On Weakly Wandering Functions

by

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Abstract

Hajian [2] showed that ergodic conservative measure preserving point transformations on infinite measure spaces always admit of a weakly wandering set of infinite measure. The notion of a weakly wandering set has been generalized by Neveu [5] for transformations acting on Lebesque spaces. Here we give a Lebesque space analogue of a weakly wandering set of infinite measure on the sequence spaces ℓ_1 and ℓ_∞. We do not know whether weakly wandering functions always exist, but we give sufficient conditions for their existence with respect to certain operators on ℓ_1 and ℓ_∞ determined by a class of positive matrices.

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1. Introduction. Let $(\Omega,\mathcal{B},\mu)$ be a measure space and let $L_1 = L_1(X,\mathcal{B},\mu)$ be the space of all $\mathcal{B}$ measurable functions f on Ω for which the L_1 norm of f, $\int_\Omega |f| d\mu$, is finite. We assume that there is a positive linear operator T acting on L_1 for which $||T|| \leq 1$. We denote the dual of T by T^*; T^* acts on functions $g \in L_\infty$ and is defined by the relationship

$$\int_\Omega f\, T^*g\, d\mu = \int_\Omega TF \cdot g\, d\mu \tag{1.1}$$

for $f \in L_1$, $g \in L_\infty$. We let the sum $\sum_0^\infty T^n f$ be denoted by $T_\infty f$. Throughout this paper, we assume that all operators are conservative; that is, $T_\infty f\ (x)$ has the value 0 or ∞ for each non-negative function $f \in L_1$.

While studying ergodic conservative measure preserving point transformations τ on Ω, Hajian and Kakutani [3] introduced the notion of a weakly wandering set; i.e., a set $W \in \mathcal{B}$ such that there is a sequence of integers $0 = n_0 < n_1 < n_2 < \ldots$ for which the images of W under τ are all disjoint: $\tau^{n_i} W \cap \tau^{n_j} W = \phi$, if $i \neq j$. Hajian and Kakutani showed that the existence of a weakly wandering set of positive measure is equivalent to the existence of an equivalent finite invariant measure which is a basic problem in ergodic theory. This result of Hajian and Kakutani has been extended to L_1 operators by Neveu [5]. A function $h \in L_\infty$ with norm $||h||_\infty = \text{ess sup } |h| \leq 1$ is weakly wandering if and only if for any $\delta > 0$, there is a sequence $0 = n_0 < n_1 < \ldots$ such that

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(1.2) $$\|\sum_{i=0}^{\infty} T^{*n_i} h\|_\infty < 1 + \delta .$$

For ergodic conservative measure preserving point transformations τ on Ω, Hajian [2] showed that if μ is an infinite (σ-finite) measure, then there exists a weakly wandering set W of infinite measure. We also note that in case $\mu(\Omega) = 1$, Sucheston [6] showed that for ergodic conservative point transformations τ which admit no equivalent finite invariant measure, for any $\varepsilon > 0$ there exists a weakly wandering set W of measure $\mu(W) > 1-\varepsilon$. Here, we construct function analogues of Hajian's weakly wandering set of infinite measure. However, our results are not completely general; we restrict our attention to operators on ℓ_1 (defined below) determined by certain matrices. It would be interesting to see these results in a general setting.

We now let $\Omega = \{1,2,...\}$. The σ-field $\mathcal{B}$ is the set of all subsets of Ω and μ is assumed to be the counting measure ascribing to each set in $\mathcal{B}$, the number of its points. The space ℓ_1 is the set of all absolutely convergent sequences $\{f_i\}_{i=1}^{\infty} : \sum_{i=1}^{\infty} |f_i| < \infty$. The dual space ℓ_∞ is the set of all bounded sequences $\{g_i\}_{i=1}^{\infty}: |g_i| \leq M$, $i = 1,2,...$ for some fixed M. Our operators T and T^* acting on ℓ_1 and ℓ_∞ are defined by an infinite matrix $T = [t_{ij}]$ which transforms a function $f \in \ell_1$ with $f = (f_i)_{i=1}^{\infty}$ into a function Tf whose i'th coordinate $(Tf)_i$ is given by

(1.3) $$(Tf)_i = \sum_{j=1}^{\infty} f_j t_{ji} .$$

It may be easily varified that the dual operator T^* of T, determined by (1.1), maps a function $g = (g_i)_{i=0}^{\infty} \in \ell_\infty$ into a function T^*g whose i'th coordinate is given by

$$(1.4) \qquad (T^*g)_i = \sum_{j=1}^{\infty} t_{ij}\, g_j \quad .$$

We remark that the action of T^* on ℓ_∞ may be easily described by considering elements g of ℓ_∞ as column vectors and noting that T^*g is obtained by matrix multiplication of g by T on the left. Similarly T acts on ℓ_1 by a matrix multiplication on the right. We shall use the symbol T to denote both the matrix $[t_{ij}]$ and the ℓ_1 operator determined by $[t_{ij}]$. When $t_{ij} \geq 0$ for $i,j = 1,2,\ldots$ the operator T is _positive_: $Tf \geq 0$ if $f \geq 0$. If $\sum_{j=1}^{\infty} t_{ij} \leq 1$ for each $i = 1,2,\ldots,$ then the matrix $[t_{ij}]$ is _sub-stochastic_; if moreover, $\sum_{i=1}^{\infty} t_{ij} \leq 1$ for each $j = 1,2,\ldots,$ then $[t_{ij}]$ is _doubly sub-stochastic_. It is easy to see that if $[t_{ij}]$ is sub-stochastic, then the operator T maps ℓ_1 into ℓ_1 and $\int|Tf|d\mu \leq \int|f|d\mu$ for any $f \in \ell_1$; i.e., $||T|| \leq 1$. We further remark that in case $[t_{ij}]$ is doubly sub-stochastic, the operator T acting on ℓ_1 actually may be extended to ℓ_∞ by (1.3). Thus T is both an ℓ_1 and ℓ_∞ contraction. For $n = 0,1,\ldots,$ let $t_{ij}^{(n)}$ $i, j = 1,2,\ldots$ be the entries of the n'th power $[t_{ij}]^n$ of the matrix $[t_{ij}]$. Throughout this paper, we assume that the operator T is conservative which means that for any $f = (f_i)_{i=1}^{\infty} \in \ell_1$ with $f_i \geq 0$, $i = 1,2,\ldots$ and any $j \geq 1$,

$$(1.5) \qquad (T_\infty f)_j = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} f_i\, t_{ij}^{(n)} = 0 \text{ or } \infty.$$

This assumption is not difficult to varify for certain classes of matrices.

2. _Main results_. In this section, we construct a function h which is, in two ways, analogous to a weakly wandering set of infinite measure: (1) the set on which $h > 0$ has infinite measure and (2) $\int h = \Sigma h_i = \infty$. It would

seem that the result should extend to general operators, but the author has not been able to obtain such an extension.

Theorem 2.1. Let $[t_{ij}]$ be a doubly sub-stochastic matrix which determines a conservative operator on ℓ_1. Suppose further that $\liminf_{n \to \infty} \sup_{i,j} t_{ij}^{(n)} = 0$. Then for any $\delta > 0$, there exists a function h whose coordinates are 0's and 1's, and a sequence of integers $0 = n_0 < n_1 < \ldots$ such that h has infinitely many 1's and

(2.1) $$\left\| \sum_{k=0}^{\infty} T^{*^{n_k}} h \right\|_\infty \le 1 + \delta.$$

Proof. Let $a_n = \sup_{i,j} t_{ij}^{(n)}$, then since $\liminf a_n = 0$, there is a subsequence $0 = n_0 < n_1 < \ldots$ such that $\sum_{k=0}^{\infty} a_{n_k} \le 1 + \delta/2$. Let ε_i, $i = 1,2,\ldots$ be a sequence such that $\varepsilon_i > 0$ for all $i \ge 1$ and $\sum_{i=1}^{\infty} \varepsilon_i \le \delta/2$. Since both the row and column sums of $[t_{ij}^{(n)}]$ are convergent series, the following conditions hold:

(2.2) $$\lim_{i \to \infty} t^{(n)}_{ij} = 0$$

for all fixed n,j and

(2.3) $$\lim_{j \to \infty} t^{(n)}_{ij} = 0$$

for all fixed n,i. We shall construct an infinite sequence of vectors $h^{(i)}$, $i = 1,2,\ldots$ for which only one coordinate is one and all others are zero. The vectors $h^{(i)}$ will have the further property that no two of them have a one in the same position. The required function will then be $h = \sum_{i=1}^{\infty} h^{(i)}$. Set $N_0 = 1$ and define $h^{(1)} = 1_{\{N_0\}}$ (the indicator function of the singleton set $\{N_0\}$). Then

$$(2.4) \qquad \left\|\sum_{k=0}^{\infty} T^{*^{n_k}} h^{(1)}\right\|_\infty = \sup_i \sum_{k=0}^{\infty} t_{i1}^{(n_k)} \leq \sum_{k=0}^{\infty} a_{n_k} \leq 1 + \delta/2 .$$

Let K_1 be such that $\sum_{k=K_1}^{\infty} a_{n_k} \leq \varepsilon_1/2$. Since for each n, $\lim t^{(n)}_{i1} = 0$, we can find an M_1 such that $i > M_1$ implies that $t_{i1}^{(n_k)} \leq \varepsilon_1/2K_1$ for $k = 0,1,\ldots, K_1 - 1$. Such an M_1 is found by determining M_{1k} such that $t_{i1}^{(n_k)} \leq \varepsilon_1/2K_1$ for $i > M_{1k}$ and setting $M_1 = \max_{0\leq k\leq K_1-1} M_{1k}$. Combining estimates we find that for $i > M_1$

$$(2.5) \qquad \sum_{k=0}^{\infty} t_{i1}^{(n_k)} = \sum_{k=0}^{K_1-1} t_{i1}^{(n_k)} + \sum_{k=K_1}^{\infty} t_{i1}^{(n_k)} \leq \varepsilon_1/2 + \sum_{k=K_1}^{\infty} a_{n_k} \leq \varepsilon_1$$

choose N_1 so large that $j \geq N_1$ implies that for each $i = 1,2,\ldots, M_1$, $\sum_{k=0}^{K_1-1} t_{ij}^{(n_k)} \leq \varepsilon_1/2$. Such an N_1 can be chosen in the following way: for each fixed $i = 1,2,\ldots, M_1$ select N_{1i} such that $j \geq N_{1i}$ implies that $t_{ij}^{(n_k)} \leq \varepsilon_1/2K_1$ for $k = 0,1,\ldots, K_1 - 1$, then set $N_1 = \max_{0<i\leq M_1} N_{1i}$. Next $h^{(2)} = 1_{\{N_1\}}$ is defined. We note that for $0 < i \leq M_1$ we have that

$$(2.6) \qquad \left(\sum_{k=0}^{\infty} T^{*^{n_k}} h^{(2)}\right)_i = \sum_{k=0}^{K_1-1} t^{(n_k)}_{i,N_1} + \sum_{k=K_1}^{\infty} t^{(n_k)}_{i,N_1} \leq \varepsilon_1 ,$$

while for $i > M_1$ we have

$$(2.7)\qquad \Big(\sum_{k=0}^{\infty} T^{*^{n_k}} h^{(2)}\Big)_i \le \sum_{k=0}^{\infty} a_{n_k} \le 1 + \delta/2 \quad .$$

Combining the above with (2.4) and (2.5) we see that for all i, $\Big(\sum_{k=0}^{\infty} T^{*^{n_k}} (h^{(1)} + h^{(2)})\Big)_i \le 1 + \delta/2 + \varepsilon_1$ (there are two cases: $i \le M_1$ and $i > M_1$). Assuming that K_{r-1}, M_{r-1}, N_{r-1}, and $h^{(r-1)}$ have been defined, we define K_r, M_r, N_r and $h^{(r)}$ in the following way: K_r is such that $K_r \ge K_{r-1}$, and $\sum_{k=K_r}^{\infty} a_{n_k} \le \varepsilon_r/2$. In analogy to the case $r = 1$, M_r is defined in such a way that $M_r > M_{r-1}$ and $i > M_r$ implies that $t^{(n_k)}_{i,N_{r-1}} < \varepsilon_r/2K_r$ for $k = 0,1,\ldots, K_r-1$. It immediately follows that for $i > M_r$,

$$(2.8)\qquad \sum_{k=0}^{\infty} t^{(n_k)}_{i,N_{r-1}} = \sum_{k=0}^{K_r-1} t^{(n_k)}_{i,N_{r-1}} + \sum_{k=K_r}^{\infty} t^{(n_k)}_{i,N_{r-1}} \le \varepsilon_r/2 + \sum_{k=K_r}^{\infty} a_{n_k} \le \varepsilon_r \ .$$

We next define N_r in such a way that $j \ge N_r$ implies that for each $i = 1,2,\ldots, M_r$, $\sum_{k=1}^{K_r-1} t^{(n_k)}_{ij} \le \varepsilon_r/2$. Indeed, for each fixed $i = 1,2,\ldots,M_r$, select N_{ri} such that $j \ge N_{ri}$ implies that $t^{(n_k)}_{ij} \le \varepsilon_r/2K_r$ for $k = 0,1,\ldots, K_r-1$, then set $N_r = \max_{0<i\le M_r} N_{ri}$. The function $h^{(r)}$ is defined by means of the formula $h^{(r)} = 1_{\{N_r\}}$. For $i \le M_r$ it follows that

$$(2.9)\qquad \Big(\sum_{k=0}^{\infty} T^{*^{n_k}} h^{(r)}\Big)_i = \sum_{k=0}^{\infty} t^{(n_k)}_{iN_r} \le \varepsilon_r \quad ,$$

while for $i > M_r$;

$$(2.10)\qquad \Big(\sum_{k=0}^{\infty} T^{*^{n_k}} h^{(r)}\Big)_i = \sum_{k=0}^{\infty} t_{iN_r}^{(n_k)} \le \sum_{k=0}^{\infty} a_{n_k} \le 1 + \delta/2 \ .$$

We assert that the function $h = \sum_{r=1}^{\infty} h^{(r)}$ is the required function. It is only necessary to check that $\|\sum_{k=0}^{\infty} T^{*^{n_k}} h\|_{\infty} \le 1 + \delta$. Let r be a given positive integer. For convenience in notation, set $g^{(r)} = \sum_{n=0}^{r} h^{(n)}$. For each i, it is clear that

$$(2.11)\qquad \Big(\sum_{k=0}^{\infty} T^{*^{n_k}} g^{(r)}\Big)_i = \sum_{k=0}^{\infty} \sum_{j=1}^{r} t_{iN_j}^{(n_k)} = \sum_{j=0}^{r} \sum_{k=0}^{\infty} t_{iN_j}^{(n_k)} \ .$$

The interchange of the order of summation is valid because all terms are non-negative. The following tabulation indicates a bound on the terms appearing in the inner sum of the last double sum. These bounds are a consequence of the construction employed.

(2.12)

	$0<i\le M_1$	$M_1<i\le M_2$	$M_2<i\le M_3$	...	$M_r<i\le M_{r+1}$	$M_{r+1}<i$
$\sum_{k=0}^{\infty} t_{iN_0}^{(n_k)}$	$1+\delta/2$	ε_1	ε_1	...	ε_1	ε_1
$\sum_{k=0}^{\infty} t_{iN_1}^{(n_k)}$	ε_1	$1+\delta/2$	ε_2	...	ε_2	ε_2
$\sum_{k=0}^{\infty} t_{iN_2}^{(n_k)}$	ε_2	ε_2	$1+\delta/2$	...	ε_3	ε_3
		$\cdot$	$\cdot$	$\cdot$		
$\sum_{k=0}^{\infty} t_{iN_r}^{(n_k)}$	ε_r	ε_r	ε_r	...	$1+\delta/2$	ε_{r+1}

From the above array, it is clear that for any $i > 0$, Letting $r \to \infty$ in (2.11) and interchanging the order of summation which is justified because all terms are positive, we obtain that for all $i > 0$ $(\sum_{k=0}^{\infty} T^{*^{n_k}} h)_i \leq 1 + \delta$, which completes the proof of the theorem.

3. An Example. In this section, we give a simple example of a matrix satisfying the conditions of Theorem 2.1. For our example, we choose the unrestricted random walk on the integers $0, \pm 1, \pm 2, \ldots$ (see [1] p.342,ff.). We map the integers $n = 0, \pm 1, \ldots$ onto the non-negative integers $m = 0, 1, \ldots$ by means of the mapping

$$m = \begin{cases} -2n & \text{if} \quad n \leq 0 \\ 2n-1 & \text{if} \quad n \geq 0 \end{cases} . \tag{3.1}$$

The transition matrix $T = [t_{ij}]$ will then have the form

$$\begin{aligned} t_{0j} &= \begin{cases} 1/2 & j = 1,2 \\ 0 & \text{otherwise} \end{cases} \\ t_{1j} &= \begin{cases} 1/2 & j = 0,3 \\ 0 & \text{otherwise} \end{cases} \\ t_{ij} &= \begin{cases} 1/2 & j = i-1,\ i+1 \\ 0 & \text{otherwise} \end{cases} . \end{aligned} \tag{3.2}$$

We note that this matrix is doubly stochastic. Moreover, this random walk is recurrent (see [4], p. 49). It also has only one closed communicating class. These two facts imply that for every i,j $\sum_{n=0}^{\infty} t_{ij}^{(n)} = \infty$. This ensures that the operator T on ℓ_1 is conservative in the sense of (1.5).

We next show that the second assumption of Theorem 2.1 is satisfied. Since the n'th power $[t_{ij}^{(n)}]$ of the matrix $[t_{ij}]$ has the probabilistic interpretation that the random walk is in state j given that it started in state i, the rows in $[t_{ij}^{(n)}]$ are binomial probabilities and it is easy to see that

$$(3.3) \qquad \sup_{i,j} t_{ij}^{(n)} = \begin{cases} \binom{2n}{n}/2^n & \text{if } n \text{ even} \\ \binom{2n+1}{n}/2^n & \text{if } n \text{ odd} \end{cases} .$$

It is well known that

$$(3.4) \qquad \lim_{n\to\infty} \binom{2n}{n}/2^{2n} = \lim_{n\to\infty} \binom{2n+1}{n}/2^{2n} = 0 .$$

This shows that all of the conditions of Theorem 2.1 are satisfied.

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