OPTIMAL PARI-MUTUEL WAGERING

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A procedure is proposed to enable a bettor to optimally place a bet on a pari-mutuel event. The problem is essentially one of multivariate classification given data on each contestant. It is shown that one can always decide optimally among the alternatives, (1) bet on any one horse and (2) do not bet at all.

1. *Introduction*. Perhaps the first explicit solution to a non-linear programming problem was presented in Isaacs [1953]. His algorithm enabled one with the prescience of *a priori* probabilities to wager optimally on a pari-mutuel event. His optimal solution determines which contestants should be played, and the amount to be wagered on them. Unfortunately, the result had little practical relevance since obtaining valid *a priori* probabilities remained a problem. Also involved are possible computational difficulties in actually implementing the algorithm. Also his technique was essentially a no data problem. What follows is an attempt to treat pari-mutuel wagering as a problem in statistical

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decision theory. However, first let us digress to explain pari-mutuel wagering.

2. *Pari-Mutuel Wagering*. Approximately half of the fifty states have legalized pari-mutuel wagering on thoroughbred racing. In addition, several states permit pari-mutuel wagering on harness racing, greyhound racing and quarter-horse racing, while Florida includes jai-alai. In what follows attention will be focused on thoroughbred horse racing, however, the technique is applicable to all pari-mutuel events.

The essence of pari-mutuel wagering is that a number of bettors place bets on various horses, the "house" deducts a fixed proportion of the betting pool, and distributes the balance among the winners. The deducted proportion is typically between .14 and .16. Assume there is a total of S dollars wagered in a race, and Y_j dollars wagered on a horse of interest. Let r denote the proportion withheld by the "house". The odds, o_j, on this horse are given by

$$o_j = B[(1-r)S/(B\,Y_j)] - 1 \tag{2.1}$$

where $[x]$ denotes the greatest integer in x, and B is called the "breakage". Typically, $B = \$0.10$. We will subsequently be interested in J discrete odds levels.

The above description is for win pari-mutuel wagering. For a discussion of place pari-mutuel wagering, and another betting algorithm, see Willis [1964].

3. *The Classification Problem*. Data on pari-mutuel wagering for thoroughbred horses is almost as plentiful as data for the stock market. In fact there is so much data that one must reduce it to some manageable statistic to make

one's decisions. Most serious handicappers do this in a highly subjective fashion, one they claim was learned by years of experience (and presumably years of financial losses too). The following is perhaps a more objective way to obtain a decision on wagering.

Excluding the possibility of a tie (called a dead-heat), every race of k horses has one winner and $(k-1)$ losers. The problem then is to classify each of the k horses as a potential winner or a potential loser. In fact let us assume that we are using p quantitative handicapping factors to classify the horse. Let X_i, $i = 1,\ldots, k$ be $p \times 1$ vectors denoting the observations on these p factors for horses $1,\ldots, k$. Furthermore, let Z be a $pk \times 1$ vector, $Z' = (X_1, X_2,\ldots, X_k)'$. While selection of these p factors is outside the scope of the present paper, one could use such factors as speed, class, or other commonly used factors (see Epstein [1967], da Silva and Dorcus [1961]).

Next let us assume we are interested in horses of odds at J levels, say $o_1,\ldots, o_J$. Actually one would probably pool several odds levels so that J would not be too large. Order the odds so that $o_1 < o_2 < \ldots < o_J$.

Then there are k states of nature $S_1,\ldots, S_k$, S_i indicating that the ith horse wins. Let us restrict ourselves to strategies which bet at most one horse, and exactly one dollar on each selected horse (never mind the fact that no race track allows less than a two dollar bet!). Then we have $k + 1$ possible actions $a_1,\ldots, a_k, a_{k+1}$, with a_i denoting betting one dollar on the ith horse, and a_{k+1} denoting placing no bet. Then the <u>loss</u> function can be described as follows: $L(a_i|S_i) = -o_{j_i}$, $i = 1,\ldots, k$ where

LOSS TABLE FOR BETTOR'S ACTIONS

State of Nature / Action	a_1	a_2	. . .	a_k	a_{k+1}
S_1	$-o_{j_1}$	1	. . .	1	0
S_2	1	$-o_{j_2}$	. . .	1	0
⋮	⋮	⋮			
S_k	1		. . .	$-o_{j_k}$	0

TABLE I

$$(3.2)\quad \rho(q,\phi) = \sum_{i=1}^{k} q_i \int \Big(-o_{j_i}\phi(a_i|z) + \sum_{\substack{i^*=1\\ i^*\neq i}}^{k} \phi(a_{i^*}|z)\Big) dF(z|S_i)$$

where the integral is over the pk dimensional space of Z. Noting that the odds are finite, condition II follows after exchanging the integration and summation. Condition I follows since the Bayes risk using a_{k+1} is zero.

Note that Theorem 1 could also include the discrete case for Z, except the Bayes procedure may not be unique (this is irrelevant, since if there is more than one, a bettor can achieve the same Bayes risk selecting any one of the procedures).

We note that Theorem 1 generalizes a result of Blackwell and Girschick [1954], Section 6.4. They considered the case $k = 2$, $o_{j_1} = o_{j_2} = 1$, $q_1 = q_2 = 1/2$.

One might have difficulty applying (3.1) without the following two seemingly reasonable assumptions.

Assumption 1. $f(Z|S_i) = \Pi_{\ell=1}^{k} f_{o_{j_\ell}}(X_\ell|S_i)$, $i = 1,\ldots,k$, *that is, the observations on the* k *horses are independent given the state of nature.*

Also, $f_{o_{j_\ell}}(\cdot|\cdot)$ indicates the possible dependence on the odds. It appears as if most handicapping factors do depend on the odds of the horse.

Assumption 2. $f_{o_{j_\ell}}(X_\ell|S_{\ell^*}) = f_{o_{j_\ell}}(X_\ell|S_\ell^c)$, $\ell = 1,\ldots,k$, $\ell^*\neq\ell$ *where* S_ℓ^c *indicates the state of nature is not* S_ℓ.

In other words, the observations on the ℓth horse only depend on whether the ℓth horse wins or loses, and not on which other horse won. With this assumption, we can let

$f_{o_{j_\ell}}(X_\ell|S_\ell) \equiv f_{o_{j_\ell}}(X_\ell|W)$, and $f_{o_{j_\ell}}(X_\ell|S_{\ell^*}) \equiv f_{o_{j_\ell}}(X_\ell|L)$, $\ell = 1,\ldots, k$, $\ell^* \neq \ell$ where W and L denote the horse is a winner or loser respectively.

Theorem 2. *Let* $\lambda_i = \dfrac{f_{o_{j_i}}(X_i|W)}{f_{o_{j_i}}(X_i|L)}$ *, and the assumptions of* Theorem 1, Assumption 1, *and* Assumption 2 hold. *Then the Bayes procedure is given by: Let* $\phi(a_i|Z) = 1$, $1 \leq i \leq k$ *if the following two conditions hold,*

$$\text{I.}\quad -o_{j_i}q_i\lambda_i + \sum_{\substack{i^*=1\\ i^*\neq i}}^{k} q_{i^*}\lambda_{i^*} < 0,$$

(3.3) *and*

$$\text{II.}\quad \lambda_{i^*}/\lambda_i \leq \big((o_{j_i}+1)q_i\big)/\big((o_{j_{i^*}}+1)q_{i^*}\big)$$

for all $i^* \neq i$. *If* I *fails to hold for some* $1 \leq i \leq k$, *let* $\phi(a_{k+1}|Z) = 1$.

Proof. From (3.1) and Assumption 1,

$$y_i = -o_{j_i}q_i \prod_{\ell=1}^{k} f_{o_{j_\ell}}(X_\ell|S_i) + \sum_{\substack{i^*=1\\ i^*\neq i}}^{k} q_{i^*}\prod_{\ell=1}^{k} f_{o_{j_\ell}}(X_\ell|S_{i^*})$$

$$= -o_{j_i}q_i\lambda_i\prod_{\ell=1}^{k} f_{o_{j_\ell}}(X_\ell|L) + \sum_{\substack{i^*=1\\ i^*\neq i}}^{k} q_{i^*}\lambda_{i^*}\prod_{\ell=1}^{k} f_{o_{j_\ell}}(X_\ell|L)$$

using Assumption 2 and the definition of λ_i. Hence,

$$y_i = \Big(-o_{j_i}q_i\lambda_i + \sum_{\substack{i^*=1\\ i^*\neq i}}^{k} q_{i^*}\lambda_{i^*}\Big) \prod_{\ell=1}^{k} f_{o_{j_\ell}}(X_\ell|L) \quad ,$$

and since the last coefficient is positive (we are tacitly assuming all densitites have the same support set) Condition I in (3.3) follows from Condition I in (3.1). Condition II of (3.3) follows from Condition II of (3.1) using the above representation for y_i.

Note that Theorem 2 could also include the discrete case for Z, and yield a (non-unique) Bayes procedure. Unfortunately (3.3) still does not have enough structure to enable a bettor to determine how well he is doing, that is to calculate the Bayes risk in (3.2). Let us make the following assumptions concerning the distribution of Z. Recall that we ordered the odds so that $o_1 < o_2 < \ldots < o_J$.

Assumption 3. *Let* X *stand for the* $p \times 1$ *observation vector on a horse of odds* o_j, *then assume*

$$X|W \sim N_p\left(\mu_j^{(1)}, \Sigma\right) \quad ,$$

$$X|L \sim N_p\left(\mu_j^{(2)}, \Sigma\right) \quad , \qquad j = 1,\ldots, J,$$

where N_p *denotes the* p-*variate normal distribution, and* Σ *is a positive definite covariance matrix.*

Interestingly enough, Assumption 3 appears a reasonable approximation in practice. Moreover, we felt that this assumption was necessary, and that qualitative classification techniques (see Cochran and Hopkins [1961]) required too large a data base to estimate parameters.

Subsequently we will also use Assumption 4.

$\mu_j^{(1)} - \mu_j^{(2)} = \mu,\ j = 1,\ldots, J.$

That is the difference between the mean vector for winners and losers at each odds level is independent of the odds level. Again this assumption appears reasonable in practice, especially if J is not too large.

Theorem 3. *With the assumptions of* Theorem 2 *and* Assumption 3, *the Bayes procedure is given by: let* $\phi(a_i|Z)=1, 1 \le i \le k$

if the following two conditions hold,

$$\text{I.}\quad -o_{j_i} q_i \exp\Big(X_i'\Sigma^{-1}(\mu_{j_i}^{(1)}-\mu_{j_i}^{(2)}) - \tfrac{1}{2}(\mu_{j_i}^{(1)}+\mu_{j_i}^{(2)})'\Sigma^{-1}(\mu_{j_i}^{(1)}-\mu_{j_i}^{(2)})\Big) + \sum_{\substack{i^*=1\\ i^*\neq i}}^{k} q_{i^*}\exp\Big(X_{i^*}'\Sigma^{-1}(\mu_{j_{i^*}}^{(1)}-\mu_{j_{i^*}}^{(2)}) - \tfrac{1}{2}(\mu_{j_{i^*}}^{(1)}+\mu_{j_{i^*}}^{(2)})'\Sigma^{-1}(\mu_{j_{i^*}}^{(1)}-\mu_{j_{i^*}}^{(2)})\Big) < 0,$$

(3.4) and

$$\text{II.}\quad X_{i^*}'\Sigma^{-1}(\mu_{j_{i^*}}^{(1)}-\mu_{j_{i^*}}^{(2)})-\tfrac{1}{2}(\mu_{j_{i^*}}^{(1)}+\mu_{j_{i^*}}^{(2)})'\Sigma^{-1}(\mu_{j_{i^*}}^{(1)}-\mu_{j_{i^*}}^{(2)}) - X_i'\Sigma^{-1}(\mu_{j_i}^{(1)}-\mu_{j_i}^{(2)})+\tfrac{1}{2}(\mu_{j_i}^{(1)}+\mu_{j_i}^{(2)})'\Sigma^{-1}(\mu_{j_i}^{(1)}-\mu_{j_i}^{(2)}) \le \ln\big(((o_{j_i}+1)q_i)/((o_{j_{i^*}}+1)q_{i^*})\big)$$

for all $i^* \neq i$. *If* I *fails to hold for some* $1 \le i \le k$, *let* $\phi(a_{k+1}|Z) = 1$.

Proof. The proof follows immediately after noting that with Assumption 3,

$$\lambda_i = \frac{\exp-\frac{1}{2}(X_i-\mu_{j_i}^{(1)})'\Sigma^{-1}(X_i-\mu_{j_i}^{(1)})}{\exp-\frac{1}{2}(X_i-\mu_{j_i}^{(2)})\ \Sigma^{-1}(X_i-\mu_{j_i}^{(2)})}$$

$$= \exp(X_i'\Sigma^{-1}(\mu_{j_i}^{(1)}-\mu_{j_i}^{(2)})-\tfrac{1}{2}(\mu_{j_i}^{(1)}+\mu_{j_i}^{(2)})\Sigma^{-1}(\mu_{j_i}^{(1)}-\mu_{j_i}^{(2)}) .$$

The following is stated without proof.

Corollary 4. *With the assumptions of* Theorem 3 *and* Assumption 4, *the Bayes procedure is given by:* *let* $\phi(a_i|Z) = 1$,

$1 \leq i \leq k$ *if the following two conditions hold,*

$$\text{I.}\quad -o_{j_i} q_i \exp\left(X_i' \Sigma^{-1}\mu - \tfrac{1}{2}(\mu_{j_i}^{(1)} + \mu_{j_i}^{(2)})'\Sigma^{-1}\mu\right) + \sum_{\substack{i^*=1 \\ i^* \neq i}}^{k} q_{i^*} \exp\left(X_{i^*}' \Sigma^{-1}\mu - \tfrac{1}{2}(\mu_{j_{i^*}}^{(1)} + \mu_{j_{i^*}}^{(2)})'\Sigma^{-1}\mu\right) < 0 ,$$

(3.5) *and*

$$\text{II.}\quad (X_{i^*} - X_i)'\Sigma^{-1}\mu - \tfrac{1}{2}(\mu_{j_{i^*}}^{(1)} + \mu_{j_{i^*}}^{(2)} - \mu_{j_i}^{(1)} - \mu_{j_i}^{(2)})\Sigma^{-1}\mu \leq \ln\left(((o_{j_i}+1)q_i)/((o_{j_{i^*}}+1)q_{i^*})\right)$$

for all $i^* \neq i$. *If* I *fails to hold for some* $1 \leq i \leq k$, *let* $\phi(a_{k+1}|Z) = 1$.

Note that Condition II of (3.5) is easy to apply at the race track. The same linear combination of the observation vector is used for all odds, and a table of the other two terms for all $\binom{J}{2}$ odds pairs can be readily made. Unfortunately, Condition I, appears most difficult to implement at the race track. Perhaps a first order expansion of each of the exp functions would be a good approximation.

The problem of obtaining the *a priori* probabilities $q = (q_1, \ldots, q_k)$ still remains. There are two seemingly reasonable choices.

Assumption 5. *Choose as prior odds*, $q_i = (1-r)/(o_{j_i}+1)$, $i=1,\ldots,k$. These prior odds are suggested by (2.1), not taking account of the breakage factor. Note that with this assumption, the right hand side of Condition II in (3.4) and (3.5) becomes zero. Da Silva and Dorcus [1961] show that in large samples of races, these q_i's are close to the actual proportion of horses that win at odds o_{j_i}.

However, the simplication of Condition II with Assumption 5, results in making Condition I even more complicated. Perhaps the following is a better assumption.

Assumption 6. *Choose as prior odds*, $q_i = k^{-1}$, $i = 1,\ldots,k$.

That is, include all possible information in your $p \times 1$ observation vector on each horse so that this is a reasonable prior.

Let us now examine the Bayes risk for (3.5) in a special case. Let $P(a_i|S_\ell)$ denote the conditional probability of taking action a_i when the state of nature is S_ℓ. Furthermore, let us assume there are k horses in a race, each with the same amount of money wagered on them, and that the breakage factor does not enter (2.1). Then

$o_{j_1} = \ldots = o_{j_k} = \big((1-r)k-1\big)$, and (3.5) reduces to let

$\phi(a_i|Z) = 1$, $1 \le i \le k$ if the following hold,

$$\text{(3.6)}\qquad \begin{array}{ll} \text{I.} & -\big((1-r)k-1\big)e^{U_i} + \sum_{\substack{i^*=1\\ i^*\neq i}}^{k} e^{U_{i^*}} < 0, \\[2ex] \text{II.} & U_{i^*} - U_i < 0 \end{array}$$

for all $i^* \neq i$, where $U_i = X_i'\Sigma^{-1}\mu - \frac{1}{2}(\mu^{(1)}+\mu^{(2)})'\Sigma^{-1}\mu$, $i = 1,\ldots,k$, and $\mu^{(1)},\mu^{(2)}$ are the mean vectors of winners and losers respectively of odds $\big((1-r)k-1\big)$. If I fails to hold for some $1 \le i \le k$, let $\phi(a_{k+1}|Z) = 1$.

Note that $U' = (U_1,\ldots,U_k)'$ has a multivariate normal distribution in k dimensions with mean vector given by $E(U') = (-\alpha/2,\ldots,-\alpha/2,+\alpha/2,-\alpha/2,\ldots,-\alpha/2)$ if S_ℓ the ℓth component, where $\alpha = \mu'\Sigma^{-1}\mu$ is the Mahalanobis distance between winners and losers. The covariance matrix is $\alpha \times I$, where I is the $k \times k$ identity matrix. From Assumptions 1-4, and Anderson [1958], Ch. 6, one can readily obtain this distribution for U.

One can now calculate $P(a_i|S_i)$, and $P(a_i|S_{i^*})$, $i \neq i^*$. First, by the assumption of equal odds, we can let $i = 1$ without loss of generality. Then

$$P(a_1|S_1) = P(\textstyle\sum_{i^*=2}^{k} e^{V_{i^*}} < (1-r)k-1,\ V_2 < 0,\dots,V_k < 0)$$

where $V_{i^*} = U_{i^*} - U_1$, $i^* = 2,\dots,k$. Since S_1 is true $V' = (V_2,\dots,V_k)'$ has a multivariate normal distribution in $(k-1)$ dimensions with mean vector $E(V') = (-\alpha,\dots,-\alpha)$, and covariance matrix,

$$\mathrm{Cov}(V) = \begin{pmatrix} 2\alpha & \alpha & \dots & \alpha \\ \alpha & 2\alpha & \dots & \alpha \\ \vdots & \vdots & & \vdots \\ \alpha & \alpha & \dots & 2\alpha \end{pmatrix}.$$

After normalization, one obtains

$$(3.7)\quad P(a_1|S_1) = P(\textstyle\sum_{i^*=2}^{k} e^{(2\alpha)^{1/2}W_{i^*}} < ((1-r)k-1)e^{\alpha},$$
$$W_2 < (\alpha/2)^{1/2},\dots,W_k < (\alpha/2)^{1/2})\ ,$$

where $W_2,\dots,W_k$ are standard normal with $\mathrm{corr}(W_i,W_j) = 1/2$, $i \neq j$. Also, if S_2 is the state of nature, $E(V') = (\alpha,0,\dots,0)$, with the same covariance matrix as above. Hence after normalization, one obtains

$$P(a_1|S_\ell) = P(a_1|S_2) = P(e^{(2\alpha)^{1/2}W_2+\alpha}$$

(3.8) $$+ \sum_{i^*=3}^{k} e^{(2\alpha)^{1/2}W_3} < (1-r)k-1, W_2 < -(\alpha/2)^{1/2},$$
$$W_3 < 0, \ldots, W_k < 0) \; ,$$

$\ell \neq 1$, $W' = (W_2, \ldots, W_k)'$ having the same distribution as in (3.7).

Thus in the case of equal odds, k horses, a "take" of r, and a Mahalanobis distance of α, the Bayes risk from (3.2), now denoted by $B(k,r,\alpha)$, is given by

(3.9) $$B(k,r,\alpha) = k^{-1}k\{-((1-r)k-1)P(a_1|S_1)+(k-1)P(a_1|S_2)\}$$
$$= -((1-r)k-1)P(a_1|S_1)+(k-1)P(a_1|S_2) \; ,$$

$P(a_1|S_1)$, $P(a_1|S_2)$ as in (3.7), (3.8).

For $k = 2$, (3.7) - (3.9) are easy to calculate. Also, for k arbitrary,

(3.10) $$P(W_2<w_2,\ldots,W_k<w_k) = \int_{-\infty}^{\infty} [\Pi_{i^*=2}^{k} \Phi(2^{1/2}w_{i^*}-y)]\phi(y)dy$$

where Φ and ϕ are the standard normal c.d.f. and density function respectively. This representation, and similar identities may be found in Gupta [1963]. Expression (3.10) may be evaluated on a computer (see Gupta [1963]). Hence one needs to calculate

(3.11) $$P\,(\sum_{i^*=2}^{k} e^{(2\alpha)^{1/2}W_{i^*}} < ((1-r)k-1)e^{\alpha},$$
$$W_2 < (\alpha/2)^{1/2}, \ldots, W_k < (\alpha/2)^{1/2}),$$

and

(3.12) $$P(e^{(2\alpha)^{1/2}W_2+\alpha}+\sum_{i^*=3}^{k} e^{(2\alpha)^{1/2}W_{i^*}}>(1-r)k-1,\quad W_2<-(\alpha/2)^{1/2}, W_3<0,\ldots,W_k<0).$$

Unfortunately, expressions (3.11) and (3.12) could be readily evaluated on a computer only when $k = 3$. In that case, (3.11) becomes

(3.13) $$\int_a^b \phi(x)\{\Phi\big(((\alpha/2)^{1/2}-x/2)/(3/4)^{1/2}\big) - \Phi\big([(\ln((2-3r)e^{\alpha}-e^{(2\alpha)^{1/2}x}))/(2\alpha)^{1/2}-x/2]/(3/4)^{1/2}\big)\}dx$$

and (3.12) becomes

(3.14) $$\int_c^d \phi(x)\{\Phi\big((-x/2)/(3/4)^{1/2}\big) - \Phi\big([(\ln(2-3r-e^{(2\alpha)^{1/2}x+\alpha}))/(2\alpha)^{1/2}-x/2]/(3/4)^{1/2}\big)\}dx \quad ,$$

where $a = (\alpha/2)^{1/2}+(\ln(1-3r))/(2\alpha)^{1/2}$, $b = (\alpha/2)^{1/2}$, $c = -(\alpha/2)^{1/2}+(\ln(1-3r))/(2\alpha)^{1/2}$, $d = -(\alpha/2)^{1/2}$, and where $3r < 1$ for problems of interest.

A table of $B(k,r,\alpha)$ for $k = 2,3$, $r = .15, .16$, and several values of α is given in Table II. Also included are $P(a_1|S_1)$, $(k-1)\,P(a_1|S_2)$, and $P(a_{k+1})$. The calculations were done on the Purdue University CDC 6500 computer.

From Table II it is interesting to conjecture that $B(k,r,\alpha)$ is monotone in all three arguments. Also, one

BAYES RISK

$r = .15$

α	k = 2 Bayes Risk	$P(a_1 \mid S_1)$	$P(a_1 \mid S_2)$	$P(a_3)$	k = 3 Bayes Risk	$P(a_1 \mid S_1)$	$2P(a_1 \mid S_2)$	$P(a_4)$
.05	-.02	.166	.099	.735	-.05	.213	.284	.504
.10	-.05	.283	.154	.563	-.10	.314	.383	.304
.15	-.07	.353	.178	.470	-.15	.368	.417	.215
.20	-.09	.402	.189	.408	-.20	.405	.430	.165
.25	-.11	.440	.196	.365	-.24	.433	.434	.133
.30	-.13	.471	.198	.331	-.27	.456	.433	.111

$r = .16$

α	k = 2 Bayes Risk	$P(a_1 \mid S_1)$	$P(a_1 \mid S_2)$	$P(a_3)$	k = 3 Bayes Risk	$P(a_1 \mid S_1)$	$2P(a_1 \mid S_2)$	$P(a_4)$
.05	-.01	.144	.084	.772	-.04	.193	.254	.553
.10	-.04	.262	.139	.600	-.09	.299	.359	.342
.15	-.06	.334	.164	.503	-.14	.356	.399	.245
.20	-.08	.385	.172	.443	-.19	.396	.415	.189
.25	-.10	.424	.184	.392	-.23	.425	.422	.153
.30	-.12	.456	.188	.356	-.26	.449	.423	.128

TABLE II

should note that the conditional Bayes risk given that a bet was made, call it $BC(k,r,\alpha)$, is given by

$$(3.15) \qquad BC(k,r,\alpha) = B(k,r,\alpha)/(1-P(a_{k+1})).$$

Finally, note that a purist might object to our tacit assumption that "winners" and "losers" comprise two populations. They certainly are not two populations in the standard statistical sense. Nevertheless, we feel that this distinction is only of philosophical importance, for if there is a positive Mahalanobis distance $\alpha = \mu'\Sigma^{-1}\mu$, we are willing to act as if we in fact had two populations.

4. *Discussion.* In discussing our procedure as given in (3.5) with that of Isaacs [1953], two interesting points are noted. First, both procedures can select a null subset to play. Also our procedure selects at most one horse per race to play, while his may possibly select more. We realize that a bettor may eliminate mathematically admissible strategies by playing at most one horse per race. Nevertheless the authors feel that in practice it makes little sense to "bet against oneself" by playing more than one horse per race. Also, the mathematics of (3.5) would become more complicated.

Second, Isaac's procedure has an advantage in that it tells the bettor how much he should wager. We feel this is a small point since one can bet very substantial amounts at the large race tracks without seriously affecting the pari-mutuel odds.

Finally, the authors are currently working on estimation problems involving (3.5). Interesting problems in estimating ordered multivariate parameters arise.

References

1. Anderson, T. W. [1958]. *An Introduction to Multivariate Analysis*, John Wiley and Sons, New York

2. Blackwell, D., and Girshick, M. A. [1954]. *Theory of Games and Statistical Decisions*, John Wiley and Sons, New York.

3. Cochran, W. G., and Hopkins, C. E. [1961]. "Some classification problems with multivariate qualitative data", *Biometrics*, (17), 10-32.

4. da Silva, E. R., and Dorcus, R. M. [1961]. *Science in Betting*, Dolphin Books, Doubleday and Company, Garden City, N.Y.

5. Epstein, R. A. [1967]. *The Theory of Gambling and Statistical Logic*, Academic Press, New York.

6. Gupta, S. S. [1963]. "Probability integrals of multivariate normal and multivariate t", *Ann. Math. Statist.*, 34, 792-828.

7. Isaacs, R. [1953]. "Optimal horse race bets", *Amer. Math. Monthly*, 60, 310-315.

8. Willis, K. E. [1964]. "Optimum no-risk strategy for win-place pari-mutuel betting", *Management Sci.* 10, 574-577.