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On Some Contributions to Multiple Decision Theory*

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ON SOME CONTRIBUTIONS TO MULTIPLE DECISION THEORY*

By Shanti S. Gupta and Klaus Nagel
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Summary. In this paper we discuss the multiple decision (*selection and ranking*) rules in a general decision theoretic framework. More specifically, we discuss the subset selection problem. The earlier part of the paper describes the general framework and gives some known results for sake of completeness; in the latter part of the paper we give some new results dealing with the subset selection problem for a class of discrete distributions (Section 2). Some relevant tables for these procedures are included. The derivation of rules with some desirable property is made in Section 3 using the likelihood ratio criterion.

1. *Preliminary Definitions and General Formulation.* We are given k populations $\Pi_1, \Pi_2, \ldots, \Pi_k$ where the population Π_i is described by the probability space $(\mathcal{X}, \mathcal{B}, P_i)$, where P_i belongs to some family $\mathcal{P}$. We assume that there is a partial order relation $(\succ)$ defined in $\mathcal{P}$. $P_i \succ P_j$ is equivalent to saying that P_i is better than or equal to P_j; or, in other words P_i is preferred over P_j. For example,

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(1.2) $$\delta(x,d) \geq 0 \quad \text{and} \quad \sum_{d \in \mathcal{D}} \delta(x,d) = 1 \quad ,$$

where $\delta(x,d)$ *denotes the probability that the subset* d *is selected when* x *is observed. The individual selection probability* $p_i(x)$ *for the population* Π_i *is then given by*

(1.3) $$p_i(x) = \sum_{d \ni i} \delta(x,d) \quad ,$$

where the summation is over all d *containing* i. *If the selection probabilities* $p_1(x), p_2(x), \ldots, p_k(x)$ *take on only the values* 0 *and* 1, *then the selection procedure* $\delta(x,d)$ *is completely specified.*

In general, we can assume that the selection of a subset $d \in \mathcal{D}$ results in a loss. Let us consider the situation where $P_i = P(\theta_i, x)$ and assume the loss $L(\theta,d) = L((\theta_1,\theta_2,\ldots,\theta_k),d) = \sum_{i \in d} L_i(\theta)$ where $L_i(\theta)$ is the loss if the ith population is selected. We may assume an additional loss L if a correct selection is not made. The overall risk for the nonrandomized rule δ is:

(1.4) $$R(\theta,\delta) = \sum_{i=1}^{k} L_i(\theta)\, E_\theta p_i(x) + L[1-P_\theta(CS|\delta)] \quad .$$

In many problems it has been assumed that $L_i(\theta) = 1$ and $L = 0$, in which case, $R(\theta,\delta)$ gives the expected size of the selected subset. In general, our aim is to minimize the risk $R(\theta,\delta)$ which will be done under the usual symmetry condition.

The subset selection problems investigated earlier have been concerned with obtaining selection rules δ which select non empty subsets and guarantee a correct selection with probability at least equal to P^* i.e.

$$\inf_{\Omega} P_\omega(CS|\delta) \geq P^* \tag{1.5}$$

where Ω is the space of joint probability measures. The points of Ω are denoted by $\omega = (P_1, P_2, \ldots, P_k)$, $P_i \in \mathcal{P}$. The condition in (1.5) has been called the basic probability requirement.

In general, we wish rules with large probability of a correct selection and a small value of the expected size. The ratio $\eta_\omega(\delta) = k\, P_\omega(CS|\delta)/E_\omega(S|\delta)$ can, among others, be considered as a measure of the efficiency of the procedure δ at ω. It should be pointed out that both $P_\omega(CS|\delta)$ and $E_\omega(S|\delta)$ depend on δ only through the individual selection probabilities and hence if we restrict our attention to these quantities, we can define two rules δ and δ' as equivalent if they have the same individual selection probabilities $p(x)$ and $p'(x)$ for all x. Hence, we can use the following simplified definition, replacing δ by R.

Def. 2. A *subset selection rule* R *is a measurable mapping from* $\mathcal{X}^k$ *into* E^k(k dimensional Euclidean space), *namely,*

$$R: x \to (p_1(x), p_2(2), \ldots, p_k(x)),\ 0 \leq p_i(x) \leq 1,\quad i = 1,2,\ldots,k\ .$$

If p_i's *are 0 or 1, the rule is nonrandomized; in this case,* δ *can also be defined by the sets* $A_i = \{x \in \mathcal{X}^k | p_i(x) = 1\}$, $i = 1,2,\ldots,k$. A_i *is the set of observations for which* Π_i *is selected.*

Def. 3. R *is unbiased iff*

$$\Pi_j \succ \Pi_i,\ i=1,2,\ldots,k \Rightarrow P_{\omega,j} \geq P_{\omega,i} \text{ for all } \omega \in \Omega$$

where $P_{\omega,i} = E_\omega p_i(x)$ = *probability that* Π_i *is selected.*

if $\mathcal{P}$ is a one-parameter family, $P_i(x) = P(\theta_i, x)$, we may define: $P_i \succ P_j$ iff $\theta_i \geq \theta_j$. In many problems $\succ$ denotes stochastic ordering. Other partial orderings that have been considered are: star-shaped ordering, convex ordering, tail ordering.

In the above set-up, we assume that there exists a population Π_j such that $\Pi_j \succ \Pi_i$ for all i. This population Π_j will be referred to as the 'best' population. In case of more than one population satisfying the condition we will consider one of them to be tagged as the best.

From each population we observe a random element X_i. The space of observations is: $\mathcal{X}^k = \{x=(x_1,x_2,\ldots,x_k),\ x_i \in \mathcal{X},$ $i = 1,2,\ldots,k\}$. In most applications $\mathcal{X}^k$ will be a real vector space.

The decision space $\mathcal{D}$ consists of the 2^k subsets d of the set $\{1,2,\ldots,k\}$: to put it formally,

$$\mathcal{D} = \{d \mid d \subseteq \{1,2,\ldots,k\}\} \quad . \tag{1.1}$$

In other words, a decision d corresponds to the selection of a subset of k populations.

A decision $d \in \mathcal{D}$ is called a correct selection (CS) if $j \in d$ which means that the best population Π_j is included in the selected subset d. It should be pointed out that in many subset selection procedures investigated earlier, the null set ϕ is excluded from $\mathcal{D}$ to guarantee the selection of a non empty subset.

Def. 1. *A measurable function* δ *defined on* $\mathcal{X}^k \times \mathcal{D}$ *is called a selection procedure provided that for each* $x \in \mathcal{X}^k$, *we have,* Def. 4. R *is monotone iff*

$$\Pi_j > \Pi_i \Rightarrow P_{\omega,j} \geq P_{\omega,i} \text{ for all } i,j \text{ and all } \omega \in \Omega .$$

We shall restrict ourselves to selection rules which are invariant under permutation.

Def. 5. *A rule* R *is invariant under permutation* (or R is symmetric) *iff*

$$(p_1(gx),\ldots,p_k(gx)) = g(p_1(x),\ldots,p_k(x)) \text{ for all } x \in \mathcal{X}^k, g \in G$$

where G *denotes the group of permutations* g *of the integers* 1,2,...,k. The minimization of the risk under the symmetry condition imposed by G is also discussed in [6].

In addition to the several desirable properties and criteria for selection rules given above, one important concept is that of "just" selection rules investigated in [5]. This concept is examined in some detail in the present paper.

Let $(\mathcal{X},\mathcal{B},P)$ be a probability space where a partial order $>$ is defined on $\mathcal{X}$ [$y > x$ or, equivalently, $x < y$ means that y is better than x].

Def. 6. *A selection rule* R, *defined by its individual selection probabilities* $p_i(x_1,\ldots,x_k)$, $i = 1,2,\ldots,k$ *is said to be just iff*

$$\left.\begin{array}{l} x_i < y_i \\ x_j > y_j, j \neq i \end{array}\right\} \Rightarrow p_i(y_1,\ldots,y_k) \geq p_i(x_1,\ldots,x_k) .$$

For nonrandomized rules determined by acceptance regions $A_1,A_2,\ldots,A_k$, we can give an equivalent definition of a just rule in terms of increasing sets and general stochastic ordering. A subset $A \subset \mathcal{X}^k$ is said to be *increasing* iff $x \in A$ and $y > x \Rightarrow y \in A$. P is stochastically better than Q ($P \underset{St}{>} Q$) iff $P(A) \geq Q(A)$ for all increasing sets $A \in \mathcal{B}$.

We note that if $\mathcal{X}$ is the real line and $\succ$ stands for $>$ (or $\geq$) then the increasing sets are the intervals $[a,\infty)$ and (a,∞) which induce the usual stochastic ordering on the distribution functions.

Def. 7. R *is just iff*

$$\left.\begin{array}{l} x \varepsilon A_i \\ x_i \prec y_i \\ x_j \succ y_j,\ j \neq i \end{array}\right\} \textit{implies}\ \ y \varepsilon A_i \quad .$$

As mentioned earlier, frequently we require a selection rule to satisfy the basic probability requirement (1.5). Hence, a central problem in the subset selection theory is to determine $\inf_{\omega\varepsilon} P_\omega(CS|R)$. For many rules investigated in the literature, this infimum is attained in $_0$ where $_0 \subseteq \Lambda$ is the set of ω where P_i are identical. This could reasonably be expected of a good rule, because in $_0$, no statisical information can be employed to find the arbitrarily tagged population. It has been proved in [5] that this property holds for a just selection rule i.e.

(1.6) $\quad \inf_{\omega\varepsilon} P(CS|R) = \inf_{\omega\varepsilon\ _0} P_\omega(CS|R)$, if R is just.

The above result enables us to restrict our attention to $_0$ for determining the infimum of the probability of a correct selection. Thus, in the case of a one-parameter family of distributions the problem is reduced to finding the infimum of a univariate function. This problem is even more simplified in some cases; for example the rule studied in [3] for selecting a subset of normal populations with means $\sigma_1,\sigma_2,\ldots,\sigma_k$ and a common known variance σ^2 is: Select Π_i iff $\bar{x}_i \geq \bar{x}_{max} - D\sigma/\sqrt{n}$ where $D = D(k,P^*)$ is determined to satisfy the P^* condition. It can easily be seen that this rule is just and that it is invariant under shift in location. Since Ω_0 also is invariant under shift in location, this implies that $P_\omega(CS|R)$ is constant for $\omega \in \Omega_0$. Hence $P_{\omega_0}(CS|R) \geq P^*$ for some $\omega_0 \in \Omega_0$ implies the P^*-condition. It is also a reasonable requirement that $P_\omega(CS|R)$ be constant over Ω_0 because in stating the P^*-condition, we express that we are content if $P_\omega(CS|R)$ is at least P^* and we are not interested in exceeding P^*, at least not in Ω_0 where it can be achieved only by increasing the expected number of populations in the selected subset.

Now we state a lemma which can be applied to construct just subset selection rules with constant probability of a correct selection in Ω_0.

Lemma 1.1. *Let* $X_1, X_2, \ldots, X_k$ *be independent and identically distributed random variables with joint distribution* P_θ. *Let* $T(X_1, X_2, \ldots, X_k)$ *be a sufficient statistic for* θ.

(i) *If* $E(\delta(X_1, \ldots, X_k)|T) = P^*$ *for all* T *then* $E_\theta\delta = P^*$ *for all* θ.

(ii) *If* T *is complete w.r.t.* $\{P_\theta(x)\}$, *then* $E_\theta(\delta(X_1, \ldots, X_k)|T) = P^*$ *is also necessary for* $E_\theta\delta = P^*$ *for all* θ.

The proof is simple and is omitted. This lemma plays a role in some selection procedures discussed in the next section.

2. *Some Selection Rules for Discrete Distributions.* In this section we discuss some new selection rules in the cases of binomial, Poisson, and negative binomial distributions. Very little work has been done under subset selection

formulation in the cases of Poisson and negative binomial distriburions. For the binomial distributions, a subset selection rule was proposed and studied in [4].

Binomial Case: Let $\Pi_1, \Pi_2, \ldots, \Pi_k$ be k binomial populations $B(\theta_i, n)$, $i = 1, \ldots, k$. Since θ_i's are the unknown parameters, $\Omega = \{\omega: \omega = (\theta_1, \theta_2, \ldots, \theta_k)$ and $\Omega_0 = \{\omega: \omega = (\theta, \theta, \ldots, \theta)\}$. We will construct a just selection rule R for selecting the population with the largest θ_i, which is also stochastically the largest, such that $P_{\omega_0}(CS|R) = P^*$ holds for all $\omega \varepsilon \Omega_0$. It is clear that this goal cannot be achieved with a nonrandomized rule, because when $\omega = (0,0,\ldots,0)$ or $\omega = (1,1,\ldots,1)$ the observations will be $x = (0,0,\ldots,0)$ or $x = (n,n,\ldots,n)$ with probability one, requiring the use of individual selection probabilities $p_i = P^*$, $i = 1,2,\ldots,k$, in these cases.

The joint density for $\omega \varepsilon \Omega_0$ is

$$(2.1) \quad f_\omega(x_1, x_2, \ldots, x_k) = (1-\theta)^{nk} \exp[(\sum_1^k x_i) \log \frac{\theta}{1-\theta}] \prod_1^k \binom{n}{x_i} \quad .$$

We see that $T = \sum_{i=1}^k X_i$ is a sufficient statistic for θ.

Since we are interested in symmetric rules R it is sufficient to know one of the individual selection probabilities, say, p_k. From the lemma of the previous section it follows

$$(2.2) \qquad E(p_k(X)|T) = P^* \quad \text{for} \quad T = 0,1,\ldots,kn.$$

The requirement that R be just leads to

$$(2.3) \quad \left.\begin{matrix} y_i \le x_i, i=1,2,\ldots,k-1 \\ y_k \ge x_k \end{matrix}\right\} \Rightarrow p_k(x_1, x_2, \ldots, x_k) \le p_k(y_1, y_2, \ldots, y_k).$$

Figure 1 shows the partial ordering induced by (2.3) among the observation vectors for the case $k = 3$, $n = 2$. The individual selection probability $p_3(x_1,x_2,x_3)$ defines a just rule if its values are nondecreasing in the direction of the arrows. Because of symmetry only one of the two permutations (x_1,x_2,x_3) and (x_2,x_1,x_3) is plotted. The numbers underneath the observation vectors denote the corresponding T values.

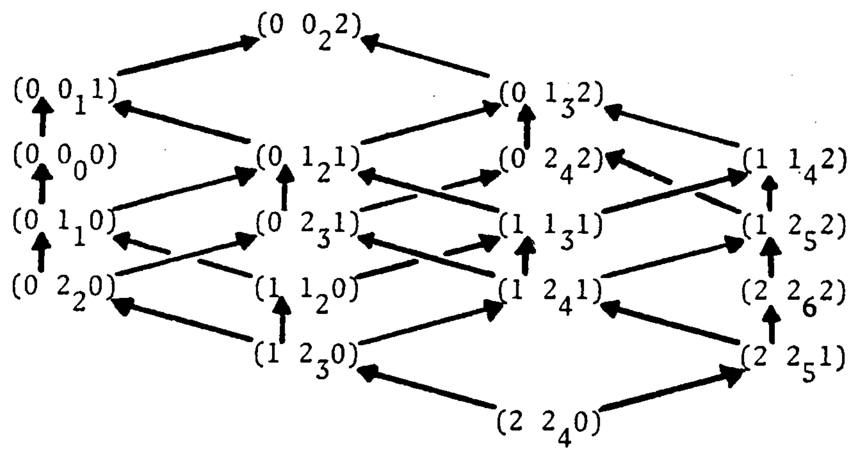

Figure 1. Partial Ordering for Binomial Observations $k = 3$, $n = 2$.

The conditions (2.2) and (2.3) do not determine a rule uniquely. We propose the following rule R_0:

$$(2.4) \qquad p_k(x) = \begin{cases} 1 & \text{if} \quad x_k > c_T \\ \rho & \text{if} \quad x_k = c_T \\ 0 & \text{if} \quad x_k < c_T \end{cases}$$

where $\rho = \rho(T,P^*,k)$ and $c_T = c_T(P^*,k)$ are determined to satisfy

$$(2.5) \qquad E(p_k(X)\,|\,T) = P\{X_k > c_T\,|\,T\} + \rho P\{X_k = c_T\,|\,T\} = P^* .$$

The conditional distribution of X_k given T is hypergeometric.

$$(2.6) \qquad P\{X_k = i|T\} = \frac{\binom{n}{i}\binom{(k-1)n}{T-i}}{\binom{kn}{T}} \quad .$$

Let Z_T have the same distribution as X_k given T. Then (2.5) becomes

$$(2.7) \qquad P\{Z_T > c_T\} + \rho P\{Z_T = c_T\} = P^*$$

and the constant c_T is the smallest integer determined from the inequalities

$$(2.8) \qquad P\{Z_T > c_T\} \leq P^*$$

and

$$(2.9) \qquad P\{Z_T \geq c_T\} > P^* \quad .$$

From (2.7), we have

$$(2.10) \qquad \rho = \frac{P^* - P\{Z_T > c_T\}}{P\{Z_T = c_T\}} \quad .$$

Now, we show that R_0 is just.

Theorem 2.1. R_0 *is just.*

Proof: *Let* $x = (x_1, x_2, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_k)$ *and denote the preference relation induced* (2.3) *by* $y \underset{k}{\succ} x$. Let $T_x = \Sigma x_i$.

Case 1. $T_x = T_y$. In this case $y \underset{k}{\succ} x$ implies $y_k \geq x_k$ and the assertion follows from (2.4).

Case 2. $T_x \neq T_y$. It suffices to show that $p_k(y) \geq p_k(x)$ for those pairs (x,y) where y ranks immediately above x i.e. $y \underset{k}{\succ} x$ and there is no y' such that $y \underset{k}{\succ} y' \underset{k}{\succ} x$ holds. There are two types of such y's for each x:

Type 1: $y_k = x_k + 1$, $y_i = x_i$, $i \neq k$; Type 2; $y_k = y_k, y_j = x_j - 1$, for some $j \neq k$ and $y_j = x_j$ for all other j.

Type 1. Here $T_y = T_x + 1$. If $p_k(x) = 1$, then by (2.8) $P\{Z_T > x_k\} \le p^*$ holds, therefore,

$$P\{Z_{T_y} > y_k\} = P\{Z_{T_x+1} > x_k + 1\} \le P\{Z_{T_x} > x_k\} \le P^*, \tag{2.11}$$

hence $p_k(y) = 1 \ge p_k(x)$.

If $p_k(x) = \rho > 0$ then by (2.8), (2.9)

$$P\{Z_{T_x} > x_k\} \le P^* \text{ and } P\{Z_{T_x} \ge x_k\} = P^*, \text{ holds.} \tag{2.12}$$

From (2.11), we get

$$P\{Z_{T_y} > y_k\} \le P^* \quad . \tag{2.13}$$

If also

$$P\{Z_{T_y} \ge y_k\} \le P^* \tag{2.14}$$

holds, then

$$p_k(y) = 1 \ge p_k(x) \quad .$$

If (2.13) holds but (2.14) does not, we have

$$\begin{aligned} p_k(y) &= \frac{P^* - P\{Z_{T_y} > y_k\}}{P\{Z_{T_y} > y_k - 1\} - P\{Z_{T_y} > y_k\}} \\ &\ge \frac{P^* - P\{Z_{T_x} > x_k\}}{P\{Z_{T_x} > x_k - 1\} - P\{Z_{T_k} > x_k\}} = p_k(x) \end{aligned} \tag{2.15}$$

where the inequality is of the kind $\frac{P^*-a}{b-a} \ge \frac{P^*-A}{B-A}$ with $0 < a \le A \le P^* < b \le B$ and is seen to be true as follows: $\frac{P^*-a}{b-a} \ge \frac{P^*-a}{B-a} \ge \frac{P^*-A}{B-A}$ where the second inequality holds because the expression in the middle is a decreasing function in a.

The third possibility $p_k(x) = 0$ is trivially true.

Type 2. Here $T_y = T_x - 1$. The proof is analogous to that for type 1 with (2.11) replaced by

(2.16) $P\{Z_{T_y} > y_k\} = P\{Z_{T_x-1} > x_k\} < P\{Z_{T_x} > x_k\}$.

This concludes the proof of Theorem 2.1.

Table 1 gives the values of c_T and ρ for several selected values of P^*, k and n. Since T takes on the values $0, 1, \ldots, kn$ these tables become very extensive for large values of k and n. Therefore it is desirable to find approximations for c_T and ρ. The normal approximation for the hypergeometric distribution gives good results when n is large and T is not extreme (close to 0 or kn). The expectation and variance of Z_T are $\mu = \frac{T}{k}$ and $\sigma^2 = \frac{T(kn-T)(k-1)}{(kn-1)k^2}$ respectively. Using the fact that asymptotically Z_T is $N(\mu,\sigma^2)$, we obtain approximate value $\tilde{c}_T$ given by

$$\tilde{c}_T = [\tfrac{1}{2} + \mu - \sigma\, \Phi^{-1}(P^*)]$$

where Φ^{-1} is the inverse of the standard normal cdf and $[x]$ is the integral part of x. For ρ we get the approximate value $\tilde{\rho} = \tilde{c}_T + 0.5 - (\mu - \sigma\Phi^{-1}(P^*))$. The exact and approximate values of c_T and ρ were compared for $k = 2,3,5,10$; $n = 5,10,20$; and some selected values of T and P^*. The results showed no change in the values of c_T and $\tilde{c}_T$ and small derivations in the values of ρ and $\tilde{\rho}$.

The nonrandomized version R_0' of R_0,

R_0': Select Π_i iff $x_i \geq c_T$, is conservative in the sense of meeting the basic probability requirement. However, R_0' may not be just and it selects large subsets if the θ_i's are close to zero or one.

The performance of a rule R,

$$R: \text{ Select } \Pi_i \text{ iff } x_i \geq \max_{1\leq j\leq k} x_j - D \quad ,$$

was studied in [4] and a table was given for the expected proportion under various slippage configurations $(\theta,\theta,\ldots,\theta,\theta+\delta)$. A comparison of R_0 and R is difficult because $\inf_{\Omega} P_\omega(CS|R)$ is not known. Since it takes place near $\theta = \frac{1}{2}$, the P*-value for R_0 was chosen to satisfy $P_\omega(CS|R) = P^*$ with $\omega = (\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ which makes the comparison slightly more favorable for R. The numerical computations showed that R_0 yields better results for small values of δ, while R is better for large δ. Hence R_0 should be applied if small differences in the success probabilities are expected. This disadvantage of R_0 becomes more evident in the case of equally spaced configurations, where almost surely more than half of the populations will be retained in the selected subset if the number of observations is increased indefinitely, whereas R will eventually select only the best one. In Section 3 a rule will be proposed, which combines the advantages of both R and R_0.

Poisson Case:

A selection rule similar to the rule R_0 for the binomial case has been constructed for the Poisson case. The conditional probability in this case is

$$P\{X_k = x|T\} = \binom{T}{x} \frac{(k-1)^{T-x}}{k^T} \quad . \tag{2.17}$$

The rule is of the same type as R_0 defined in (2.4) and the constant c_T and ρ are determined as before except that the conditional distribution is now given by (2.17). Table 2 gives the values of c_T and ρ for various k, T and P^*.

Negative Binomial Case:

A selection rule for large θ for the negative binomial distribution with probability function $\binom{x-1}{r-1}\theta_i^r(1-\theta_i)^{x-r}$ is similar to R_0 except that $p_i(x) = 1$ or 0 according as $x_i < c_T$ or $> c_T$. The evaluation of the constants c_T and ρ is accomplished as before. Table 3 gives values of these for selected values of k, P^* and r.

Similar selection rules have also been computed for Fisher's logarithmic distribution [5].

Remark 2.1. It should be pointed out that the rules discussed in this section overcame the difficulty in the evaluation of the infimum of the probability of a correct selection encountered in rules of the type R for the binomial case that was studied in [4]. The conditional rules of the type R_0 lead to $P(CS|R_0)$ which is constant in Ω_0 which is not the case for rules of type R.

3. *Some Rules with Constant* $P(CS|R)$ *in* Ω_0 *derived from the Likelihood Ratio Criterion.* From a likelihood ratio test under slippage hypotheses a derivation was given in [1] for the following rule for selecting a subset containing the one with highest mean from several normal populations. This derivation can be generalized for Koopman-Darmois families and more general hypotheses.

Let X_i, $i = 1,2,\dots,k$, have the probability densities

$$f(\theta_i, X_i) = c(\theta_i)e^{Q(\theta_i)T(x_i)}h(x_i) \quad . \tag{3.1}$$

If we make the usual assumption that $Q(\theta_i)$ is strictly monotone, say increasing, so that we can consider $Q(\theta_i)$ as the parameter and rename it θ_i, simplifying (3.1) to

$$f(\theta_i, X_i) = c(\theta_i)e^{\theta_i T(x_i)}h(x_i) \quad . \tag{3.2}$$

Let us assume we know that the θ_i take on the values $\theta_1' \leq \theta_2' \leq \ldots \leq \theta_k'$, but that the correct pairing is not known. Consider the set of hypotheses

$$(3.3) \qquad H_i: \ \theta_i = \theta_k'; \ \ i = 1,\ldots,k,$$

i.e. H_i is the hypothesis that θ_i corresponds to θ_k' without specifying the parameters of the remaining $(k-1)$ populations. If Ω_i, $i = 1,2,\ldots,k$, denotes the subset of Ω where H_i is true, then the likelihood ratio test of H_{k} against the alternatives $H_1,\ldots,H_{k-1}$ yields the region of acceptance:

$$(3.4) \quad \lambda = \frac{\max\limits_{\omega\varepsilon\Omega_k} \prod\limits_{i=1}^{k} f(\theta_i',x_i)}{\max\limits_{\omega\varepsilon\Omega_k} \prod\limits_{i=1}^{k} f(\theta_i',x_i)} = e^{\Sigma\theta_i'[T_{[i]}' - T_{[i]}]} \geq c$$

where the $T_{[i]}$ are the ordered values of $T_i = T(x_i)$, $i = 1,2,\ldots,k$, and $T_{[i]}'$ are the ordered values of T_i, $i = 1,2,\ldots,k-1$, $T_{[k]}' = T_k$. Let r be the rank of T_k among the T_i's, i.e. $T_{[r]} = T_k$. Then (3.4) becomes

$$(3.5) \quad \sum_{i=1}^{k} \theta_i(T_{[i]}'-T_{[i]}) = \sum_{j=r+1}^{k} (\theta_{j-1}-\theta_j)T_{[j]}+(\theta_k-\theta_r)T_k \geq c_1 \ .$$

Under slippage configuration $\omega_k = (\theta_1',\ldots,\theta_k')=(\theta,\ldots,\theta,\theta+\delta)$, (3.5) simplifies to

$$(3.6) \qquad -\delta T_{[k]} + \delta T_k \geq c_1$$

or

$$(3.7) \qquad T_k \geq T_{[k]} - c_2 \ .$$

If θ and δ are known this gives rise to the selection rule

(3.8) R: Select Π_i if $T_i \geq T_{[k]} - c_2$

where $c_2 = c_2(k,P^*,\theta,\delta)$ is determined from the P^*-condition

(3.9) $$P_{\omega_k}\{T_k \geq T_{[k]} - c_2\} = P^* .$$

The rule given in (3.8) was introduced by Gupta [1,3]. It can easily be seen that this rule is just, hence if we keep θ fixed the minimum of $P(CS)$ takes place of when $\delta = 0$, in which case (3.9) becomes

(3.10) $$\int_{-\infty}^{\infty} F_\theta^{k-1}(t+c_2)dF_\theta(t) = P^* ,$$

where F_θ is the cumulative distribution function of T. For normal distributions with θ as location parameter, c_2 in (3.10) does not depend on θ. For this case the constants c_2 are tabulated in [2]. In general c_2 depends on θ and if θ is not known an estimator for θ may be used in (3.9). Since ΣT_i is a sufficient statistic for θ, this leads to a selection rule of the form

(3.11) Select Π_i, if $T_i \geq T_{[k]} - c(\Sigma T_i,P^*)$.

By Lemma 1.1 this rule has constant $P(CS)$ in Ω_0, if $c(\Sigma T_i,P^*)$ is determined to satisfy:

(3.12) $$P_{\omega_0}\{T_i \geq T_{[k]} - c(\Sigma T_i,P^*)|\Sigma T_j\} = P^* \quad \text{for all } \Sigma T_i,\ \omega_0 \varepsilon\ \Omega_0 .$$

However it is not known whether (4.11) is a just rule.

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Table 1. c_T and ρ_T for Binomial Distributions

K = 2 N = 5

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
0	0	.75	0	.90	0	.95	0	.99
1	0	.50	0	.80	0	.90	0	.98
2	1	.95	0	.55	0	.78	0	.96
3	1	.60	1	.96	0	.40	0	.88
4	1	.05	1	.68	1	.89	0	.58
5	2	.63	1	.03	1	.54	1	.94
6	2	.05	2	.68	2	.89	1	.58
7	3	.60	3	.96	2	.40	2	.88
8	4	.95	3	.55	3	.78	3	.96
9	4	.50	4	.80	4	.90	4	.98
10	5	.75	5	.90	5	.95	5	.99

K = 3 N = 5

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
0	0	.75	0	.90	0	.95	0	.99
1	0	.63	0	.85	0	.93	0	.98
2	0	.42	0	.77	0	.88	0	.98
3	0	.05	0	.62	0	.81	0	.96
4	1	.78	0	.35	0	.68	0	.94
5	1	.53	1	.95	0	.40	0	.88
6	1	.17	1	.77	1	.97	0	.76
7	2	.83	1	.50	1	.81	0	.46
8	2	.54	1	.00	1	.54	1	.97
9	2	.15	2	.78	2	.99	1	.82
10	3	.79	2	.45	2	.78	1	.42
11	3	.48	3	.93	2	.37	2	.91
12	4	.98	3	.65	3	.87	2	.55
13	4	.68	4	.99	3	.48	3	.90
14	4	.25	4	.70	4	.85	4	.97
15	5	.75	5	.90	5	.95	5	.99

K = 5 N = 5

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
0	0	.75	0	.90	0	.95	0	.99
1	0	.69	0	.87	0	.94	0	.99
2	0	.61	0	.84	0	.92	0	.98
3	0	.50	0	.80	0	.90	0	.98
4	0	.35	0	.74	0	.87	0	.97
5	0	.14	0	.66	0	.83	0	.97
6	1	.93	0	.54	0	.77	0	.95
7	1	.78	0	.38	0	.69	0	.94
8	1	.63	0	.14	0	.57	0	.91
9	1	.46	1	.94	0	.39	0	.88
10	1	.25	1	.83	0	.12	0	.82
11	2	.99	1	.70	1	.94	0	.73
12	2	.82	1	.53	1	.84	0	.59

K = 5 N = 5 (Cont'd.)

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
13	2	.65	1	.30	1	.71	0	.33
14	2	.45	2	.98	1	.53	1	.99
15	2	.22	2	.85	1	.24	1	.91
16	3	.95	2	.69	2	.95	1	.80
17	3	.77	2	.47	2	.81	1	.60
18	3	.58	2	.13	2	.63	1	.19
19	3	.34	3	.89	2	.31	2	.93
20	3	.01	3	.71	3	.94	2	.77
21	4	.81	3	.44	3	.78	2	.39
22	4	.62	4	.98	3	.48	3	.94
23	4	.35	4	.80	4	.95	3	.70
24	5	.94	4	.50	4	.75	4	.95
25	5	.75	5	.90	5	.95	5	.99

K = 2 N = 10

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
0	0	.75	0	.90	0	.95	0	.99
1	0	.50	0	.80	0	.90	0	.98
2	1	.98	0	.58	0	.79	0	.96
3	1	.63	0	.05	0	.53	0	.91
4	1	.17	1	.77	1	.97	0	.77
5	2	.72	1	.38	1	.75	0	.38
6	2	.26	2	.88	1	.31	1	.93
7	3	.77	2	.51	2	.85	1	.69
8	3	.31	3	.94	2	.47	2	1.00
9	4	.79	3	.57	3	.90	2	.77
10	4	.33	4	.96	3	.51	2	.14
11	5	.79	4	.57	4	.90	3	.77
12	5	.31	5	.94	4	.47	4	1.00
13	6	.77	5	.51	5	.85	4	.69
14	6	.26	6	.88	5	.31	5	.93
15	7	.72	6	.38	6	.75	5	.38
16	7	.17	7	.77	7	.97	6	.77
17	8	.63	7	.05	7	.53	7	.91
18	9	.98	8	.58	8	.79	8	.96
19	9	.50	9	.80	9	.90	9	.98
20	10	.75	10	.90	10	.95	10	.99

K = 3 N = 10

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
0	0	.75	0	.90	0	.95	0	.99
1	0	.63	0	.85	0	.93	0	.98
2	0	.43	0	.77	0	.89	0	.98
3	0	.11	0	.64	0	.82	0	.96
4	1	.82	0	.43	0	.72	0	.94
5	1	.58	0	.08	0	.54	0	.91

Table 1 (Continued)

K = 3 N = 10 (Cont'd.)								
	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
6	1	.29	1	.87	0	.23	0	.85
7	2	.94	1	.67	1	.94	0	.74
8	2	.68	1	.41	1	.78	0	.54
9	2	.38	2	1.00	1	.57	0	.15
10	2	.00	2	.80	1	.22	1	.93
11	3	.73	2	.54	2	.91	1	.80
12	3	.43	2	.18	2	.70	1	.56
13	3	.05	3	.86	2	.38	1	.11
14	4	.75	3	.60	3	.96	2	.89
15	4	.44	3	.23	3	.74	2	.67
16	4	.05	4	.87	3	.42	2	.26
17	5	.74	4	.60	4	.96	3	.90
18	5	.42	4	.19	4	.73	3	.66
19	5	.00	5	.84	4	.36	3	.15
20	6	.70	5	.53	5	.91	4	.85
21	6	.36	5	.04	5	.63	4	.50
22	7	.94	6	.75	5	.13	5	.94
23	7	.63	6	.37	6	.79	5	.67
24	7	.24	7	.90	6	.39	6	.98
25	8	.84	7	.57	7	.88	6	.72
26	8	.50	8	.98	7	.52	7	.97
27	8	.01	8	.68	8	.91	7	.66
28	9	.68	8	.03	8	.52	8	.90
29	9	.25	9	.70	9	.85	9	.97
30	10	.75	10	.90	10	.95	10	.99
K = 5 N = 10								
0	0	.75	0	.90	0	.95	0	.99
1	0	.69	0	.88	0	.94	0	.99
2	0	.61	0	.84	0	.92	0	.98
3	0	.50	0	.80	0	.90	0	.98
4	0	.37	0	.75	0	.87	0	.97
5	0	.20	0	.68	0	.84	0	.97
6	1	.98	0	.59	0	.79	0	.96
7	1	.84	0	.46	0	.73	0	.95
8	1	.69	0	.30	0	.65	0	.93
9	1	.54	0	.08	0	.54	0	.91
10	1	.37	1	.93	0	.39	0	.88
11	1	.17	1	.83	0	.19	0	.84

K = 5 N = 10 (Cont'd.)								
	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
12	2	.96	1	.72	1	.98	0	.78
13	2	.80	1	.58	1	.90	0	.71
14	2	.64	1	.41	1	.80	0	.60
15	2	.46	1	.20	1	.69	0	.44
16	2	.27	2	.97	1	.54	0	.22
17	2	.04	2	.85	1	.36	1	.98
18	3	.86	2	.72	1	.11	1	.92
19	3	.69	2	.56	2	.94	1	.85
20	3	.51	2	.36	2	.82	1	.75
21	3	.31	2	.12	2	.69	1	.61
22	3	.08	3	.92	2	.51	1	.41
23	4	.88	3	.78	2	.29	1	.13
24	4	.70	3	.61	3	1.00	2	.95
25	4	.52	3	.41	3	.88	2	.85
26	4	.31	3	.16	3	.73	2	.72
27	4	.07	4	.94	3	.56	2	.54
28	5	.87	4	.78	3	.32	2	.27
29	5	.68	4	.60	3	.01	3	.97
30	5	.49	4	.39	4	.87	3	.86
31	5	.27	4	.11	4	.71	3	.71
32	5	.00	5	.90	4	.50	3	.50
33	6	.82	5	.73	4	.22	3	.16
34	6	.63	5	.52	5	.95	4	.93
35	6	.42	5	.26	5	.79	4	.78
36	6	.17	6	.96	5	.59	4	.57
37	7	.93	6	.80	5	.31	4	.22
38	7	.74	6	.59	6	.97	5	.93
39	7	.53	6	.33	6	.81	5	.77
40	7	.30	7	.98	6	.59	5	.50
41	8	1.00	7	.81	6	.28	5	.01
42	8	.81	7	.60	7	.94	6	.86
43	8	.60	7	.30	7	.75	6	.62
44	8	.36	8	.94	7	.48	6	.09
45	8	.04	8	.75	8	.99	7	.87
46	9	.82	8	.49	8	.81	7	.56
47	9	.62	9	.99	8	.52	8	.96
48	9	.35	9	.81	9	.96	8	.73
49	10	.94	9	.50	9	.75	9	.95
50	10	.75	10	.90	10	.95	10	.99

Table 2. c_T and ρ_T for Poisson Distributions

K = 2

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
0	0	.75	0	.90	0	.95	0	.99
1	0	.50	0	.80	0	.90	0	.98
2	0	0.00	0	.60	0	.80	0	.96
3	1	.67	0	.20	0	.60	0	.92
4	1	.25	1	.85	0	.20	0	.84
5	2	.80	1	.56	1	.88	0	.68
6	2	.40	1	.10	1	.63	0	.36
7	3	.91	2	.77	1	.23	1	.96
8	3	.52	2	.41	2	.86	1	.81
9	3	.02	3	.94	2	.57	1	.54
10	4	.62	3	.61	2	.11	1	.08
11	4	.15	3	.16	3	.79	2	.85
12	5	.71	4	.78	3	.43	2	.58
13	5	.26	4	.38	4	.96	2	.13
14	6	.79	5	.92	4	.65	3	.84
15	6	.35	5	.56	4	.22	3	.55
16	7	.87	5	.08	5	.83	3	.07
17	7	.43	6	.70	5	.46	4	.80
18	8	.94	6	.27	6	.97	4	.47
19	8	.51	7	.83	6	.65	5	.98
20	8	.01	7	.43	6	.21	5	.72
21	9	.58	8	.94	7	.80	5	.34
22	9	.10	8	.57	7	.42	6	.91
23	10	.65	8	.09	8	.94	6	.61
24	10	.18	9	.69	8	.59	6	.17
25	11	.72	9	.24	8	.12	7	.81
26	11	.25	10	.80	9	.74	7	.46
27	12	.78	10	.38	9	.32	8	.97
28	12	.32	11	.91	10	.87	8	.68
29	13	.83	11	.50	10	.48	8	.26
30	13	.38	11	.00	11	.99	9	.85
31	14	.89	12	.62	11	.63	9	.50
32	14	.44	12	.15	11	.17	9	.00
33	15	.95	13	.72	12	.76	10	.70
34	15	.50	13	.27	12	.34	10	.28
35	16	1.00	14	.82	13	.88	11	.86
36	16	.55	14	.39	13	.48	11	.50
37	16	.06	15	.91	14	.99	11	.01
38	17	.61	15	.50	14	.62	12	.69
39	17	.12	16	1.00	14	.15	12	.26
40	18	.66	16	.60	15	.74	13	.84

K = 2 (Cont'd.)

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
41	18	.18	16	.12	15	.30	13	.47
42	19	.71	17	.69	16	.85	14	.98
43	19	.23	17	.23	16	.44	14	.63
44	20	.76	18	.78	17	.95	14	.20
45	20	.29	18	.34	17	.56	15	.80
46	21	.80	19	.86	17	.08	15	.41
47	21	.34	19	.43	18	.68	16	.94
48	22	.85	20	.94	18	.22	16	.58
49	22	.39	20	.53	19	.78	16	.11
50	23	.90	20	.03	19	.35	17	.73

K = 3

T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
0	0	.75	0	.90	0	.95	0	.99
1	0	.63	0	.85	0	.93	0	.98
2	0	.44	0	.77	0	.89	0	.98
3	0	.16	0	.66	0	.83	0	.97
4	1	.87	0	.49	0	.75	0	.95
5	1	.64	0	.24	0	.62	0	.92
6	1	.38	1	.95	0	.43	0	.89
7	1	.07	1	.80	0	.15	0	.83
8	2	.80	1	.61	1	.93	0	.74
9	2	.54	1	.37	1	.80	0	.62
10	2	.25	1	.05	1	.62	0	.42
11	3	.93	2	.84	1	.40	0	.14
12	3	.68	2	.64	1	.09	1	.95
13	3	.39	2	.39	2	.89	1	.85
14	3	.07	2	.07	2	.71	1	.73
15	4	.79	3	.84	2	.49	1	.55
16	4	.51	3	.62	2	.21	1	.30
17	4	.21	3	.35	3	.93	2	.99
18	5	.90	3	.02	3	.75	2	.88
19	5	.62	4	.80	3	.52	2	.73
20	5	.32	4	.57	3	.24	2	.53
21	6	.99	4	.28	4	.95	2	.27
22	6	.72	5	.97	4	.76	3	.97
23	6	.43	5	.74	4	.52	3	.84
24	6	.11	5	.48	4	.24	3	.66
25	7	.81	5	.18	5	.94	3	.43
26	7	.52	6	.90	5	.74	3	.13
27	7	.21	6	.66	5	.49	4	.91
28	8	.89	6	.38	5	.20	4	.74

Table 2 (Continued)

K = 3 (Cont'd.)								
	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
29	8	.61	6	.05	6	.92	4	.53
30	8	.30	7	.81	6	.70	4	.25
31	9	.97	7	.55	6	.44	5	.96
32	9	.69	7	.25	6	.13	5	.79
33	9	.39	8	.94	7	.87	5	.58
34	9	.06	8	.70	7	.64	5	.32
35	10	.77	8	.42	7	.37	6	.99
36	10	.47	8	.10	7	.05	6	.83
37	10	.15	9	.83	8	.81	6	.62
38	11	.84	9	.57	8	.57	6	.36
39	11	.55	9	.27	8	.28	6	.03
40	11	.23	10	.95	9	.96	7	.84
41	12	.91	10	.70	9	.74	7	.63
42	12	.62	10	.42	9	.48	7	.38
43	12	.31	10	.10	9	.17	7	.05
44	13	.98	11	.83	10	.89	8	.85
45	13	.69	11	.56	10	.65	8	.63
46	13	.38	11	.26	10	.37	8	.37
47	13	.05	12	.94	10	.04	8	.05
48	14	.75	12	.69	11	.80	9	.84
49	14	.45	12	.40	11	.54	9	.62
50	14	.13	12	.08	11	.25	9	.35
K = 5								
0	0	.75	0	.90	0	.95	0	.99
1	0	.69	0	.87	0	.94	0	.99
2	0	.61	0	.84	0	.92	0	.98
3	0	.51	0	.80	0	.90	0	.98
4	0	.39	0	.76	0	.88	0	.98
5	0	.24	0	.69	0	.85	0	.97
6	0	.05	0	.62	0	.81	0	.96
7	1	.89	0	.52	0	.76	0	.95
8	1	.75	0	.40	0	.70	0	.94
9	1	.62	0	.25	0	.63	0	.93
10	1	.47	0	.07	0	.53	0	.91
11	1	.31	1	.94	0	.42	0	.88
12	1	.12	1	.85	0	.27	0	.85
13	2	.94	1	.75	0	.09	0	.82

K = 5 (Cont'd.)								
	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
14	2	.79	1	.64	1	.96	0	.77
15	2	.64	1	.51	1	.89	0	.72
16	2	.48	1	.36	1	.81	0	.64
17	2	.31	1	.19	1	.71	0	.56
18	2	.12	2	.99	1	.61	0	.44
19	3	.94	2	.89	1	.48	0	.31
20	3	.79	2	.77	1	.33	0	.13
21	3	.63	2	.65	1	.16	1	.98
22	3	.46	2	.51	2	.98	1	.94
23	3	.28	2	.36	2	.89	1	.88
24	3	.09	2	.18	2	.79	1	.81
25	4	.91	3	.99	2	.68	1	.74
26	4	.76	3	.87	2	.55	1	.64
27	4	.59	3	.74	2	.41	1	.54
28	4	.42	3	.61	2	.24	1	.40
29	4	.24	3	.46	2	.05	1	.25
30	4	.04	3	.29	3	.93	1	.06
31	5	.87	3	.10	3	.82	2	.95
32	5	.71	4	.94	3	.70	2	.88
33	5	.54	4	.81	3	.57	2	.80
34	5	.36	4	.67	3	.42	2	.71
35	5	.17	4	.52	3	.25	2	.60
36	6	.98	4	.36	3	.06	2	.47
37	6	.81	4	.18	4	.93	2	.32
38	6	.65	5	.99	4	.81	2	.14
39	6	.47	5	.86	4	.69	3	.98
40	6	.29	5	.72	4	.55	3	.90
41	6	.09	5	.57	4	.39	3	.81
42	7	.91	5	.41	4	.22	3	.71
43	7	.74	5	.23	4	.02	3	.59
44	7	.57	5	.03	5	.90	3	.46
45	7	.39	6	.89	5	.77	3	.30
46	7	.20	6	.75	5	.64	3	.12
47	7	.01	6	.60	5	.49	4	.96
48	8	.84	6	.43	5	.33	4	.88
49	8	.67	6	.26	5	.14	4	.78
50	8	.49	6	.06	6	.96	4	.66

Table 3. c_T and ρ_T for Negative Binomial Distributions

K = 2 r = 5								
	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
10	5	.75	5	.90	5	.95	5	.99
11	6	.50	6	.80	6	.90	6	.98
12	7	.08	7	.63	7	.82	7	.96
13	7	.73	8	.37	8	.69	8	.94
14	8	.38	8	.99	9	.49	9	.90
15	8	.95	9	.79	10	.21	10	.84
16	9	.61	10	.54	10	.94	11	.76
17	10	.22	11	.22	11	.77	12	.65
18	10	.82	11	.91	12	.56	13	.51
19	11	.45	12	.66	13	.31	14	.32
20	12	.04	13	.37	13	.99	15	.08
21	12	.67	14	.03	14	.81	15	.94
22	13	.28	14	.77	15	.60	16	.84
23	13	.88	15	.49	16	.35	17	.71
24	14	.51	16	.17	17	.05	18	.57
25	15	.11	16	.87	17	.84	19	.40
26	15	.72	17	.60	18	.62	20	.20
27	16	.34	18	.30	19	.37	20	.98
28	16	.93	18	.97	20	.08	21	.87
29	17	.56	19	.70	20	.85	22	.74
30	18	.16	20	.41	21	.63	23	.59
31	18	.77	21	.09	22	.38	24	.42
32	19	.39	21	.80	23	.10	25	.23
33	19	.98	22	.52	23	.86	26	.01
34	20	.60	23	.20	24	.63	26	.88
35	21	.21	23	.90	25	.38	27	.74
36	21	.82	24	.62	26	.11	28	.59
37	22	.43	25	.31	26	.87	29	.42
38	23	.03	25	.99	27	.64	30	.23
39	23	.65	26	.71	28	.39	31	.02
40	24	.26	27	.42	29	.12	31	.88
41	24	.86	28	.10	29	.87	32	.74
42	25	.48	28	.81	30	.64	33	.58
43	26	.08	29	.52	31	.39	34	.41
44	26	.69	30	.21	32	.12	35	.22
45	27	.30	30	.91	32	.87	36	.02
46	27	.90	31	.62	33	.64	36	.88
47	28	.52	32	.32	34	.39	37	.73
48	29	.12	32	1.00	35	.12	38	.57
49	29	.73	33	.72	35	.87	39	.40
50	30	.34	34	.42	36	.64	40	.21
51	30	.94	35	.11	37	.39	41	.01
52	31	.56	35	.81	38	.12	41	.87
53	32	.16	36	.52	38	.87	42	.71
54	32	.77	37	.21	39	.64	43	.55
55	33	.39	37	.91	40	.39	44	.38
56	33	.99	38	.62	41	.12	45	.20
57	34	.60	39	.32	41	.87	45	1.00
58	35	.21	40	.00	42	.64	46	.85
59	35	.82	40	.71	43	.38	47	.70
60	36	.43	41	.42	44	.12	48	.53

K = 3 r = 5								
	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
15	5	.75	5	.90	5	.95	5	.99
16	6	.25	6	.70	6	.85	6	.97
17	6	.70	7	.20	7	.60	7	.92
18	7	.10	7	.78	8	.03	8	.81
19	7	.58	8	.33	8	.76	9	.56
20	7	.95	8	.83	9	.35	10	.08
21	8	.43	9	.38	9	.88	10	.86
22	8	.83	9	.85	10	.51	11	.60
23	9	.26	10	.40	10	.98	12	.18
24	9	.68	10	.87	11	.63	12	.86
25	10	.08	11	.41	12	.14	13	.58
26	10	.53	11	.88	12	.73	14	.16
27	10	.92	12	.42	13	.28	14	.83
28	11	.36	12	.89	13	.82	15	.52
29	11	.76	13	.42	14	.40	16	.09
30	12	.18	13	.89	14	.91	16	.79
31	12	.60	14	.42	15	.50	17	.45
32	12	.99	14	.89	15	.99	17	1.00
33	13	.43	15	.41	16	.60	18	.72
34	13	.84	15	.89	17	.10	19	.36
35	14	.26	16	.41	17	.68	19	.94
36	14	.67	16	.88	18	.22	20	.64
37	15	.07	17	.40	18	.77	21	.25
38	15	.50	17	.87	19	.32	21	.87
39	15	.90	18	.39	19	.85	22	.55
40	16	.33	18	.87	20	.41	23	.14
41	16	.74	19	.38	20	.93	23	.80
42	17	.15	19	.86	21	.50	24	.45
43	17	.57	20	.37	22	.00	25	.02
44	17	.96	20	.85	22	.59	25	.71
45	18	.39	21	.36	23	.11	26	.34
46	18	.80	21	.84	23	.67	26	.94
47	19	.21	22	.35	24	.20	27	.61
48	19	.63	22	.83	24	.75	28	.22
49	20	.03	23	.33	25	.30	28	.85
50	20	.46	23	.82	25	.83	29	.51
51	20	.86	24	.32	26	.39	30	.10
52	21	.28	24	.81	26	.91	30	.76
53	21	.69	25	.31	27	.47	31	.40
54	22	.10	25	.80	27	.99	31	.98
55	22	.52	26	.29	28	.56	32	.66
56	22	.92	26	.78	29	.08	33	.28
57	23	.34	27	.28	29	.64	33	.89
58	23	.75	27	.77	30	.17	34	.55
59	24	.16	28	.27	30	.72	35	.16
60	24	.58	28	.76	31	.26	35	.80
61	24	.98	29	.25	31	.80	36	.44
62	25	.41	29	.75	32	.35	37	.04
63	25	.81	30	.24	32	.88	37	.70
64	26	.23	30	.73	33	.43	38	.32
65	26	.64	31	.22	33	.95	38	.93

Table 3 (Continued

K = 5 r = 5

T	P=.75 c_T	P=.75 ρ_T	P=.90 c_T	P=.90 ρ_T	P=.95 c_T	P=.95 ρ_T	P=.99 c_T	P=.99 ρ_T
25	5	.75	5	.90	5	.95	5	.99
26	5	.94	6	.50	6	.75	6	.95
27	6	.34	6	.83	6	.99	7	.78
28	6	.62	7	.14	7	.63	8	.16
29	6	.84	7	.59	7	.92	8	.81
30	7	.10	7	.87	8	.40	9	.24
31	7	.42	8	.22	8	.77	9	.78
32	7	.67	8	.60	9	.07	10	.18
33	7	.90	8	.87	9	.55	10	.72
34	8	.17	9	.21	9	.86	11	.05
35	8	.46	9	.57	10	.24	11	.63
36	8	.70	9	.86	10	.64	11	.96
37	8	.93	10	.18	10	.93	12	.52
38	9	.20	10	.53	11	.35	12	.89
39	9	.48	10	.82	11	.71	13	.37
40	9	.72	11	.12	11	.99	13	.79
41	9	.95	11	.48	12	.42	14	.21
42	10	.22	11	.78	12	.76	14	.68
43	10	.49	12	.06	13	.08	15	.02
44	10	.73	12	.42	13	.48	15	.54
45	10	.96	12	.73	13	.81	15	.91
46	11	.24	12	.99	14	.15	16	.38
47	11	.50	13	.36	14	.54	16	.79
48	11	.74	13	.67	14	.86	17	.20
49	11	.97	13	.94	15	.21	17	.66
50	12	.25	14	.28	15	.58	18	.01
51	12	.51	14	.60	15	.90	18	.51
52	12	.75	14	.89	16	.26	18	.89
53	12	.98	15	.21	16	.62	19	.34
54	13	.25	15	.54	16	.93	19	.75
55	13	.51	15	.83	17	.31	20	.15
56	13	.75	16	.13	17	.66	20	.61
57	13	.99	16	.46	17	.97	20	.97
58	14	.26	16	.76	18	.35	21	.44
59	14	.51	17	.05	18	.70	21	.84
60	14	.76	17	.39	19	.01	22	.26
61	14	.99	17	.70	19	.40	22	.69
62	15	.26	17	.98	19	.74	23	.07
63	15	.51	18	.31	20	.05	23	.53
64	15	.76	18	.62	20	.43	23	.91
65	15	.99	18	.91	20	.77	24	.36
66	16	.26	19	.23	21	.10	24	.77
67	16	.51	19	.55	21	.47	25	.18
68	16	.76	19	.84	21	.80	25	.62
69	16	.99	20	.15	22	.14	25	.99
70	17	.26	20	.47	22	.50	26	.45
71	17	.52	20	.77	22	.83	26	.84
72	17	.76	21	.07	23	.18	27	.27
73	17	1.00	21	.40	23	.54	27	.69
74	18	.26	21	.70	23	.86	28	.08
75	18	.51	21	.99	24	.21	28	.53

K = 2 r = 10

T	P=.75 c_T	P=.75 ρ_T	P=.90 c_T	P=.90 ρ_T	P=.95 c_T	P=.95 ρ_T	P=.99 c_T	P=.99 ρ_T
20	10	.75	10	.90	10	.95	10	.99
21	11	.50	11	.80	11	.90	11	.98
22	12	.05	12	.62	12	.81	12	.96
23	12	.70	13	.30	13	.65	13	.93
24	13	.32	13	.92	14	.38	14	.88
25	13	.88	14	.69	14	.98	15	.79
26	14	.51	15	.37	15	.81	16	.65
27	15	.06	15	.96	16	.57	17	.42
28	15	.68	16	.70	17	.24	18	.09
29	16	.27	17	.37	17	.91	18	.92
30	16	.84	17	.97	18	.68	19	.78
31	17	.44	18	.68	19	.39	20	.59
32	17	.99	19	.34	19	1.00	21	.33
33	18	.61	19	.95	20	.77	21	1.00
34	19	.18	20	.65	21	.49	22	.85
35	19	.76	21	.30	22	.13	23	.67
36	20	.35	21	.92	22	.84	24	.43
37	20	.92	22	.61	23	.57	25	.13
38	21	.52	23	.25	24	.23	25	.90
39	22	.08	23	.88	24	.90	26	.72
40	22	.67	24	.56	25	.63	27	.49
41	23	.25	25	.19	26	.31	28	.21
42	23	.82	25	.84	26	.96	28	.94
43	24	.41	26	.51	27	.69	29	.75
44	24	.97	27	.13	28	.38	30	.52
45	25	.57	27	.79	29	.02	31	.25
46	26	.14	28	.45	29	.75	31	.96
47	26	.72	29	.07	30	.45	32	.77
48	27	.30	29	.73	31	.10	33	.54
49	27	.87	30	.39	31	.80	34	.28
50	28	.46	30	1.00	32	.50	34	.98
51	29	.02	31	.68	33	.17	35	.78
52	29	.61	32	.32	33	.85	36	.56
53	30	.19	32	.95	34	.56	37	.29
54	30	.77	33	.62	35	.23	37	1.00
55	31	.35	34	.25	35	.90	38	.79
56	31	.92	34	.89	36	.61	39	.57
57	32	.50	35	.55	37	.28	40	.31
58	33	.07	36	.18	37	.94	41	.01
59	33	.66	36	.83	38	.65	41	.80
60	34	.23	37	.49	39	.34	42	.57
61	34	.81	38	.11	39	.99	43	.31
62	35	.39	38	.77	40	.70	44	.02
63	35	.96	39	.42	41	.39	44	.80
64	36	.54	40	.04	42	.04	45	.57
65	37	.11	40	.70	42	.75	46	.31
66	37	.70	41	.35	43	.44	47	.02
67	38	.27	41	.97	44	.10	47	.80
68	38	.84	42	.64	44	.79	48	.57
69	39	.43	43	.28	45	.48	49	.31
70	39	.99	43	.91	46	.15	50	.03

Table 3 (Continued)

K = 3   r = 10

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
30	10	.75	10	.90	10	.95	10	.99
31	11	.25	11	.70	11	.85	11	.97
32	11	.69	12	.15	12	.58	12	.92
33	12	.07	12	.75	12	.97	13	.77
34	12	.56	13	.23	13	.70	14	.43
35	12	.92	13	.75	14	.17	14	.95
36	13	.38	14	.22	14	.76	15	.72
37	13	.77	14	.72	15	.27	16	.28
38	14	.17	15	.17	15	.80	16	.87
39	14	.59	15	.68	16	.32	17	.54
40	14	.95	16	.10	16	.82	17	.99
41	15	.40	16	.62	17	.35	18	.71
42	15	.78	17	.02	17	.83	19	.25
43	16	.18	17	.55	18	.36	19	.83
44	16	.59	17	.96	18	.84	20	.45
45	16	.96	18	.48	19	.36	20	.94
46	17	.38	18	.90	19	.84	21	.59
47	17	.77	19	.39	20	.35	22	.08
48	18	.16	19	.83	20	.83	22	.72
49	18	.57	20	.30	21	.34	23	.26
50	18	.94	20	.75	21	.82	23	.82
51	19	.36	21	.20	22	.33	24	.41
52	19	.75	21	.67	22	.81	24	.92
53	20	.13	22	.09	23	.31	25	.53
54	20	.54	22	.58	23	.79	26	.01
55	20	.92	22	.99	24	.28	26	.65
56	21	.33	23	.48	24	.77	27	.16
57	21	.72	23	.91	25	.26	27	.75
58	22	.10	24	.38	25	.75	28	.30
59	22	.51	24	.82	26	.23	28	.84
60	22	.89	25	.28	26	.72	29	.42
61	23	.29	25	.73	27	.20	29	.93
62	23	.68	26	.17	27	.70	30	.53
63	24	.06	26	.64	28	.17	31	.02
64	24	.47	27	.06	28	.67	31	.63
65	24	.85	27	.54	29	.14	32	.15
66	25	.25	27	.96	29	.64	32	.73
67	25	.64	28	.44	30	.10	33	.27
68	26	.02	28	.87	30	.61	33	.82
69	26	.43	29	.34	31	.07	34	.38
70	26	.81	29	.78	31	.58	34	.90
71	27	.20	30	.23	32	.03	35	.49
72	27	.60	30	.68	32	.55	35	.98
73	27	.98	31	.12	32	1.00	36	.58
74	28	.38	31	.58	33	.52	37	.09
75	28	.77	32	.00	33	.97	37	.68
76	29	.16	32	.48	34	.49	38	.21
77	29	.56	32	.91	34	.94	38	.76
78	29	.94	33	.37	35	.45	39	.32
79	30	.34	33	.81	35	.91	39	.85
80	30	.73	34	.27	36	.41	40	.42

K = 5   r = 10

	P=.75		P=.90		P=.95		P=.99	
T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T	c_T	ρ_T
50	10	.75	10	.90	10	.95	10	.99
51	10	.94	11	.50	11	.75	11	.95
52	11	.34	11	.82	11	.98	12	.77
53	11	.62	12	.10	12	.60	12	1.00
54	11	.84	12	.56	12	.89	13	.75
55	12	.08	12	.83	13	.29	13	.99
56	12	.40	13	.12	13	.69	14	.65
57	12	.65	13	.52	13	.94	14	.94
58	12	.87	13	.79	14	.38	15	.48
59	13	.12	14	.03	14	.72	15	.84
60	13	.41	14	.43	14	.97	16	.23
61	13	.65	14	.72	15	.39	16	.69
62	13	.87	14	.95	15	.72	16	.97
63	14	.12	15	.31	15	.97	17	.47
64	14	.40	15	.62	16	.37	17	.82
65	14	.64	15	.87	16	.70	18	.17
66	14	.86	16	.16	16	.95	18	.62
67	15	.10	16	.49	17	.33	18	.92
68	15	.37	16	.76	17	.66	19	.36
69	15	.62	17	.00	17	.92	19	.73
70	15	.84	17	.35	18	.28	20	.02
71	16	.07	17	.64	18	.61	20	.49
72	16	.34	17	.89	18	.88	20	.82
73	16	.58	18	.19	19	.21	21	.18
74	16	.81	18	.50	19	.55	21	.59
75	17	.04	18	.77	19	.84	21	.90
76	17	.30	19	.02	20	.13	22	.30
77	17	.55	19	.35	20	.49	22	.68
78	17	.78	19	.63	20	.78	22	.97
79	17	1.00	19	.89	21	.05	23	.41
80	18	.26	20	.18	21	.41	23	.76
81	18	.51	20	.48	21	.71	24	.07
82	18	.74	20	.76	21	.98	24	.49
83	18	.96	20	1.00	22	.33	24	.83
84	19	.22	21	.32	22	.64	25	.18
85	19	.47	21	.61	22	.92	25	.57
86	19	.70	21	.87	23	.24	25	.89
87	19	.92	22	.15	23	.57	26	.27
88	20	.17	22	.45	23	.85	26	.64
89	20	.42	22	.72	24	.15	26	.95
90	20	.66	22	.97	24	.49	27	.35
91	20	.88	23	.28	24	.78	27	.71
92	21	.12	23	.57	25	.06	28	.01
93	21	.37	23	.83	25	.40	28	.42
94	21	.61	24	.10	25	.70	28	.77
95	21	.84	24	.41	25	.97	29	.09
96	22	.07	24	.68	26	.31	29	.49
97	22	.32	24	.94	26	.62	29	.82
98	22	.57	25	.23	26	.90	30	.17
99	22	.80	25	.52	27	.21	30	.55
100	23	.08	25	.79	27	.53	30	.88

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13. ABSTRACT

In this paper we discuss the multiple decision (*selection and ranking*) rules in a general decision theoretic framework. More specifically, we discuss the subset selection problem. The earlier part of the paper describes the general framework and gives some known results for sake of completeness; in the latter part of the paper we give some new results dealing with the subset selection problem for a class of discrete distributions (Section 2). Some relevant tables for these procedures are included. The derivation of rules with some desirable property is made in Section 3 using the likelihood ratio criterion.

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