# A NOTE ON SRINIVASAN'S GOODNESS OF FIT TEST

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## SUMMARY

It is shown that for testing goodness of fit to the normal and exponential families, a goodness of fit test proposed by Srinivasan is asymptotically equivalent to a standard test of Kolmogorov-Smirnov type under the null hypothesis and under all non-trivial alternatives. This result appears to be true in general, though no proof is known.

## 1. INTRODUCTION AND DISCUSSION

Let $X_1,\ldots,X_n$ be a random sample of observations from an absolutely continuous distribution. We wish to test the composite null hypothesis that the distribution function is of known functional form $F(x|\theta)$, where the parameter $\theta$ (which may be a vector) is not specified. If $F_n(x)$ is the empiric distribution function, a standard statistic of Kolmogorov-Smirnov type is

$$\hat{D}_n = \sup_x |F_n(x) - F(x|\hat{\theta}_n)|$$

where $\hat{\theta}_n = \hat{\theta}_n(X_1,\ldots,X_n)$ is a maximum likelihood estimator of $\theta$.

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Srinivasan (1970) proposed a test based on an estimate of $F(x|\theta)$ other than $F(x|\hat{\theta}_n)$. He defined

$$\tilde{F}_n(x) = P[X_1 \leq x | T_n]$$

where $T_n$ (which may be a vector) is a sufficient statistic for $\theta$. The goodness of fit test statistic then becomes

$$\tilde{D}_n = \sup_x |F_n(x) - \tilde{F}_n(x)|.$$

Srinivasan computed critical points of $\tilde{D}_n$ by simulation for the exponential and normal families. His numerical results were in error, as was pointed out by Schafer, Finkelstein and Collins (1972). These authors observed that in the exponential case the statistics $\tilde{D}_n$ and $\hat{D}_n$ should have about the same large sample performance. In this note it is shown that in both the exponential and normal cases the two statistics satisfy

$$\sup_x n^{1-\delta} |F(x|\hat{\theta}_n) - \tilde{F}_n(x)| \to 0 \tag{1}$$

with probability 1, for any $\delta > 0$ and for all non-trivial alternatives as well as under the null hypothesis. Thus the performance of $\hat{D}_n$ and $\tilde{D}_n$ should be practically identical for quite small samples.

Two remarks are in order. First Durbin (1971) has shown that $\sqrt{n}\,(F_n(x) - F(x|\hat{\theta}_n))$ converges weakly to a certain Gaussian process. If $\delta = 1/2$ in (1), this convergence (even in probability) is sufficient to show that $\sqrt{n}\,(F_n(x) - \tilde{F}(x))$ has the same limit. Since Durbin treats convergence under sequences of alternatives as well as under the null hypothesis, $\hat{D}_n$ and $\tilde{D}_n$ have identical limiting laws in both situations.

The statement (1) is of course much stronger than this.

Second, the truth of (1) does not depend on the regularity present in the exponential and normal cases. To illustrate this, suppose that the $X_i$ are uniformly distributed on $(0,\theta)$, $\theta > 0$ being unknown. Then

$$\hat{\theta}_n = T_n = \max X_i$$

and calculation shows that

$$F(x|\hat{\theta}_n) = \begin{cases} 0 & x \leq 0 \\ x/\max X_i & 0 < x < \max X_i \\ 1 & x \geq \max X_i \end{cases}$$

and

$$\tilde{F}_n(x) = \begin{cases} 0 & x \leq 0 \\ \dfrac{x}{\max X_i}\left(1 - \dfrac{1}{n}\right) & 0 < x < \max X_i \\ 1 & x \geq \max X_i . \end{cases}$$

So (1) holds here for <u>any</u> distribution of the $X_i$ for which $\max X_i > 0$ always. Of course, the task of testing goodness of fit to the uniform distribution on $(0,\theta)$ is trivial for distributions not satisfying this. Unfortunately, it seems quite difficult to construct a general proof of (1).

## 2. THE EXPONENTIAL CASE

If $X_1, X_2, \ldots$ have density under the null hypothesis

$$
\begin{aligned}
f(x|\theta) &= 0 && x \le 0 \\
&= \frac{1}{\theta} e^{-x/\theta} && x > 0
\end{aligned}
$$

for some $\theta > 0$, then we know that

$$\hat{\theta}_n = T_n = \bar{X}_n$$

$$
\begin{aligned}
F(x|\hat{\theta}_n) &= 0 && x \le 0 \\
&= 1-e^{-x/\bar{X}} && x > 0
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}_n(x) &= 0 && x \le 0 \\
&= 1-(1-\frac{x}{n\bar{X}})^{n-1} && 0 < x < n\bar{X} \\
&= 1 && x \ge n\bar{X} .
\end{aligned}
$$

Therefore

$$\sup_x |F(x|\hat{\theta}_n) - \tilde{F}_n(x)| = \max \{ \sup_{0<x<n\bar{X}} |e^{-x/\bar{X}} - (1 - \frac{x}{n\bar{X}})^{n-1}|, 1 - e^{-n}\} .$$

We need only show that for any $\delta > 0$

$$\sup_{0<u<n} n^{1-\delta}|e^{-u} - (1 - \frac{u}{n})^n| \to 0 \tag{2}$$

to establish (1) for <u>any</u> distribution of the $X_i$ for which $\bar{X} > 0$ always. Distributions not satisfying this are trivial alternatives to the exponential null hypothesis. Suppose indeed that $u = y$ is an extremum of

$e^{-u} - (1 - \frac{u}{n})^n$ in $0 < u < n$. Then the derivative of this function at $y$ vanishes, so that

$$e^{-y} = (1 - \frac{y}{n})^{n-1}$$

and hence

$$|e^{-y} - (1 - \frac{y}{n})^n| = |\frac{y}{n} e^{-y}| < \frac{e^{-1}}{n} \quad . \tag{3}$$

(2) follows immediately from (3).

## 3. THE NORMAL CASE

If $X_1, X_2, \ldots$ are hypothesized to have density

$$f(x|\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \qquad -\infty < x < \infty$$

for some $\theta = (\mu, \sigma^2)$ with $-\infty < \mu < \infty$ and $\sigma^2 > 0$, then

$$T_n = \hat{\theta}_n = (\overline{X}_n, s_n^2) \quad \text{where} \quad s_n^2 = \frac{1}{n} \sum_1^n (X_i - \overline{X})^2 \quad .$$

From Lieberman and Resnikoff (1955) the conditional density of $X_1$ given $T_n = (\overline{x}, s^2)$ is

$$f(x|\overline{x}, s^2) = \frac{\Gamma(\frac{n-1}{2})}{\pi\sqrt{n-1}\,\Gamma(\frac{n-2}{2})} \frac{1}{s} [1 - \frac{1}{n-1}(\frac{x-\overline{x}}{s})^2]^{(n-4)/2}$$

for

$$|\frac{x-\overline{x}}{s}| \leq \sqrt{n-1}$$

and 0 elsewhere. Let $\Phi$ denote the standard normal distribution function and $\Phi_o(t) = \Phi(t) - 1/2$. Standard estimates of the tails of $\Phi$ and symmetry of $f(x|\overline{x}, s^2)$ about $x = \overline{x}$ imply that to establish (1) under <u>any</u> distribution of the $X_i$ we need only study

$$\sup_{0 \le a \le \sqrt{n-1}} \left| C_n \int_0^a \left[1 - \frac{u^2}{n-1}\right]^{\frac{n-4}{2}} du - \Phi_o(a) \right|$$

where $C_n$ denotes the normalizing constant in $f(x|\bar{x},s^2)$. Elementary estimates (see Chu (1956)) give

$$C_n \sqrt{2\pi} \sqrt{\frac{n-1}{n-2}}\, \Phi_o\left(a\sqrt{\frac{n-2}{n-1}}\right) \le C_n \int_0^a \left[1 - \frac{u^2}{n-1}\right]^{\frac{n-4}{2}} du \le C_n \sqrt{2\pi} \sqrt{\frac{n-1}{n-4}}\, \Phi_o\left(a\sqrt{\frac{n-4}{n-1}}\right).$$

It is easily checked using the first few terms of Stirling's series for $\Gamma(x)$ and the bound (3) that

$$C_n \sqrt{2\pi} \left(\frac{n-1}{n-2}\right)^{1/2} = 1 + 0(1/n)$$

and it is almost trivial that

$$\Phi_o\left(a\left(\frac{n-2}{n-1}\right)^{1/2}\right) - \Phi_o(a) = 0(1/n),$$

with similar results for the upper bound. Consequently (1) holds in this case also.

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Security Classification

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Goodness of Fit<br>Normal distribution<br>Exponential distribution<br>Estimation of distribution functions | | | | | | |

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