A NOTE ON THE NON-EXISTENCE OF SUBSET SELECTION PROCEDURE FOR POISSON POPULATIONS*

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A NOTE ON THE NON-EXISTENCE OF SUBSET SELECTION PROCEDURE FOR POISSON POPULATIONS*

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1. SUMMARY. The problem of selecting a subset of **k** Poisson populations, which includes the best, i.e., the one having the largest value of the parameter, is considered. Gupta and Nagel (1971) propose a randomized selection rule R_0 for Poisson distribution, and also compare the performance of an identical randomized rule R_0 with a rule R, used for a location parameter problem, for Binomial distribution. However, no similar comparision is possible in case of Poisson distribution because it is shown here that the rule R does not exist for some values of the probability level P* of correct selection. The procedures suggested by Gupta (1965) for location and scale parameters are investigated and it is shown that these procedures do not exist for Poisson populations for some values of probability P* of correct selection. If unrestricted sampling is allowed, one can use the subset selection procedure for smallest of the scale parameter of Gamma populations with some shape parameter, see Gupta and Sobel (1962). A criterion is suggested to choose the shape parameter for this procedure which puts a bound in probability on sampling costs.

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2. Introduction. The motivation for this multiple decision problem is given in a series of papers by Gupta and others on subset selection procedures. Therefore, we start with the formulation of the problem for brevity sake.

Let us assume that $\pi_1, \pi_2, \ldots, \pi_k$ are k Poisson populations, i.e., π_i follows a Poisson distribution with parameter λ_i, i=1, 2,..., k. The observation could possibly be number of occurrences of a Poisson process, with rate of occurrence λ_i, during a unit time interval. Suppose that we have equal sample size from each population. Without loss of generality, one can assume the sample size to be one. Let $\lambda_{[1]} \leq \lambda_{[2]} \leq \cdots \leq \lambda_{[k]}$ and let us assume that the order relationship of $\lambda_1, \ldots, \lambda_k$ is not known.

Given any P*, 0 < P* < 1, we want to select a subset of these k populations such that the subset contains population corresponding to the parameter $\lambda_{[k]}$ with probability at least p*, no matter what the configuragion of $\lambda_1, \lambda_2, \ldots, \lambda_k$ is. The usual notation for this is CS. Therefore we are interested in a selection rule R such that

$$\inf_{\Omega} P(CS|R) \geq P^*$$

where, Ω is the set of all k-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\lambda_1 > 0$.

Let $X_1, X_2, \ldots, X_k$ denote the sample from $\pi_1, \pi_2, \ldots,$ and π_k respectively. The selection rules for location and scale parameters suggested by Gupta (1965) are described below.

Rule R. Select the population π_i in the subset if

$$X_i \geq \max_{j=1,\ldots,k} X_j - d, \qquad (2.1)$$

where $0 \leq d < \infty$ is to be chosen such that

(2.2) $$\inf_{\Omega} P(CS|R) \geq P^* .$$

If the population associated with $\lambda_{[k]}$ is π_k, then (2.2) can be written as

(2.3) $$\inf_{\Omega} P\{X_k \geq \max_{j=1,2,\ldots,(k-1)} X_j - d\} \geq P^* .$$

Without loss of generality d can be assumed to be an integer. This rule is also obtained by the likelihood principle, Gupta and Nagel (1971), under the slippage configuration $(\lambda, \lambda, \ldots, \lambda, \sigma\lambda)$, $\sigma > 1$.

Rule R*. Select the population π_i in the subset if

(2.4) $$X_i \geq c \max_{j=1,\ldots,k} X_j,$$

where $0 < c \leq 1$ is to be chosen such that

(2.5) $$\inf_{\Omega} P\{CS|R^*\} \geq P^*.$$

If the population associated with $\lambda_{[k]}$ is π_k, then (2.5) can be written as

$$\inf_{\Omega} P\{X_k \geq c \max_{j=1,\ldots,(k-1)} X_j\} \geq P^*.$$

3. Non-existence of the rules R and R*.

(a) Rule R.

We have

$$P(CS|R) = P\{X_k \geq \max_{j=1,\ldots,(k-1)} X_j - d\}.$$

Therefore,

(3.1) $$P(CS|R) = \sum_{x=0}^{\infty} e^{-\lambda_{[k]}} \frac{\lambda_{[k]}^x}{x!} \{ \prod_{j=1}^{k-1} \sum_{i=0}^{x+d} e^{-\lambda_{[j]}} \frac{\lambda_{[j]}^i}{i!} \}.$$

Now (3.1) can be written as

$$(3.2)\qquad P(CS|R) = \sum_{x=0}^{\infty} e^{-\lambda_{[k]}} \frac{\lambda_{[k]}^{x}}{x!} \{ \prod_{j=1}^{k-1} \int_{\lambda_{[j]}}^{\infty} \frac{1}{(x+d)!} z^{x+d} e^{-z} dz \}$$

It follows from equation (3.2) that for a fixed $\lambda_{[k]}=\lambda$, the expression inside the braces is a decreasing function of $\lambda_{[j]}$ for each j. Therefore $P(CS|R)$ is minimized, if we set $\lambda_{[1]}=\ldots=\lambda_{[k-1]}=\lambda^*$ and let λ^* go to λ in the limit. Hence,

$$(3.3)\qquad \inf_{\Omega} P(CS|R) = \inf_{\lambda} [\sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^{x}}{x!} (\sum_{i=0}^{x+d} e^{-\lambda} \frac{\lambda^{i}}{i!})^{k-1}].$$

Let us denote

$$(3.4)\qquad f(x,\lambda) = e^{-\lambda} \frac{\lambda^{x}}{x!} ,$$

and

$$(3.5)\qquad F(x,\lambda) = \sum_{i=0}^{x} e^{-\lambda} \frac{\lambda^{i}}{i!} , \ x = 0, 1, 2, \ldots$$

Therefore, we can write

$$(3.6)\qquad \inf_{\Omega} P(CS|R) = \inf_{\lambda} [\sum_{x=0}^{\infty} f(x,\lambda) F^{k-1}(x+d,\lambda)].$$

First we will consider a special case k=2, and then prove that for any $k \geq 2$, the rule R does not provide a desired probability level.

Lemma 1. Let

$$(3.7)\qquad P_d(\lambda) = \sum_{x=0}^{\infty} f(x,d) F(x+d,\lambda),$$

where f and F are defined by (3.4) and (3.5). Then, for any non-negative finite integer d, $P_d(\lambda)$ is a monotone decreasing function of λ, and hence

(3.8) $$\inf_{\lambda} P_d(\lambda) = 0.5$$

Proof: Since $P_d(\lambda)$ is a differentiable function of λ, we get

(3.9) $$\frac{\partial}{\partial\lambda} P_d(\lambda) = e^{-2\lambda} \left\{ \sum_{x=0}^{\infty} \frac{\lambda^{2x+d+1}}{x!(x+d+1)!} - e^{-2\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{2x+d}}{x!(x+d)!} \right\} .$$

It follows from (3.9) that

(3.10) $$\frac{\partial}{\partial\lambda} P_d(\lambda) = P\{Y-Z = d+1\} - P\{Y-Z = d\} ,$$

where Y and Z are i.i.d. random variables each having a Poisson distribution with parameter λ. Since the random variable (Y-Z) has a unique mode at zero and since Poisson distribution is totally positive under translation, it follows that

(3.11) $$P\{Y-Z = d+1\} < P\{Y-Z = d\} \quad \text{for each non-negative integer } d.$$

An alternative proof of (3.11) can be given as follows.
For each non-negative integer d, let

(3.12) $$g_{d+1}(\lambda) = \frac{P\{Y-Z = d+1\}}{P\{Y-Z = d\}} = \lambda h_{d+1}(\lambda),$$

where

$$h_{d+1}(\lambda) = \frac{\sum_{x=0}^{\infty} \frac{\lambda^{2x}}{x!(x+d+1)!}}{\sum_{x=0}^{\infty} \frac{\lambda^{2x}}{x!(x+d)!}}$$

An application of Lemma 2.1, Alam and Thompson (1971), implies that $\frac{\partial}{\partial\lambda} h_{d+1}(\lambda) \leq 0$. However,

$$\frac{\partial}{\partial\lambda} h_{d+1}(\lambda) = -2\lambda h_{d+1}(\lambda)\{h_{d+1}(\lambda) - h_{d+2}(\lambda)\}.$$

Therefore,

(3.13) $$h_{d+1}(\lambda) \geq h_{d+2}(\lambda) \quad \text{for all} \quad \lambda > 0.$$

Now, a successive application of (3.13) implies that

(3.14) $$g_{d+1}(\lambda) \leq g_1(\lambda) \qquad \text{for } d = 0, 1, 2, \ldots \, .$$

Since (Y-Z) has a unique mode at zero, we have $g_1(\lambda) < 1$. Hence,

$$g_{d+1}(\lambda) = \frac{P\{Y-Z = d+1\}}{P\{Y-Z = d\}} < 1,$$

and this proves (3.11).

Now (3.10) and (3.11) together imply that, for every non-negative finite integer d, $P_d(\lambda)$ is a monotone decreasing function of λ.

Now,

$$\inf_{\lambda} P_d(\lambda) = \lim_{\lambda \to \infty} P\{Y \geq Z-d\},$$

where Y and Z are as above. Using the fact that

$$\lambda^{-1/2}(Y-\lambda) \xrightarrow{\mathcal{L}} N(0,1) \text{ as } \lambda \to \infty,$$

we have

$\inf_{\lambda} P_d(\lambda) = P(Y^* \geq Z^*)$, where Y^* and Z^* are i.i.d. random variables having standard normal distribution. Therefore (3.8) holds.

Lemma 2. For any $K \geq 2$ and any non-negative finite integer d,

(3.15) $$\inf_{\lambda} \sum_{x=0}^{\infty} f(x,\lambda) F^{k-1}(x+d,\lambda) \leq 0.5.$$

Proof: Since $F(x+d,\lambda) \leq 1$, $k \geq 2$ implies that

$$(3.16) \qquad \sum_{x=0}^{\infty} f(x,\lambda)F^{k-1}(x+d,\lambda) \leq \sum_{x=0}^{\infty} f(x,\lambda)F(x+d,\lambda) = P_d(\lambda).$$

Therefore,

$$(3.17) \qquad \inf_{\lambda} \sum_{x=0}^{\infty} f(x,\lambda)F^{k-1}(x+d,\lambda) \leq \inf_{\lambda} P_d(\lambda).$$

However, from Lemma 1, the r.h.s. of (3.17) is equal to 0.5. This completes the proof of this lemma.

From equation (3.6) and Lemma 2, we get the following result.

Theorem 1. For any $k \geq 2$, and any non-negative finite integer d,

$$(3.18) \qquad \inf_{\Omega} P(CS|R) \leq \frac{1}{2}.$$

Therefore, if $\frac{1}{2} < P^* < 1$, then the rule R does not exist, since there doesn't exist a $d < \infty$, satisfying (2.2). Our conjecture is that $\inf_{\Omega} P(CS|R) = \frac{1}{k}$, but we are not able to prove it.

(b) Rule R*.

We have

$$P(CS|R^*) = P\{X_k \geq c \max_{j=1,2,\ldots,k-1} X_j\}.$$

Therefore,

$$(3.19) \qquad P(CS|R^*) = \sum_{x=0}^{\infty} f(x,\lambda_{[k]}) \prod_{j=1}^{k-1} F([\tfrac{x}{c}],\lambda_{[j]}),$$

where f and F are defined by (3.4) and (3.5) and $[\frac{x}{c}]$ denotes the integer part of $\frac{x}{c}$. By arguments analogous to the ones after equation (3.2), we have

$$(3.20) \qquad \inf_{\Omega} P(CS|R^*) = \inf_{\lambda}\{\sum_{x=0}^{\infty} f(x,\lambda)F^{k-1}([\tfrac{x}{c}],\lambda)\}.$$

Again, we first consider a special case k=2, and then prove that for any $k \geq 2$, the rule R* does not provide a desired probability level.

<u>Lemma 3</u>. Let

$$(3.21) \qquad P_c(\lambda) = \sum_{x=0}^{\infty} f(x,\lambda)F([\tfrac{x}{c}],\lambda),$$

where f and F are defined by (3.4) and (3.5). Then for any c, $0 < c \leq 1$,

$$(3.22) \qquad \inf_{\lambda} P_c(\lambda) \leq .75$$

<u>Proof</u>. We can write (3.21) as

$$P_c(\lambda) = e^{-\lambda} F(0,\lambda) + \sum_{x=1}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} F([\tfrac{x}{c}],\lambda)$$

$$\leq e^{-\lambda} . e^{-\lambda} + \{1-F(0,\lambda)\}$$

$$= e^{-2\lambda} + 1-e^{-\lambda}$$

$$= 1 - e^{-\lambda} (1-e^{-\lambda}) .$$

Therefore,

$$\inf_{\lambda} P_c (\lambda) \leq \inf_{\lambda} \{1-e^{-\lambda} (1-e^{-\lambda})\} = 0.75.$$

This proves the lemma.

<u>Lemma 4</u>. For any $k \geq 2$, and any c, $0 < c \leq 1$,

$$(3.23) \qquad \inf_{\lambda} \sum_{x=0}^{\infty} f(x,\lambda)F^{k-1}([\tfrac{x}{c}],\lambda) \leq 0.75.$$

The proof is identical to the proof of Lemma 1.

From equation (3.20) and Lemma 4, we get the following result.

<u>Theorem 2</u>. For any $k \geq 2$, and any c, $0 < c \leq 1$,

$$(3.24) \qquad \inf_{\Omega} P(CS|R^*) \leq 0.75.$$

Therefore, if $0.75 < P^* < 1$, then the rule R^* does not exist, since there does not exist a $c \geq 0$, such that (2.5) holds.

4. In this section, it is assumed that instead of observing the number of Poisson events during an interval of fixed length, one can observe all the k Poisson processes at discrete time points t=0, 1, 2,... . Therefore we fix a positive integer N and observe the random variables $T_{i,N}$ (i=1, 2,..., k) where $T_{i,N}$ is the first time point such that number of events, for the ith population, occuring during $(0, T_{i,N})$ is at least N. For selecting a subset containing the 'best' population, one can use the subset selection procedure for smallest of the scale parameter for the Gamma populations with the same shape parameter N, see Gupta and Sobel (1962). If it is assumed that the sampling costs are proportional to the time, for which the processes are observed in order to select a subset, then a criterion is suggested to choose the shape parameter N which puts a probabilistic bound on the sampling costs.

Let $T_{i,N}$ be as defined above and let $X_{i,t}$ = number of events up to time t in the ith population. Define

(4.1) $$V_i = T_{i,N} - 1 + U(m_i, n_i),$$

where $n_i = X_{i,T_N} - X_{i,T_N - 1}$, $m_i = N - X_{i,T_N - 1}$, and U(m,n) is the mth order statistic in a random sample of size n from a uniform distribution on the interval (0,1). If the random samples from U(0,1) are selected independently for each population, then V_i is a sample from a Gamma population with shape parameter N and scale parameter $\theta_i = \frac{1}{\lambda_i}$, i=1,2,...,k; see Alam and Thompson (1971). For selecting a subset containing $\lambda_{[k]}$, one can use $V_1, V_2, \ldots, V_k$ as observations from k Gamma populations and select a subset containing the population with smallest scale parameter. The selection rule described in Gupta and Sobel (1962) is as follows:

Rule R_1: Select π_i if $V_i \leq V_{[1]}/C_N$, $0 < C_N < 1$, where C_N is chosen such that

$$(4.2) \qquad \inf_{\Omega} P(CS|R_1) \geq P^*$$

The tables of C_N-values for N=1(1)25, K=2(1)11 and P*=.75, .90, .95, and .99 are given in Gupta and Sobel (1962a). These C_N-values are given as percentage points of a smallest of several correlated F-statistics.

Since $V_i > T_{i,N}-1$, for all i, it is clear from the selection rule R_1 that we do not have to observe $T_{i,N}$ for those populations for which $T_{i,N} > V_{[1]}/C_N + 1$. Therefore, by putting a bound on $V_{[1]}/C_N$, we will in effect put a bound on the sampling costs. We propose the following criterion.

Choose largest N such that

$$(4.3) \qquad \max_{\Omega} P\{\frac{V_{[1]}}{C_N} > \frac{t}{\lambda_{[1]}}\} < \beta^*$$

for given values of t and β^*.

It is easily shown that the maximum in (4.3) occurs if we set $\lambda_{[k]}=\lambda_{[k-1]}=\ldots=\lambda_{[2]}=\lambda$ and let $\lambda \to \lambda_{[1]}$ in the limit. Thus, (4.3) reduces to

$$(4.4) \qquad P(N,2C_N t) \geq 1-(\beta^*)^{1/k},$$

where $P_N(x)$ is the distribution function of a χ^2- distribution with 2N degrees of freedom, i.e.,

$$P(N,x) = \frac{1}{2^N\Gamma(N)} \int_0^x z^{N-1} e^{-z/2} dz.$$

For given values of P^*, β^* and t we can easily obtain N with the help of tables of C_N-values and tables of the incomplete Gamma function ratio. The extensive tables for P(N,x) by Khamis (1965) could be used for this purpose.

A sample of N values for $P^*=0.75$, $K=2(1)5$, $\beta=.01$, .05, .10, .25, and $t=3(1)10$, 15, 20, 25 are given in Table I.

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Table I. OPTIMAL VALUE of SHAPE PARAMETER

a. Number of populations = 2

β \ t	3	4	5	6	7	8	9	10	15	20	25
.01					1	1	2	3	6	10	14
.05			1	2	2	3	4	4	8	12	16
.10		1	2	2	3	4	5	5	9	13	18
.25	1	2	3	3	4	5	6	7	11	16	20

b. Number of populations = 3

β \ t	3	4	5	6	7	8	9	10	15	20	25
.01								2	6	10	14
.05				1	2	2	3	4	8	12	16
.10			1	2	3	3	4	5	9	13	17
.25	1	1	2	3	4	5	5	6	10	15	19

c. Number of populations = 4

β \ t	3	4	5	6	7	8	9	10	15	20	25
.01								2	6	10	14
.05					1	2	3	4	8	12	16
.10				2	2	3	4	5	9	13	17
.25		1	2	3	4	4	5	6	10	15	19

d. Number of populations = 5

β \ t	3	4	5	6	7	8	9	10	15	20	25
.01								2	6	10	14
.05						2	3	4	8	12	16
.10				1	2	3	4	5	9	13	17
.25		1	2	3	4	4	5	6	10	15	19

Note: Blank space indicates that the equation (4.4) does not hold for the corresponding values of $P^*=.75$, K, t, and β.

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13. ABSTRACT

The problem of selecting a subset of k Poisson populations, which includes the one having the largest value of the parameter, is considered. The subset selection procedures for location and scale parameters are investigated and it is shown that both these procedures do not exist for some values of P*. If unrestricted sampling is allowed, the subset selection procedure for smallest of the scale parameter of Gamma populations with some shape parameter can be used. A criterion is suggested to chose the shape parameter for this procedure which puts a bound in probability on the sampling costs.

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