An Infinite Depth Dam with Poisson Input and Poisson Release

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1. INTRODUCTION. In two recent papers Hasofer ([3] and [4]) studies the content $Z(t)$ of a reservoir with infinite depth. In that model instantaneous inputs occur at random times such that the number, $N(t)$, of these inputs in the interval $(0,t]$ for $t > 0$ is a Poisson process with parameter $\lambda > 0$. The inputs form a sequence of independent random variables $\{X_n\}$, which are independent of $N(t)$ and have a common distribution $B(y)$. There is continuous release of one unit of water per unit time. The reservoir has capacity h and is therefore full when $Z(t) = h$ for some $t \geq 0$. If an input at time τ, say, of random amount X exceeds $h - Z(\tau)$, the deficiency of the reservoir, an instantaneous overflow occurs and the deterministic release continues immediately from the time τ of input. Hasofer obtains the Laplace transform of the content distribution for an initially full reservoir (in [3]) and for an arbitrary initial content (in [4]), and their inversions. These distributions, it is shown, tend as $t \to \infty$ to a limit independent of the initial conditions. The distribution of the time until first overflow is also found. Hasofer's model is characterized by a deterministic release (demand) rule. In certain

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situations, though it is not a demand of a deterministic nature that characterizes the process but rather a demand of a random nature with either random or deterministic input. An example of a situation with random demand is that of a warehouse of capacity h. At regular intervals the level of stock in the warehouse is observed. Let the level of stock at the end of the nth period be Z_n. During the nth period a demand X_n is made upon the stock in the warehouse. When Z_n falls to or below some prespecified reorder level, an order is placed to replenish the stock. Various reorder schemes are possible, and a realistic scheme will include some random delay in replenishing the stock. During the time when the warehouse is empty additional orders are not refused, but kept on record and filled as soon as the stock is replenished. Thus Z_n can also be thought of as taking values in $(-\infty,h]$.

Karlin and Fabens [5] studied a discrete version of the warehouse model above where demands possess a semi-Markov structure. A search of the literature reveals a gap concerning continuous time models in which not only inputs but also releases are allowed to vary in a random manner.

In this paper we fill this gap by considering a continuous time dam theory model in which occurrences of inputs and releases form a Poisson process.

Let $\{Z(t), t > 0\}$ be a stochastic process which represents the content of a reservoir with infinite depth. That is, $Z(t)$ takes values in the interval $(-\infty,h]$, and we may think of $Z(t)$ as the content of the reservoir at time t, measured from an arbitrary point of reference.

The process Z(t) is defined constructively in the following way. Initially Z(0) = z. Z(t) remains constant at the level z for a random length of time, whose distribution function is

$$H(t) = \begin{cases} 1-\exp\{-(\lambda+\mu)t\}, & t \geq 0 \\ 0 & , t < 0 \end{cases}$$

where $\lambda,\mu > 0$. At the end of this random length of time the process Z(t) jumps to a new level. We shall say that such a jump is, with probability $\lambda/(\lambda+\mu)$, an instantaneous input, X, to the reservoir and is, with probability $\mu/(\lambda+\mu)$, an instantaneous release, Y, from the reservoir. The reservoir is full when the content attains the value h. If the input, X, exceeds h-z, the deficiency of the reservoir, an instantaneous overflow occurs, so that Z(t) takes the value h until the occurrence of a release. The process continues in this manner, the waiting times between jumps of the process all following the same distribution H. We assume that the sequences $\{X_n\}$ and $\{Y_n\}$ are independent of each other and of the waiting times. The X_n are assumed to be independent nonnegative random variables with common distribution function B(x). The Y_n are assumed to be independent nonnegative random variables with common distribution function D(y).

2. THE PROCESS Z(t).

2.1 AN INTEGRAL EQUATION FOR THE PROCESS Z(t).

We introduce the following notation.

$$P(Z(t) \le x | Z(0)=z) = W(t,z,x), \quad x < h$$

$$\Phi(\theta,z,x) = \int_0^\infty \exp(-\theta t) W(t,z,x)\,dt, \quad \mathrm{Re}(\theta) > 0$$

$$I(x) = \begin{cases} 1 & \text{for } x \ge 0. \\ 0 & \text{otherwise} \end{cases}$$

For $x \ge h$ we have $W(t,z,x) = 1$. If we denote by $N(t)$ the number of jumps of the process $Z(t)$ in the interval $(0,t]$, then $N(t) < \infty$ almost surely. The forward Kolmogorov integral equation for $W(t,z,x)$ is then valid, and we may concentrate on the nature of the last jump of the process $Z(t)$ before time t. Either there is no jump at all in the interval $(0,t]$, or the last jump occurs at some time $0 < \tau \le t$, $Z(\tau) \le x-y$ and an input of size y occurs with no other jumps in the interval $(\tau, t]$, or the last jump occurs at time τ, $Z(\tau) \le x+y$ and a release of size y occurs with no other jumps in the interval $(\tau,t]$. Adding up the probabilities of these three events over τ and y we obtain the following equation for $W(t,z,x)$ for the case $x < h$.

$$(1)\quad W(t,z,x)=I(x-z)\exp\{-(\lambda+\mu)t\}+\lambda\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\int_0^\infty W(\tau,z,x,-y)\,dB(y)$$
$$+\mu\int_0^t \exp\{-(\lambda+\mu)(t-\tau)\}d\tau\left[\int_0^{h-x} W(\tau,z,x+y)\,dD(y)+1-D(h-x)\right]$$

Converting equation (1) into its Laplace transform we have, for $x < h$,

$$(2)\quad \Phi(\theta,z,x)(\lambda+\mu+\theta)=I(x-z)+\lambda\int_0^\infty \Phi(\theta,z,x-y)\,dB(y)$$
$$+\mu\left[\int_0^{h-x} \Phi(\theta,z,x+y)\,dD(y)+\theta^{-1}(1-D(h-x))\right]$$

where $\mathrm{Re}(\theta) > 0$.

The existence and uniqueness of the solution of (2) is proved as follows by use of the principle of contraction mappings. Let M be the metric space of all bounded complex-valued functions defined on $-\infty < x \leq h$ and integrable in any finite sub interval of $-\infty < x \leq h$. Take as the matric for this space

$$\rho(f,g) = \sup_{-\infty<x\leq h} |g(x)-f(x)|, \qquad f, g \in M.$$

Now define a mapping, $A: M \to M$, of M into itself by the equation

$$A\Phi(x)=I(x-z)(\lambda+\mu+\theta)^{-1}+\lambda(\lambda+\mu+\theta)^{-1}\int_0^\infty \Phi(x-y)\,dB(y) +$$

$$+\mu(\lambda+\mu+\theta)^{-1}[\int_0^{h-x} \Phi(x+y)\,dD(y)+\theta^{-1}(1-D(h-x))].$$

Then for all $x \leq h$

$$|A\Phi_2(x)-A\Phi_1(x)| \leq |\lambda+\mu+\theta|^{-1}\rho(\Phi_1,\Phi_2)(\lambda+\mu).$$

Hence

$$\rho(A\Phi_1,A\Phi_2) \leq |\lambda+\mu+\theta|^{-1}(\lambda+\mu)\rho(\Phi_1,\Phi_2),$$

and since $Re(\theta) > 0$, A is a contraction mapping. By Theorem 1, page 73 of Kolmogorov and Fomin [6], A has a unique fixed point which is the unique solution of equation (2). The solution of (2) appears to be rather complicated in the present general form. A tractable solution is possible for the case in which

$$(3) \qquad D(y) = \begin{cases} 1-\exp(-\beta y), & y \geq 0 \\ 0 & ; \quad y < 0, \end{cases}$$

where $\beta > 0$, and we turn to this in the next sub-section.

2.2 SOLUTION FOR THE TRANSFORM $\Phi(\theta,z,x)$.

From now on we assume D has the form (3), and first exhibit the solution of (2) in the case z=h.

THEOREM 1. *For* z=h *equation* (2) *has the unique solution, for* $x < h$,

$$\Phi(\theta,h,x) = (\beta-\gamma(\theta))(\theta\beta^{-1}\exp\{-\gamma(\theta)(h-x)\}, \tag{4}$$

with

$$\int_0^\infty \exp(-\theta t)P(Z(t)=h|Z(0)=h)dt=[1-(\beta-\gamma(\theta))\beta^{-1}]\theta^{-1}, \tag{5}$$

where $\gamma(\theta)$ *is the unique root of*

$$(\lambda+\mu+\theta)+\mu\beta(\gamma-\beta)^{-1}-\lambda\int_0^\infty \exp(-\gamma y)dB(y)=0 \tag{6}$$

with $0 < \text{Re}(\gamma(\theta)) < \beta$, $\text{Re}(\theta) > 0$.

Proof: We show that one solution of (2) is of the form $\Phi(\theta,h,x)= C(\theta)\exp(rx)$, where clearly in order that Φ be a nonconstant bounded solution of (2), r must satisfy $\text{Re}(r) > 0$. If we substitute this form of the solution into (2) we obtain an identity in x. Comparing the coefficients of $\exp(rx)$ and of $\exp(\beta x)$ on both sides of this identity we obtain the two relations

$$(\lambda+\mu+\theta)+\mu\beta(r-\beta)^{-1}-\lambda\int_0^\infty \exp(-ry)dB(y)=0, \quad r\neq\beta, \tag{7}$$

and

$$\mu\theta^{-1}+\mu\beta C(\theta)(r-\beta)^{-1}\exp(rh)=0. \tag{8}$$

It can be shown by Rouche's theorem that (7) has a unique root $r=\gamma(\theta)$, in $0 < \text{Re}(\gamma(\theta)) < \beta$ for $\text{Re}(\theta) > 0$. Once $\gamma(\theta)$ has been determined (8) then yields the term $C(\theta)$, which in trun yields (4). The uniqueness of solution of (2) guarantees that (4) is the only solution for the case $z=h$.

Finally, since

$$\int_0^\infty \exp(-\theta t)P(Z(t)=h|Z(0)=h)dt=\theta^{-1}-\lim_{x\uparrow h}\Phi(\theta,h,x),$$

(5) follows.

We shall next prove a lemma which is essential in the case $z < h$. Let H be an arbitrary function, defined on the nonnegative half of the real line, which is integrable in every finite subinterval of that half line and which can be expressed as the difference of two monotone nondecreasing functions. Let also

(9) $$U(s) = \sum_{k=0}^{\infty} H^{(k)}(s),$$

where

$$H^{(k)}(s) = \int_0^s H^{(k-1)}(s-u)dH(u), \quad 0 \le s \le h-z, \ k=1,2,\dots .$$

$$H^{(0)}(s) \equiv 1. \qquad , \ 0 \le s \le h-z.$$

<u>LEMMA 1</u>. (i) <u>The Volterra Equation</u>

(10) $$F(\xi)=a+\int_0^\xi F(\xi-y)dH(y), \quad 0 \le \xi \le b < \infty$$

<u>in</u> F, <u>where</u> H <u>is given and has the properties listed above and</u> a <u>is a given constant, has solution given by</u>

(11) $$F(\xi)=aU(\xi)$$

and this solution is unique, provided $U(\xi)$ converges uniformly in $0 \le \xi \le b < \infty$.

(ii) *Let* B *be such that* $B(s)/s \le A < \infty$ for $0 < s \le \epsilon$, *for some* $\epsilon > 0$ *and some constant* $A > 0$. *Then for*

(12) $$H(s)=[\lambda B(s)+\mu(1-\exp(\beta s))](\lambda+\mu+\theta)^{-1}, \quad 0 \le s \le h-z,$$

where z *is fixed but otherwise arbitrary*, U(s) *exists and is finite for* $0 \le s \le h-z$.

Proof: The assertion follows first by a substitution of (11) into (10). Then the interchange of summation and integration operations is justified since the series $U(\xi)$ converges uniformly in $0 \le \xi \le b < \infty$. Also since $|U| < \infty$, for a fixed $\epsilon > 0$, there is an $n_0=n_0\ (\epsilon,s)$ such that $|H^{(n)}(s)| < \epsilon$ for $n > n_0$. Now consider the difference, V, of two solutions of equations (10). V satisfies V=H*V, where * denotes the convolution operation, and hence $V=H^{(n)}*V$ for all n. But the remark above indicates that $H^{(n)}(s) \to 0$ for all s as $n \to \infty$, and hence $V(s)=0$. The solution of (10) is thus unique.

(ii) Let

$$M=|\lambda+\mu+\theta|^{-1}[\lambda A+\beta\mu\exp\{(h-z)\beta\}].$$

It can be shown that $B(s)/s \le A < \infty$ for $0 < s \le \epsilon$, $\epsilon > 0$ implies $B(s)/s \le A < \infty$ for $0 < s \le h-z$ where z is fixed but otherwise arbitrary, and A is used in a generic sense here.

Thus M is finite. The assertion of the lemma now follows from the fact that $|U(s)| \le \exp(Ms)$, which is proved below by using an induction argument.

$$H^{(0)}(s) \equiv 1$$

$$|H^{(1)}(s)| \leq |\lambda+\mu+\theta|^{-1}[\lambda As+\mu(\exp(\beta s)-1)]$$

$$\leq |\lambda+\mu+\theta|^{-1}[\lambda A+\mu\beta\exp\{(h-z)\beta\}]s=Ms,$$

where in the first inequality we have used the fact $B(s)/s \leq A < \infty$ for $0 \leq s \leq h-z$. Suppose $|H^{(k)}(s)| \leq M^{k-1}s^{k-1}/(k-1)!$ We have shown this to be true for k=2. Then

$$(14)\quad |H^{(k)}(s)| \leq |\lambda+\mu+\theta|^{-1}\{\lambda\int_0^s |H^{(k-1)}(s-u)|dB(u)+\mu\int_0^s |H^{(k-1)}(s-u)|d\{\exp(\beta u)-1\}\}$$

$$\leq |\lambda+\mu+\theta|^{-1}M^{k-1}[(k-1)!]^{-1}[\lambda\int_0^s (s-u)^{k-1}dB(u)+\mu\beta\int_0^s (s-u)^{k-1}\exp(\beta u)du].$$

Two successive integrations by parts of the first integral on the right hand side of (14) yield

$$|H^{(k)}(s)| \leq M^{k-1}[(k-1)!]^{-1}|\lambda+\mu+\theta|^{-1}[\lambda A+\mu\beta\exp\{(h-z)\beta\}]s^k/k = M^k s^k/k!$$

It follows by induction therefore that

$$(15)\qquad |H^{(k)}(s)| \leq M^k s^k/k!, \quad k=0,1,2,\ldots$$

Thus

$$|\sum_{k=0}^{\infty} H^{(k)}(s)| \leq \sum_{k=0}^{\infty} |H^{(k)}(s)| \leq \sum_{k=0}^{\infty} M^k s^k/k! = \exp(Ms) < \infty$$

for $0 \leq s \leq h-z$, which proves the lemma.

__REMARK__. In the case B has a density the condition, $B(s)/s \leq A < \infty$

for $0 < s \le \epsilon$, some $\epsilon > 0$, $A > 0$, is satisfied and the lemma holds. The condition (15) guarantees, by the Weierstrass M-test, that $U(\xi)$ converges uniformly.

THEOREM 2. *Suppose* $B(s)/s \le A < \infty$ *for* $0 < s \le \infty$ *for some* $\epsilon > 0$, $A > 0$. *Then for* $z < h$, *equation* (2) *has the unique solution, for* $z \le x < h$,

$$\Phi(\theta,z,x)=\exp\{\gamma(\theta)x\}(\beta-\gamma(\theta))\exp\{-(\gamma(\theta)-\beta)h\}\cdot \tag{16}$$

$$\cdot[\exp(-\beta h)(\beta\theta)^{-1}+(\lambda+\mu+\theta)^{-1}\int_z^h U(v-z)\exp(-\beta v)\,dv]$$

$$+(\lambda+\mu+\theta)^{-1}U(x-z),$$

and, for $x < z$,

$$\Phi(\theta,z,x)=(\beta-\gamma(\theta))\exp\{\gamma(\theta)x\}\exp\{-(\gamma(\theta)-\beta)h\}\cdot \tag{17}$$

$$\cdot[(\beta\theta)^{-1}\cdot\exp(-\beta h)+(\lambda+\mu+\theta)^{-1}\int_z^h U(v-z)\exp(-\beta v)\,dv],$$

where U *is given in* (9) *and* H *by* (12).

Proof: Let

$$\Phi(\theta,z,x) = \begin{cases} \Phi_1(\theta,z,x) & \text{for } z \le x < h \\ \Phi_2(\theta,z,x) & \text{for } x < z \end{cases}$$

Equation (2) may then be broken into the two parts

$$\Phi_1(\theta,z,x)(\lambda+\mu+\theta)=1+\lambda[\int_0^{x-z}\Phi_1(\theta,z,x-y)\,dB(y)+\int_{x-z}^{\infty}\Phi_2(\theta,z,x)\,dB(y)] \tag{18}$$

$$+\mu\exp(\beta x)[\exp(-\beta h)\theta^{-1}+\beta\int_x^h\Phi_1(\theta,z,v)\exp(-\beta v)\,dv],\quad z \le x < h$$

and, for $x < z$,

$$(19)\quad \Phi_2(\theta,z,x)(\lambda+\mu+\theta)=\lambda\int_0^\infty \Phi_2(\theta,z,x-y)\,dB(y)\ +$$

$$+\mu\exp(\beta x)\cdot\beta\cdot\left[\int_x^z \Phi_2(\theta,z,v)\exp(-\beta v)+\int_z^h \Phi_1(\theta,z,v)\exp(-\beta v)+\exp(-\beta h)(\beta\theta)^{-1}\right].$$

Set

$$(20)\qquad J(\theta,x,h)=\int_z^h \exp(-\beta v)\,\Phi_1(\theta,z,v)\,dv,$$

which is independent of x. Now when z=h we saw that Φ_2 had the form

$$(21)\qquad \Phi_2=A\ \exp(rx),\ x < h,\ \mathrm{Re}(r) > 0.$$

We shall construct a solution of (2) this time by putting Φ_2 as in (21) and setting for $\mathrm{Re}(r) > 0$ and $z \leq x < h$,

$$(22)\qquad \Phi_1(\theta,z,x)=A\ \exp(rx)+K(\theta,z,x),$$

where $K(\theta,z,x)$ is a function to be determined. Substitution of (21) into (19) produces an identity in x. Comparing the coefficients of $\exp(rx)$ and $\exp(\beta x)$ on both sides of this identity we obtain the relations (7) and

$$(23)\qquad J+(\beta\theta)^{-1}.\exp(-\beta h)+A(r-\beta)^{-1}.\exp\{(r+\beta)z\}=0,\ r\neq\beta.$$

Equation (7) has a unique root, $r=\gamma(\theta)$, in $0 < \mathrm{Re}(\gamma(\theta)) < \beta$ for $\mathrm{Re}(\theta) > 0$.

Straight substitution of Φ_2 as in (21) and Φ_1 as in (22) into equation (18) leads us to the following integral equation

$$(24)\quad K(\theta,z,\xi+z)(\lambda+\mu+\theta)=1+\lambda\int_0^\xi K(\theta,z,\xi+z-y)\,dB(y)-\mu\beta\int_0^\xi K(\theta,z,\xi+z-\eta)\exp(\beta y)\,d\eta.$$

Here we have made the change of variable $\xi = x-z,\ 0 \leq \xi \leq h-z$. Now let $L(\theta,\xi) = K(\theta,z,\xi+z)$. Equation (24) then becomes

$$L(\theta,\xi) = \frac{1}{\lambda+\mu+\theta}\int_0^{\xi} L(\theta,\xi-y)\,dH(y), \tag{25}$$

where H is defined by (12). Equation (25) is a Volterra type equation, so that from Lemma 1 it follows that $L(\theta,\xi) = (\lambda+\mu+\theta)^{-1}U(\xi)$.

Converting from L back to K we find $\Phi_1(\theta,z,x)$ in terms of A by means of (22). Then using (20) and (23) we obtain

$$J = A(\gamma-\beta)^{-1}(\exp\{(\gamma-\beta)h\}-\exp\{(\gamma-\beta)z\}) + \int_z^h U(v-z)\exp(-\beta v)\,dv, \tag{26}$$

and hence, by (22),

$$A = (\beta-\gamma)\exp\{(\beta-\gamma)h\}\cdot[\exp(-\beta h)(\beta\theta)^{-1} + \int_z^h U(v-z)\exp(-\beta v)\,dv]. \tag{27}$$

From (22) and (21) equations (16) and (17) now follow.

From Theorems 1 and 2 one could derive the moments of Z(t). Let us assume $h = 0$. We can do this without loss of generality, for such an assumption requires only a change of origin of Z(t). Let, for $\mathrm{Re}(s) > 0$,

$$c(t,x) = \int_{-\infty}^{\infty} \exp(sx)\,d_x W(t,z,x) = 1-s\int_{-\infty}^{0} \exp(sx)W(t,z,x)\,dx.$$

Then

$$\int_0^{\infty} \exp(-\theta t)c(t,x)\,dt = \theta^{-1}-s\int_{-\infty}^{0} \exp(sx)\int_0^{\infty} \exp(-\theta t)W(t,z,x)\,dt\ dx$$

by virtue of the integrability of W(t,z,x) and Fubini's theorem. From both (16) and (17) it then follows that

$$\text{(28)} \quad \int_0^\infty \exp(-\theta t)c(t,x)dt = \theta^{-1} - s\int_z^h \exp(sx)U(x-z)dx$$

$$-s \exp\{(\beta+s)h\}(\gamma+s)^{-1}[(\beta\theta)^{-1} + \exp(\beta h)\int_z^h \exp(-\beta v)U(v-z)dv].$$

Thus

$$\text{(29)} \quad \int_0^\infty \exp(-\theta t)E[Z(t)|Z(0)=z]dt = \frac{\partial}{\partial s}\{\int_0^\infty \exp(-\theta t)c(t,s)dt\}|_{s=0}$$

$$= -\{\gamma^{-1}\exp(\beta h)[(\beta\theta)^{-1} + \exp(\beta h)\int_z^h \exp(-\beta v)U(v-z)dv] + \int_z^h U(x-z)dx\}$$

and similarly

$$\text{(30)} \quad \int_0^\infty \exp(-\theta t)E[Z^2(t)|Z(0)=z]dt$$

$$= 2\{\exp(\beta h)\gamma^{-2}[1-\gamma h][(\beta\theta)^{-1} + \exp(\beta h)\int_z^h \exp(-\beta v)U(v-z)dv - \int_z^h xU(x-z)dx\}.$$

In order to investigate the limit behavior of $Z(t)$ as $t\to\infty$ we must investigate the behavior of $\gamma(\theta)$ as $\theta\to 0$. Knowledge of this behavior will then allow us to apply a standard Tauberian argument to $\Phi(\theta,z,x)$. By an application of Rouche's theorem equation (6) has a unique root $r(\theta)=\gamma(\theta)$ such that $0 < |\gamma(\theta)| \leq \alpha\ |(\mu+\theta)/(\lambda+\mu+\theta)|$. The following lemma can be proved by means of the exact same technique employed by Benes in [1].

<u>LEMMA</u> <u>2</u>. <u>For</u> $\theta > 0$,

$$\text{(31)} \quad \lim_{\theta\downarrow 0}\gamma(\theta) = \begin{cases} 0 & \text{if } E_B \leq \mu/\beta\lambda \\ \beta[1-\mu\{\mu+\lambda(1-\zeta)\}^{-1}] & \text{if } E_B > \mu/\beta\lambda, \end{cases}$$

where ζ *is the least nonnegative root of the equation* $\xi=\phi(\xi)$, $0 \leq \xi < 1$ and $\phi(\xi)=B^*(\beta[1-\mu\{\mu+\lambda(1-\xi)\}^{-1}])$, $B^*(r)=\int_0^\infty \exp(-ry)dB(y)$, and $E_B=\int_0^\infty ydB(y)$.

Remark. The root ζ exists. Its properties are discussed on p. 274 of Feller [2].

We now use Lemma 2 to prove the following theorem.

THEOREM 3. Under the conditions of Theorem 2

$$\lim_{t\to\infty} P(Z(t) \leq x) = \Psi(x), \ x \geq 0,$$

independent of the value of z, *where the distribution* Ψ *is given for* $x \geq 0$ *by*

$$\Psi(x) = \begin{cases} (\beta-\gamma^*)\beta^{-1}\exp\{-\gamma^*(h-x)\}, & E_B > \mu/\beta\lambda \\ 1 & , E_B \leq \mu/\beta\lambda, \end{cases} \tag{32}$$

where $\gamma^*=\beta[1-\mu\{\mu+\lambda(1-\zeta)\}^{-1}]$, *and* ζ *is as in Lemma 2.*

Proof: By a standard Tauberian argument (Widder [9],p. 192)

$$\Psi(x) = \lim_{\theta\downarrow 0} \theta\Phi(\theta,z,x), \text{ for } x \geq 0.$$

Applying this argument to Φ first in (4) and then in (16) we arrive at (32) with the aid of Lemma 2. Thus the limit is independent of the initial condition $Z(0) = z$.

The interpretation of this limit is straightforward. If average inputs per unit time exceed average releases per unit time, then Z(t) has a nondegenerate limiting distribution. Otherwise Z(t) degenerates to $-\infty$ as $t\to\infty$.

The important aspect of the model presented here is the random nature of the withdrawal process in addition to the random input process as opposed to a purely deterministic withdrawal process and random input process. The authors became interested in the present model while formulating a new approach to the mathematical theory of quantal response assays (see [7]). The analogous model for $Z(t)$ assuming values in $(0,\infty)$ is studied in [8].

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