1.849	2.300	2.812	3.178	4.017
17	14	2.226	2.802	3.468	3.953	5.088
17	15	2.769	3.551	4.487	5.185	6.876
17	16	3.670	4.873	6.397	7.593	10.665
17	17	5.762	8.376	12.204	15.586	25.616
18	1	0.169	0.236	0.309	0.360	0.471
18	2	0.264	0.344	0.430	0.490	0.619
18	3	0.348	0.441	0.541	0.610	0.759
18	4	0.431	0.538	0.652	0.730	0.901
18	5	0.517	0.638	0.768	0.857	1.051
18	6	0.608	0.745	0.893	0.995	1.215
18	7	0.706	0.862	1.030	1.146	1.397
18	8	0.813	0.991	1.183	1.315	1.603
18	9	0.934	1.137	1.357	1.508	1.840
18	10	1.071	1.304	1.558	1.733	2.119
18	11	1.229	1.499	1.795	2.001	2.456
18	12	1.417	1.734	2.084	2.329	2.875
18	13	1.646	2.024	2.446	2.745	3.417
18	14	1.934	2.398	2.922	3.297	4.155
18	15	2.318	2.908	3.589	4.084	5.242
18	16	2.871	3.670	4.624	5.336	7.056
18	17	3.787	5.013	6.564	7.779	10.898
18	18	5.913	8.568	12.452	15.880	26.035

(Table 2 continued)

n	k	α .50	.75	.90	.95	.99
19	1	0.165	0.229	0.300	0.348	0.455
19	2	0.256	0.333	0.415	0.472	0.595
19	3	0.337	0.425	0.520	0.585	0.725
19	4	0.415	0.516	0.624	0.698	0.857
19	5	0.496	0.610	0.732	0.815	0.996
19	6	0.580	0.709	0.847	0.941	1.145
19	7	0.671	0.816	0.972	1.078	1.309
19	8	0.769	0.933	1.109	1.230	1.491
19	9	0.878	1.064	1.264	1.401	1.699
19	10	1.000	1.211	1.439	1.596	1.939
19	11	1.139	1.380	1.643	1.824	2.222
19	12	1.300	1.579	1.884	2.095	2.563
19	13	1.490	1.817	2.177	2.428	2.988
19	14	1.723	2.112	2.545	2.850	3.538
19	15	2.016	2.492	3.028	3.411	4.286
19	16	2.407	3.010	3.705	4.210	5.389
19	17	2.969	3.784	4.755	5.480	7.229
19	18	3.900	5.147	6.724	7.958	11.121
19	19	6.058	8.753	12.690	16.162	26.437
20	1	0.161	0.223	0.291	0.338	0.440
20	2	0.249	0.323	0.402	0.456	0.573
20	3	0.326	0.411	0.501	0.563	0.696
20	4	0.401	0.497	0.599	0.669	0.819
20	5	0.477	0.585	0.700	0.778	0.947
20	6	0.556	0.677	0.807	0.894	1.084
20	7	0.640	0.776	0.921	1.020	1.233
20	8	0.731	0.883	1.046	1.157	1.397
20	9	0.830	1.002	1.185	1.310	1.581
20	10	0.940	1.133	1.340	1.482	1.791
20	11	1.064	1.282	1.518	1.680	2.033
20	12	1.204	1.454	1.724	1.911	2.320
20	13	1.368	1.655	1.969	2.186	2.666
20	14	1.561	1.897	2.266	2.524	3.097
20	15	1.797	2.196	2.640	2.952	3.653
20	16	2.096	2.582	3.130	3.521	4.413
20	17	2.492	3.108	3.817	4.331	5.530
20	18	3.063	3.894	4.882	5.618	7.394
20	19	4.008	5.277	6.877	8.129	11.336
20	20	6.198	8.931	12.918	16.432	26.822

Table 3. The product moments of the i^{th} and j^{th} order statistics from a sample of size n from the standard lognormal distribution; n = 1(1)20, i=1(1)n, j=i+1(1)n.

n	i	j	$E(X_{(i)}X_{(j)})$
2	1	2	2.71828
3	1	2	0.92315
3	1	3	2.06396
3	2	3	5.16774
4	1	2	0.50989
4	1	3	0.85519
4	1	4	1.81762
4	2	3	1.81762
4	2	4	3.76540
4	3	4	7.54397
5	1	2	0.34107
5	1	3	0.51778
5	1	4	0.82672
5	1	5	1.63788
5	2	3	1.00846
5	2	4	1.58692
5	2	5	3.19773
5	3	4	2.74213
5	3	5	5.42231
5	4	5	9.85181
6	1	2	0.25267
6	1	3	0.36417
6	1	4	0.52656
6	1	5	0.81222
6	1	6	1.60701
6	2	3	0.67158
6	2	4	0.96107
6	2	5	1.47057
6	2	6	2.88701
6	3	4	1.53747
6	3	5	2.32911
6	3	6	4.52740
6	4	5	3.68292
6	4	6	7.04999
6	5	6	12.09448
7	1	2	0.19933
7	1	3	0.27813
7	1	4	0.38195
7	1	5	0.53466
7	1	6	0.80392
7	1	7	1.55128
7	2	3	0.49388
7	2	4	0.67276
7	2	5	0.93612
7	2	6	1.40055
7	2	7	2.68880
7	3	4	1.02637
7	3	5	1.41849
7	3	6	2.11003
7	3	7	4.02666
7	4	5	2.08822
7	4	6	3.08224
7	4	7	5.83393
7	5	6	4.63044
7	5	7	8.64996
7	6	7	14.27620
8	1	2	0.16405
8	1	3	0.22382
8	1	4	0.29768
8	1	5	0.39621
8	1	6	0.54183
8	1	7	0.79880
8	1	8	1.51022
8	2	3	0.38649
8	2	4	0.51055
8	2	5	0.67628
8	2	6	0.92130
8	2	7	1.35368
8	2	8	2.55011
8	3	4	0.75404
8	3	5	0.99368
8	3	6	1.34812
8	3	7	1.97355
8	3	8	3.70309
8	4	5	1.40037
8	4	6	1.89021
8	4	7	2.75445
8	4	8	5.14246
8	5	6	2.65361
8	5	7	3.84216
8	5	8	7.12241
8	6	7	5.57833
8	6	8	10.22253
8	7	8	16.40137
9	1	2	0.13917
9	1	3	0.18675
9	1	4	0.24293
9	1	5	0.31338
9	1	6	0.40799
9	1	7	0.54814
9	1	8	0.79549
9	1	9	1.47851
9	2	3	0.31553
9	2	4	0.49808
9	2	5	0.52428
9	2	6	0.68041
9	2	7	0.91176
9	2	8	1.31998
9	2	9	2.44687
9	3	4	0.58856

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$
9	3	5	0.75300
9	3	6	0.97408
9	3	7	1.30172
9	3	8	1.87984
9	3	9	3.47487
9	4	5	1.03021
9	4	6	1.32765
9	4	7	1.76855
9	4	8	2.54639
9	4	9	4.69121
9	5	6	1.78872
9	5	7	2.37296
9	5	8	3.40344
9	5	9	6.24272
9	6	7	3.22868
9	6	8	4.60532
9	6	9	8.39402
9	7	8	6.52446
9	7	9	11.76824
9	8	9	18.47423
10	1	2	0.12077
10	1	3	0.15997
10	1	4	0.20474
10	1	5	0.25848
10	1	6	0.32641
10	1	7	0.41795
10	1	8	0.55372
10	1	9	0.79328
10	1	10	1.45316
10	2	3	0.26561
10	2	4	0.33819
10	2	5	0.42545
10	2	6	0.53579
10	2	7	0.68456
10	2	8	0.90525
10	2	9	1.29449
10	2	10	2.36656
10	3	4	0.47892
10	3	5	0.60038
10	3	6	0.75408
10	3	7	0.96133
10	3	8	1.26883
10	3	9	1.81115
10	3	10	3.30414
10	4	5	0.80387
10	4	6	1.00664
10	4	7	1.28016
10	4	8	1.68597
10	4	9	2.40166
10	4	10	4.37091
10	5	6	1.31901
10	5	7	1.67241
10	5	8	2.19679
10	5	9	3.12132
10	5	10	5.66397
10	6	7	2.18774
10	6	8	2.86376
10	6	9	4.05537
10	6	10	7.33000
10	7	8	3.80997
10	7	9	5.36916
10	7	10	9.64914
10	8	9	7.46507
10	8	10	13.23785
10	9	10	20.49867
11	1	2	0.10655
11	1	3	0.13931
11	1	4	0.17669
11	1	5	0.21957
11	1	6	0.27152
11	1	7	0.33746
11	1	8	0.42651
11	1	9	0.55859
11	1	10	0.79178
11	1	11	1.43233
11	2	3	0.22831
11	2	4	0.28783
11	2	5	0.35654
11	2	6	0.43984
11	2	7	0.54561
11	2	8	0.68853
11	2	9	0.90060
11	2	10	1.27450
11	2	11	2.30199
11	3	4	0.40171
11	3	5	0.49609
11	3	6	0.61059
11	3	7	0.75605
11	3	8	0.95252
11	3	9	1.24430
11	3	10	1.75838
11	3	11	3.17090
11	4	5	0.65336
11	4	6	0.80216
11	4	7	0.99125
11	4	8	1.24678
11	4	9	1.62603
11	4	10	2.29460
11	4	11	4.13028
11	5	6	1.03007

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$	n	i	j	$E(X_{(i)}X_{(j)})$
11	5	7	1.26993	12	4	5	0.54711
11	5	8	1.59411	12	4	6	0.66202
11	5	9	2.07531	12	4	7	0.80213
11	5	10	2.92329	12	4	8	0.98062
11	5	11	5.25093	12	4	9	1.22201
11	6	7	1.61777	12	4	10	1.58041
11	6	8	2.02572	12	4	11	2.21183
11	6	9	2.63128	12	4	12	3.94196
11	6	10	3.69815	12	5	6	0.83713
11	6	11	6.62514	12	5	7	1.01237
11	7	8	2.59472	12	5	8	1.23563
11	7	9	3.36028	12	5	9	1.53771
11	7	10	4.70867	12	5	10	1.98605
11	7	11	8.40522	12	5	11	2.77587
11	8	9	4.39502	12	5	12	4.93946
11	8	10	6.13184	12	6	7	1.26514
11	8	11	10.88804	12	6	8	1.54122
11	9	10	8.39924	12	6	9	1.91473
11	9	11	14.78230	12	6	10	2.46927
11	10	11	22.47821	12	6	11	3.44561
12	1	2	0.09549	12	6	12	6.11970
12	1	3	0.12412	12	7	8	1.92439
12	1	4	0.15529	12	7	9	2.38572
12	1	5	0.19064	12	7	10	3.07056
12	1	6	0.23214	12	7	11	4.27614
12	1	7	0.28267	12	7	12	7.57626
12	1	8	0.34701	12	8	9	3.00763
12	1	9	0.43397	12	8	10	3.86073
12	1	10	0.56316	12	8	11	5.36206
12	1	11	0.79074	12	8	12	9.46878
12	1	12	1.41486	12	9	10	4.98208
12	2	3	0.20069	12	9	11	6.89207
12	2	4	0.25001	12	9	12	12.11100
12	2	5	0.30603	12	10	11	9.32607
12	2	6	0.37184	12	10	12	16.25261
12	2	7	0.45201	12	11	12	24.41598
12	2	8	0.55411	13	1	2	0.08648
12	2	9	0.69227	13	1	3	0.11160
12	2	10	0.89711	13	1	4	0.13847
12	2	11	1.25838	13	1	5	0.16834
12	2	12	2.24870	13	1	6	0.20256
12	3	4	0.34479	13	1	7	0.24299
12	3	5	0.42090	13	1	8	0.29239
12	3	6	0.51039	13	1	9	0.35536
12	3	7	0.61946	13	1	10	0.44055
12	3	8	0.75834	13	1	11	0.56722
12	3	9	0.94617	13	1	12	0.79000
12	3	10	1.22531	13	1	13	1.39994
12	3	11	1.71635	13	2	3	0.17856
12	3	12	3.06357	13	2	4	0.22068

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$
13	2	5	0.26757
13	2	6	0.32134
13	2	7	0.38487
13	2	8	0.46249
13	2	9	0.56159
13	2	10	0.69573
13	2	11	0.89438
13	2	12	1.24515
13	2	13	2.20381
13	3	4	0.30131
13	3	5	0.36444
13	3	6	0.43628
13	3	7	0.52253
13	3	8	0.62724
13	3	9	0.76067
13	3	10	0.94152
13	3	11	1.21011
13	3	12	1.68195
13	3	13	2.97496
13	4	5	0.46868
13	4	6	0.56080
13	4	7	0.66974
13	4	8	0.80295
13	4	9	0.97297
13	4	10	1.20279
13	4	11	1.54457
13	4	12	2.14557
13	4	13	3.78997
13	5	6	0.70055
13	5	7	0.83530
13	5	8	1.00009
13	5	9	1.21036
13	5	10	1.49508
13	5	11	1.91739
13	5	12	2.66141
13	5	13	4.69504
13	6	7	1.02871
13	6	8	1.22978
13	6	9	1.48634
13	6	10	1.83370
13	6	11	2.34930
13	6	12	3.25661
13	6	13	5.73726
13	7	8	1.50746
13	7	9	1.81904
13	7	10	2.24084
13	7	11	2.86703
13	7	12	3.96855
13	7	13	6.97931
13	8	9	2.23724
13	8	10	2.75093
13	8	11	3.51339
13	8	12	4.95452
13	8	13	8.51804
13	9	10	3.42491
13	9	11	4.36373
13	9	12	6.01456
13	9	13	10.52094
13	10	11	5.56985
13	10	12	7.64392
13	10	13	13.31840
13	11	12	10.24503
13	11	13	17.69982
13	12	13	26.31481
14	1	2	0.07905
14	1	3	0.10138
14	1	4	0.12492
14	1	5	0.15066
14	1	6	0.17958
14	1	7	0.21295
14	1	8	0.25251
14	1	9	0.30096
14	1	10	0.36275
14	1	11	0.44643
14	1	12	0.57096
14	1	13	0.78945
14	1	14	1.38703
14	2	3	0.16074
14	2	4	0.19734
14	2	5	0.23740
14	2	6	0.28248
14	2	7	0.33449
14	2	8	0.39614
14	2	9	0.47166
14	2	10	0.56831
14	2	11	0.69882
14	2	12	0.89221
14	2	13	1.23411
14	2	14	2.16533
14	3	4	0.26715
14	3	5	0.32068
14	3	6	0.38093
14	3	7	0.45049
14	3	8	0.53304
14	3	9	0.63407
14	3	10	0.76292
14	3	11	0.93821
14	3	12	1.19745
14	3	13	1.65327
14	3	14	2.90031

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$
14	4	5	0.40873
14	4	6	0.48473
14	4	7	0.57250
14	4	8	0.67661
14	4	9	0.80427
14	4	10	0.96718
14	4	11	1.18740
14	4	12	1.51578
14	4	13	2.09098
14	4	14	3.66435
14	5	6	0.59954
14	5	7	0.70711
14	5	8	0.83477
14	5	9	0.99115
14	5	10	1.19108
14	5	11	1.46150
14	5	12	1.86284
14	5	13	2.56963
14	5	14	4.49736
14	6	7	0.86086
14	6	8	1.01496
14	6	9	1.20382
14	6	10	1.44499
14	6	11	1.77187
14	6	12	2.25638
14	6	13	3.10933
14	6	14	5.43641
14	7	8	1.22684
14	7	9	1.45324
14	7	10	1.74244
14	7	11	2.13407
14	7	12	2.71545
14	7	13	3.73730
14	7	14	6.52625
14	8	9	1.75575
14	8	10	2.10222
14	8	11	2.57139
14	8	12	3.26777
14	8	13	4.49176
14	8	14	7.83079
14	9	10	2.55506
14	9	11	3.12017
14	9	12	3.95875
14	9	13	5.43258
14	9	14	9.45086
14	10	11	3.84537
14	10	12	4.86825
14	10	13	6.66540
14	10	14	11.56191
14	11	12	6.15735

n	i	j	$E(X_{(i)}X_{(j)})$
14	11	13	8.40176
14	11	14	14.51065
14	12	13	11.15581
14	12	14	19.12495
14	13	14	28.17722
15	1	2	0.07282
15	1	3	0.09290
15	1	4	0.11379
15	1	5	0.13632
15	1	6	0.16124
15	1	7	0.18944
15	1	8	0.22212
15	1	9	0.26097
15	1	10	0.30859
15	1	11	0.36932
15	1	12	0.45177
15	1	13	0.57441
15	1	14	0.78901
15	1	15	1.37573
15	2	3	0.14612
15	2	4	0.17556
15	2	5	0.21318
15	2	6	0.25173
15	2	7	0.29539
15	2	8	0.34595
15	2	9	0.40599
15	2	10	0.47985
15	2	11	0.57437
15	2	12	0.70145
15	2	13	0.89060
15	2	14	1.22475
15	2	15	2.13187
15	3	4	0.23968
15	3	5	0.28588
15	3	6	0.33707
15	3	7	0.39505
15	3	8	0.46229
15	3	9	0.54224
15	3	10	0.63997
15	3	11	0.76525
15	3	12	0.93586
15	3	13	1.18643
15	3	14	1.62912
15	3	15	2.83635
15	4	5	0.36163
15	4	6	0.42574
15	4	7	0.49840
15	4	8	0.58260
15	4	9	0.68279
15	4	10	0.80589

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$
15	4	11	0.96240
15	4	12	1.17508
15	4	13	1.49206
15	4	14	2.04498
15	4	15	3.55854
15	5	6	0.52225
15	5	7	0.61062
15	5	8	0.71311
15	5	9	0.83494
15	5	10	0.98448
15	5	11	1.17598
15	5	12	1.43396
15	5	13	1.81874
15	5	14	2.49395
15	5	15	4.33371
15	6	7	0.73646
15	6	8	0.85909
15	6	9	1.00500
15	6	10	1.18393
15	6	11	1.41278
15	6	12	1.72297
15	6	13	2.18191
15	6	14	2.99106
15	6	15	5.19261
15	7	8	1.02702
15	7	9	1.20013
15	7	10	1.41260
15	7	11	1.68402
15	7	12	2.05201
15	7	13	2.59755
15	7	14	3.55645
15	7	15	6.16903
15	8	9	1.43047
15	8	10	1.68180
15	8	11	2.00309
15	8	12	2.43813
15	8	13	3.08400
15	8	14	4.21797
15	8	15	7.30773
15	9	10	2.00895
15	9	11	2.38976
15	9	12	2.90552
15	9	13	3.67076
15	9	14	5.01483
15	9	15	8.67469
15	10	11	2.87682
15	10	12	3.49250
15	10	13	4.40582
15	10	14	6.00976
15	10	15	10.37495
15	11	12	4.26808
15	11	13	5.37346
15	11	14	7.31401
15	11	15	12.59190
15	12	13	6.74386
15	12	14	9.15013
15	12	15	15.68819
15	13	14	12.05827
15	13	15	20.52898
15	14	15	30.00546
16	1	2	0.06754
16	1	3	0.08575
16	1	4	0.10449
16	1	5	0.12447
16	1	6	0.14627
16	1	7	0.17057
16	1	8	0.19820
16	1	9	0.23032
16	1	10	0.26857
16	1	11	0.31541
16	1	12	0.37523
16	1	13	0.45671
16	1	14	0.57757
16	1	15	0.78865
16	1	16	1.36576
16	2	3	0.13392
16	2	4	0.16266
16	2	5	0.19333
16	2	6	0.22684
16	2	7	0.26421
16	2	8	0.30672
16	2	9	0.35601
16	2	10	0.41477
16	2	11	0.48733
16	2	12	0.57972
16	2	13	0.70364
16	2	14	0.88959
16	2	15	1.21661
16	2	16	2.10245
16	3	4	0.21717
16	3	5	0.25763
16	3	6	0.30136
16	3	7	0.35119
16	3	8	0.40735
16	3	9	0.47270
16	3	10	0.55023
16	3	11	0.64515
16	3	12	0.76793
16	3	13	0.93393
16	3	14	1.17660

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$	n	i	j	$E(X_{(i)}X_{(j)})$
16	3	15	1.60871	16	8	15	4.00310
16	3	16	2.78074	16	8	16	6.89446
16	4	5	0.32376	16	9	10	1.63875
16	4	6	0.37883	16	9	11	1.91466
16	4	7	0.44029	16	9	12	2.26753
16	4	8	0.51021	16	9	13	2.74528
16	4	9	0.59144	16	9	14	3.45444
16	4	10	0.68855	16	9	15	4.69845
16	4	11	0.80737	16	9	16	8.08242
16	4	12	0.95826	16	10	11	2.26623
16	4	13	1.16558	16	10	12	2.68037
16	4	14	1.47171	16	10	13	3.24248
16	4	15	2.00566	16	10	14	4.07540
16	4	16	3.46800	16	10	15	5.53735
16	5	6	0.46146	16	10	16	9.51134
16	5	7	0.53573	16	11	12	3.20171
16	5	8	0.62029	16	11	13	3.86715
16	5	9	0.71854	16	11	14	4.85395
16	5	10	0.83545	16	11	15	6.58560
16	5	11	0.97969	16	11	16	11.29052
16	5	12	1.16342	16	12	13	4.69231
16	5	13	1.41091	16	12	14	5.87873
16	5	14	1.78273	16	12	15	7.95995
16	5	15	2.42993	16	12	16	13.61116
16	5	16	4.19574	16	13	14	7.32884
16	6	7	0.64112	16	13	15	9.89373
16	6	8	0.74156	16	13	16	16.85147
16	6	9	0.85837	16	14	15	12.95238
16	6	10	0.99749	16	14	16	21.91285
16	6	11	1.16811	16	15	16	31.80158
16	6	12	1.38733	17	1	2	0.06299
16	6	13	1.68267	17	1	3	0.07965
16	6	14	2.12107	17	1	4	0.09662
16	6	15	2.89366	17	1	5	0.11452
16	6	16	4.99042	17	1	6	0.13385
16	7	8	0.87861	17	1	7	0.15511
16	7	9	1.01604	17	1	8	0.17891
16	7	10	1.17982	17	1	9	0.20608
16	7	11	1.38091	17	1	10	0.23774
16	7	12	1.63787	17	1	11	0.27542
16	7	13	1.98703	17	1	12	0.32153
16	7	14	2.50250	17	1	13	0.38064
16	7	15	3.41092	17	1	14	0.46132
16	7	16	5.87905	17	1	15	0.58042
16	8	9	1.19821	17	1	16	0.78835
16	8	10	1.39003	17	1	17	1.35688
16	8	11	1.62570	17	2	3	0.12359
16	8	12	1.92689	17	2	4	0.14947
16	8	13	2.33507	17	2	5	0.17679
16	8	14	2.94067	17	2	6	0.20632

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$
17	2	7	0.23882
17	2	8	0.27525
17	2	9	0.31673
17	2	10	0.36494
17	2	11	0.42230
17	2	12	0.49418
17	2	13	0.58424
17	2	14	0.70558
17	2	15	0.89921
17	2	16	1.20932
17	2	17	2.07635
17	3	4	0.19842
17	3	5	0.23428
17	3	6	0.27304
17	3	7	0.31573
17	3	8	0.36356
17	3	9	0.41827
17	3	10	0.48187
17	3	11	0.55704
17	3	12	0.65011
17	3	13	0.77096
17	3	14	0.93178
17	3	15	1.16790
17	3	16	1.59138
17	3	17	2.73176
17	4	5	0.29274
17	4	6	0.34075
17	4	7	0.39365
17	4	8	0.45293
17	4	9	0.52049
17	4	10	0.59947
17	4	11	0.69386
17	4	12	0.80816
17	4	13	0.95521
17	4	14	1.15838
17	4	15	1.45333
17	4	16	1.97190
17	4	17	3.38945
17	5	6	0.41258
17	5	7	0.47614
17	5	8	0.54743
17	5	9	0.62881
17	5	10	0.72331
17	5	11	0.83648
17	5	12	0.97623
17	5	13	1.15198
17	5	14	1.39212
17	5	15	1.75265
17	5	16	2.37465
17	5	17	4.07769
17	6	7	0.56606
17	6	8	0.65023
17	6	9	0.74632
17	6	10	0.85831
17	6	11	0.99141
17	6	12	1.15571
17	6	13	1.36662
17	6	14	1.64810
17	6	15	2.07123
17	6	16	2.81141
17	6	17	4.81967
17	7	8	0.76469
17	7	9	0.87698
17	7	10	1.00779
17	7	11	1.16393
17	7	12	1.35516
17	7	13	1.60105
17	7	14	1.93370
17	7	15	2.42402
17	7	16	3.29119
17	7	17	5.63816
17	8	9	1.02532
17	8	10	1.17732
17	8	11	1.35866
17	8	12	1.58161
17	8	13	1.86611
17	8	14	2.25331
17	8	15	2.82475
17	8	16	3.82974
17	8	17	6.55845
17	9	10	1.37370
17	9	11	1.58399
17	9	12	1.84251
17	9	13	2.17309
17	9	14	2.62066
17	9	15	3.28536
17	9	16	4.44938
17	9	17	7.61356
17	10	11	1.85098
17	10	12	2.15113
17	10	13	2.53514
17	10	14	3.05497
17	10	15	3.82630
17	10	16	5.17851
17	10	17	8.85076
17	11	12	2.52688
17	11	13	2.97489
17	11	14	3.58167
17	11	15	4.48117
17	11	16	6.05897

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$	n	i	j	$E(X_{(i)}X_{(j)})$
17	11	17	10.34097	18	3	5	0.21469
17	12	13	3.52907	18	3	6	0.24907
17	12	14	4.24352	18	3	7	0.28651
17	12	15	5.30260	18	3	8	0.32792
17	12	16	7.15972	18	3	9	0.37460
17	12	17	12.19773	18	3	10	0.42807
17	13	14	5.11745	18	3	11	0.48972
17	13	15	6.38356	18	3	12	0.56307
17	13	16	8.60290	18	3	13	0.65535
17	13	17	14.61992	18	3	14	0.77337
17	14	15	7.91188	18	3	15	0.92903
17	14	16	10.63234	18	3	16	1.16056
17	14	17	18.00094	18	3	17	1.57650
17	15	16	13.83817	18	3	18	2.68814
17	15	17	23.27744	18	4	5	0.26691
17	16	17	33.56742	18	4	6	0.30929
18	1	2	0.05905	18	4	7	0.35547
18	1	3	0.07438	18	4	8	0.40660
18	1	4	0.08987	18	4	9	0.46396
18	1	5	0.10606	18	4	10	0.52970
18	1	6	0.12337	18	4	11	0.60704
18	1	7	0.14220	18	4	12	0.69808
18	1	8	0.16303	18	4	13	0.80848
18	1	9	0.18643	18	4	14	0.95393
18	1	10	0.21324	18	4	15	1.15242
18	1	11	0.24448	18	4	16	1.43614
18	1	12	0.28159	18	4	17	1.94304
18	1	13	0.32707	18	4	18	3.32041
18	1	14	0.38569	18	5	6	0.37253
18	1	15	0.46565	18	5	7	0.42775
18	1	16	0.58295	18	5	8	0.48891
18	1	17	0.78812	18	5	9	0.55773
18	1	18	1.34892	18	5	10	0.63619
18	2	3	0.11475	18	5	11	0.72766
18	2	4	0.13824	18	5	12	0.83841
18	2	5	0.16283	18	5	13	0.97266
18	2	6	0.18913	18	5	14	1.14151
18	2	7	0.21778	18	5	15	1.37774
18	2	8	0.24949	18	5	16	1.72629
18	2	9	0.28508	18	5	17	2.32637
18	2	10	0.32561	18	5	18	3.97542
18	2	11	0.37307	18	6	7	0.50566
18	2	12	0.43031	18	6	8	0.57749
18	2	13	0.50026	18	6	9	0.65828
18	2	14	0.58793	18	6	10	0.75083
18	2	15	0.70755	18	6	11	0.85835
18	2	16	0.88938	18	6	12	0.98649
18	2	17	1.20258	18	6	13	1.14659
18	2	18	2.05303	18	6	14	1.34806
18	3	4	0.18258	18	6	15	1.61849

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$	n	i	j	$E(X_{(i)}X_{(j)})$
18	6	16	2.03031	18	12	15	3.92257
18	6	17	2.74021	18	12	16	4.83765
18	6	18	4.67341	18	12	17	6.57941
18	7	8	0.67489	18	12	18	11.16376
18	7	9	0.76877	18	13	14	3.85835
18	7	10	0.87613	18	13	15	4.62110
18	7	11	1.00159	18	13	16	5.75134
18	7	12	1.15074	18	13	17	7.73184
18	7	13	1.33403	18	13	18	13.09674
18	7	14	1.57165	18	14	15	5.54305
18	7	15	1.88760	18	14	16	6.88754
18	7	16	2.35899	18	14	17	9.24262
18	7	17	3.19045	18	14	18	15.61845
18	7	18	5.43439	18	15	16	8.49266
18	8	9	0.89238	18	15	17	11.36584
18	8	10	1.01636	18	15	18	19.13707
18	8	11	1.16092	18	16	17	14.71573
18	8	12	1.33380	18	16	18	24.62360
18	8	13	1.54546	18	17	18	35.30464
18	8	14	1.81697	19	1	2	0.05559
18	8	15	2.18668	19	1	3	0.06978
18	8	16	2.72823	19	1	4	0.08402
18	8	17	3.68705	19	1	5	0.09878
18	8	18	6.27895	19	1	6	0.11442
18	9	10	1.17599	19	1	7	0.13129
18	9	11	1.34239	19	1	8	0.14973
18	9	12	1.54101	19	1	9	0.17020
18	9	13	1.78554	19	1	10	0.19330
18	9	14	2.09723	19	1	11	0.21979
18	9	15	2.52144	19	1	12	0.25060
18	9	16	3.14849	19	1	13	0.28715
18	9	17	4.24780	19	1	14	0.33217
18	9	18	7.23181	19	1	15	0.39048
18	10	11	1.55292	19	1	16	0.46969
18	10	12	1.78145	19	1	17	0.58514
18	10	13	2.06249	19	1	18	0.78799
18	10	14	2.42211	19	1	19	1.34173
18	10	15	2.90842	19	2	3	0.10710
18	10	16	3.63125	19	2	4	0.12858
18	10	17	4.89524	19	2	5	0.15088
18	10	18	8.32687	19	2	6	0.17454
18	11	12	2.06657	19	2	7	0.20007
18	11	13	2.39067	19	2	8	0.22803
18	11	14	2.80537	19	2	9	0.25905
18	11	15	3.36673	19	2	10	0.29381
18	11	16	4.19919	19	2	11	0.33363
18	11	17	5.65792	19	2	12	0.38073
18	11	18	9.61305	19	2	13	0.43728
18	12	13	2.79035	19	2	14	0.50540
18	12	14	3.27127	19	2	15	0.59098

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$
19	2	16	0.70986
19	2	17	0.88992
19	2	18	1.19615
19	2	19	2.03209
19	3	4	0.16904
19	3	5	0.19805
19	3	6	0.22884
19	3	7	0.26206
19	3	8	0.29840
19	3	9	0.33884
19	3	10	0.38464
19	3	11	0.43659
19	3	12	0.49636
19	3	13	0.56903
19	3	14	0.66092
19	3	15	0.77597
19	3	16	0.92572
19	3	17	1.15487
19	3	18	1.56345
19	3	19	2.64894
19	4	5	0.24512
19	4	6	0.28292
19	4	7	0.32373
19	4	8	0.36844
19	4	9	0.41796
19	4	10	0.47372
19	4	11	0.53847
19	4	12	0.61366
19	4	13	0.70080
19	4	14	0.80941
19	4	15	0.95445
19	4	16	1.14635
19	4	17	1.42005
19	4	18	1.91853
19	4	19	3.25901
19	5	6	0.33919
19	5	7	0.38778
19	5	8	0.44100
19	5	9	0.50020
19	5	10	0.56674
19	5	11	0.64249
19	5	12	0.73259
19	5	13	0.84036
19	5	14	0.96790
19	5	15	1.13326
19	5	16	1.36710
19	5	17	1.70175
19	5	18	2.28421
19	5	19	3.88579
19	6	7	0.45615
19	6	8	0.51839
19	6	9	0.58751
19	6	10	0.66559
19	6	11	0.75487
19	6	12	0.85802
19	6	13	0.98385
19	6	14	1.13905
19	6	15	1.32996
19	6	16	1.59462
19	6	17	1.99580
19	6	18	2.67738
19	6	19	4.54669
19	7	8	0.60256
19	7	9	0.68250
19	7	10	0.77257
19	7	11	0.87599
19	7	12	0.99659
19	7	13	1.13902
19	7	14	1.31812
19	7	15	1.54635
19	7	16	1.84[illegible]66
19	7	17	2.30567
19	7	18	3.10353
19	7	19	5.25953
19	8	9	0.78747
19	8	10	0.89091
19	8	11	1.00935
19	8	12	1.14798
19	8	13	1.31326
19	8	14	1.51482
19	8	15	1.77803
19	8	16	2.12939
19	8	17	2.64712
19	8	18	3.56742
19	8	19	6.04214
19	9	10	1.02371
19	9	11	1.15923
19	9	12	1.31733
19	9	13	1.50667
19	9	14	1.73877
19	9	15	2.03501
19	9	16	2.44113
19	9	17	3.03391
19	9	18	4.08162
19	9	19	6.91390
19	10	11	1.33013
19	10	12	1.51076
19	10	13	1.72644
19	10	14	1.99225
19	10	15	2.33093

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$	n	i	j	$E(X_{(i)}X_{(j)})$
19	10	16	2.79117	20	1	12	0.22581
19	10	17	3.47339	20	1	13	0.25615
19	10	18	4.66527	20	1	14	0.29221
19	10	19	7.89974	20	1	15	0.33697
19	11	12	1.73537	20	1	16	0.39508
19	11	13	1.98194	20	1	17	0.47338
19	11	14	2.28519	20	1	18	0.58700
19	11	15	2.67351	20	1	19	0.78799
19	11	16	3.19803	20	1	20	1.33519
19	11	17	3.97798	20	2	3	0.10042
19	11	18	5.34059	20	2	4	0.12019
19	11	19	9.03473	20	2	5	0.14057
19	12	13	2.28503	20	2	6	0.16202
19	12	14	2.63279	20	2	7	0.18497
19	12	15	3.07778	20	2	8	0.20989
19	12	16	3.68016	20	2	9	0.23728
19	12	17	4.57278	20	2	10	0.26759
19	12	18	6.13646	20	2	11	0.30165
19	12	19	10.36947	20	2	12	0.34116
19	13	14	3.05613	20	2	13	0.38803
19	13	15	3.56954	20	2	14	0.44349
19	13	16	4.26478	20	2	15	0.50955
19	13	17	5.29448	20	2	16	0.59373
19	13	18	7.09841	20	2	17	0.71269
19	13	19	11.97984	20	2	18	0.89055
19	14	15	4.18911	20	2	19	1.18992
19	14	16	4.99947	20	2	20	2.01323
19	14	17	6.19983	20	3	4	0.15734
19	14	18	8.30172	20	3	5	0.18375
19	14	19	13.98772	20	3	6	0.21157
19	15	16	5.96871	20	3	7	0.24134
19	15	17	7.39038	20	3	8	0.27359
19	15	18	9.87893	20	3	9	0.30906
19	15	19	16.60699	20	3	10	0.34887
19	16	17	9.07093	20	3	11	0.39358
19	16	18	12.09416	20	3	12	0.44370
19	16	19	20.26030	20	3	13	0.50248
19	17	18	15.58518	20	3	14	0.57550
19	17	19	25.95210	20	3	15	0.66629
19	18	19	37.01475	20	3	16	0.77670
20	1	2	0.05253	20	3	17	0.92231
20	1	3	0.06575	20	3	18	1.15100
20	1	4	0.07890	20	3	19	1.55158
20	1	5	0.09245	20	3	20	2.61349
20	1	6	0.10670	20	4	5	0.22650
20	1	7	0.12194	20	4	6	0.26052
20	1	8	0.13845	20	4	7	0.29695
20	1	9	0.15657	20	4	8	0.33651
20	1	10	0.17677	20	4	9	0.37988
20	1	11	0.19963	20	4	10	0.42794

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$
20	4	11	0.48301
20	4	12	0.54667
20	4	13	0.61843
20	4	14	0.70280
20	4	15	0.81206
20	4	16	0.95586
20	4	17	1.13915
20	4	18	1.40552
20	4	19	1.89775
20	4	20	3.20380
20	5	6	0.31106
20	5	7	0.35427
20	5	8	0.40116
20	5	9	0.45278
20	5	10	0.51021
20	5	11	0.57426
20	5	12	0.64891
20	5	13	0.73808
20	5	14	0.84044
20	5	15	0.96271
20	5	16	1.12856
20	5	17	1.35846
20	5	18	1.67793
20	5	19	2.24777
20	5	20	3.80636
20	6	7	0.41493
20	6	8	0.46955
20	6	9	0.52956
20	6	10	0.59652
20	6	11	0.67228
20	6	12	0.75762
20	6	13	0.85878
20	6	14	0.98341
20	6	15	1.13021
20	6	16	1.31324
20	6	17	1.57693
20	6	18	1.96490
20	6	19	2.62149
20	6	20	4.43579
20	7	8	0.54323
20	7	9	0.61235
20	7	10	0.68926
20	7	11	0.77625
20	7	12	0.87648
20	7	13	0.99115
20	7	14	1.13029
20	7	15	1.30648
20	7	16	1.52149
20	7	17	1.81168
20	7	18	2.26188
20	7	19	3.02666
20	7	20	5.10783
20	8	9	0.70287
20	8	10	0.79080
20	8	11	0.89000
20	8	12	1.00396
20	8	13	1.13776
20	8	14	1.29437
20	8	15	1.49077
20	8	16	1.74610
20	8	17	2.07744
20	8	18	2.57993
20	8	19	3.46480
20	8	20	5.83857
20	9	10	0.90337
20	9	11	1.01628
20	9	12	1.14565
20	9	13	1.29710
20	9	14	1.47862
20	9	15	1.69819
20	9	16	1.98551
20	9	17	2.37297
20	9	18	2.93633
20	9	19	3.94257
20	9	20	6.64422
20	10	11	1.15826
20	10	12	1.30519
20	10	13	1.47668
20	10	14	1.68213
20	10	15	1.93474
20	10	16	2.25514
20	10	17	2.69665
20	10	18	3.34095
20	10	19	4.47518
20	10	20	7.54370
20	11	12	1.48732
20	11	13	1.68204
20	11	14	1.91456
20	11	15	2.20132
20	11	16	2.56693
20	11	17	3.06230
20	11	18	3.79912
20	11	19	5.08221
20	11	20	8.56263
20	12	13	1.92062
20	12	14	2.18504
20	12	15	2.51023
20	12	16	2.92689
20	12	17	3.48922
20	12	18	4.32528

(Table 3 continued)

n	i	j	$E(X_{(i)}X_{(j)})$
20	12	19	6.78528
20	12	20	9.73739
20	13	14	2.50594
20	13	15	2.87709
20	13	16	3.35198
20	13	17	3.99490
20	13	18	4.94679
20	13	19	6.61392
20	13	20	11.12020
20	14	15	3.32384
20	14	16	3.86934
20	14	17	4.60792
20	14	18	5.70136
20	14	19	7.61577
20	14	20	12.78936
20	15	16	4.52097
20	15	17	5.37830
20	15	18	6.64776
20	15	19	8.86919
20	15	20	14.87085
20	16	17	6.39414
20	16	18	7.89180
20	16	19	10.51169
20	16	20	17.58581
20	17	18	9.64652
20	17	19	12.81726
20	17	20	21.37105
20	18	19	16.44666
20	18	20	27.26368
20	19	20	38.69915

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13. ABSTRACT

This paper deals with order statistics of the lognormal distribution. References are given to a variety of situations where this distribution is found applicable. The moments, the product moments and the percentage points of the various order statistics are computed and tabulated for all samples of size 20 or less. The methods of computation and the accuracy of the tabulated numerical values are discussed. Application is made to the estimation of parameters of a linear transform of a standard lognormal process. Equivalently, the best linear unbiased estimates of the threshold and the mean of a three-parameter lognormal distribution can be computed from the tables of this paper.