ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF CHARACTERISTIC ROOTS IN MANOVA AND CANONICAL CORRELATION*

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1. Introduction. An asymptotic expansion of the distribution of a sample covariance matrix (one-sample case) was studied by Anderson [1] and James [7], and extending their work, an asymptotic representation was obtained by Chang [2] in the two-sample case when the population roots are all distinct. Li, Pillai **and Chang** [10] generalized Chang's results [2] to cover the case of a single extreme multiple population root. Li and Pillai [9], [10], have further obtained the second term of the expansion in the two-sample case and also extended the results to the complex case. In this paper, asymptotic expansions are derived in the MANOVA and canonical correlation situations both in the real and complex cases.

2. Asymptotic expansion for canonical correlation -- population roots all distinct.

Let $x_1,\ldots, x_p, x_{p+1},\ldots, x_{p+q}$, $p \leq q$ be distributed $N(0,\underset{\sim}{\Sigma})$, where

$$(1)\qquad \underset{\sim}{\Sigma} = \begin{array}{cc} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} \end{array} .$$

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Let $\underset{\sim}{P}^2 = \mathrm{diag}(\rho_1^2, \ldots, \rho_p^2)$, where ρ_i^2, $i = 1, \ldots, p$, be the roots of

$$(2) \qquad |\underset{\sim}{\Sigma}_{12}\, \underset{\sim}{\Sigma}_{22}^{-1}\, \underset{\sim}{\Sigma}'_{12} - \rho^2\, \underset{\sim}{\Sigma}_{11}| = 0 .$$

and let $\underset{\sim}{\hat{P}}^2 = \mathrm{diag}(\hat{\rho}_1^2, \ldots, \hat{\rho}_p^2)$, where $\hat{\rho}_i^2$, $i = 1, \ldots, p$, be the maximum likelihood estimates of ρ_i^2, $i = 1, \ldots, p$, from a sample of size $n \geq p+q$ from the above population. Then the joint density of

$$(3) \qquad \underset{\sim}{\hat{P}}^2 = R = \mathrm{diag}(r_1, \ldots, r_p)$$

is given by [4]

$$(4) \qquad D_1 \int_{O(p)} {}_2F_1\left(\tfrac{1}{2}n, \tfrac{1}{2}n, \tfrac{1}{2}q, \underset{\sim}{H}'\, \underset{\sim}{A}\, \underset{\sim}{H}\, \underset{\sim}{R}\right) d(\underset{\sim}{H}),$$

where

$$(5) \qquad \underset{\sim}{\hat{P}}^2 = \underset{\sim}{A} = \mathrm{diag}(\ell_1, \ldots, \ell_p), \quad 1 > \ell_1 > \ldots > \ell_p \geq 0,$$

$d(\underset{\sim}{H})$ is the invariant or Haar measure defined on the group $O(p)$ of $p \times p$ orthogonal matrices,

$$(6) \qquad D_1 = \left\{\pi^{\frac{p^2}{2}}\, \Gamma_p(\tfrac{1}{2}n) \big/ \Gamma_p(\tfrac{1}{2}q)\, \Gamma_p(\tfrac{1}{2}(n-q))\, \Gamma_p(\tfrac{1}{2}p)\right\} |\underset{\sim}{I}-\underset{\sim}{A}|^{\frac{n}{2}}$$

$$|\underset{\sim}{R}|^{\frac{1}{2}(q-p-1)}\, |\underset{\sim}{I} - \underset{\sim}{R}|^{\frac{1}{2}(n-q-p-1)} \prod_{i<j} (r_i - r_j) \quad ,$$

where

$$\Gamma_p(t) = \Pi^{\frac{p(p-1)}{4}} \prod_{j=1}^{p} (t - \frac{1}{2}(j-1)) \quad ,$$

and the hypergeometric function of the symmetric matrix $\underset{\sim}{Q}$ is given by [4]

$${}_{\mu}F_{\eta}(a_1,\ldots, a_{\mu}, b_1,\ldots, b_{\eta}, \underset{\sim}{Q}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_{\kappa} \ldots (a_{\mu})_{\kappa}}{(b_1)_{\kappa} \ldots (b_{\eta})_{\kappa}} \frac{C_{\kappa}(\underset{\sim}{Q})}{k!} \quad ,$$

where $a_1,\ldots, a_{\mu}, b_1,\ldots, b_{\eta}$ are real or complex constants and the multivariate hypergeometric coefficient $(a)_{\kappa}$ is given by

$$(a)_{\kappa} = \prod_{i=1}^{p} (a - \frac{1}{2}(i-1))_{k_i} \quad ,$$

where

$$(a)_k = a(a+1) \ldots (a+k-1) \quad .$$

The group $0(p)$ has volume

$$v(p) = \int_{0(p)} d(\underset{\sim}{H}) = 2^p \, \Pi^{\frac{p^2}{2}} \{\Gamma_p(\tfrac{1}{2}p)\}^{-1} \quad .$$

Let us order the r_i's in $\underset{\sim}{R}$ as

(7) $$1 > r_1 > \ldots > r_p > 0 \quad .$$

The density (4) involves an integral and following Anderson [1], Chang [2], Li and Pillai, [9], [10], our main objective is to maximize this integral. Let us now denote the integral by

$$E = \int_{0(p)} {}_2F_1\,(s,s,t,\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R})\,d(\underset{\sim}{H}) \quad , \tag{8}$$

where, for notational simplicity we put $s = \frac{1}{2}n$ and $t = \frac{1}{2}q$.

Now with mild restrictions on s by theorem 1 of [3], we find that for variation of $\underset{\sim}{H}\varepsilon 0(p)$, ${}_2F_1(s,s,t,\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R})$ is maximized when $\underset{\sim}{H}$ is given by (ii) in lemma 3 of [3], namely, ${}_2F_1(s,s,t,\underset{\sim}{A}\,\underset{\sim}{R})$. But here we proceed to obtain an alternate form which is more useful. First we use Kummer's formula and get

$$ {}_2F_1\,(s,s,t,\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = |\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-(2s-t)}\,{}_2F_1((t-s),(t-s),t,\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R})). \tag{9}$$

Now following earlier results [3], varying $\underset{\sim}{H}$ over $N(\underset{\sim}{I})$, the neighborhood of $\underset{\sim}{I}(p \times p)$, i.e. varying $\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}$ around $\underset{\sim}{A}\,\underset{\sim}{R}$, we get

$$ {}_2F_1\,(t-s,t-s,t,\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = {}_2F_1(t-s,t-s,t,\underset{\sim}{A}\,\underset{\sim}{R}) + 0(\varepsilon) \quad . \tag{10}$$

We prove below a more general result.

Lemma 1. If $\underset{\sim}{H}\varepsilon N(\underset{\sim}{I})$, $a_i \geq \frac{1}{2}(p-1)$, $b_j \geq \frac{1}{2}(p-1)$, $i = 1,\ldots,\mu$ $j = 1,\ldots,\eta$, then

$$ {}_\mu F_\eta(a_1,\ldots, a_\mu, b_1,\ldots, b_\eta, \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) = {}_\mu F_\eta(a_1,\ldots,a_\mu, b_1,\ldots,b_\eta,\underset{\sim}{A}\,\underset{\sim}{R}) + 0(\varepsilon), $$

provided

$$ t_i - \varepsilon \leq ch_i(\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}) \leq t_i+\varepsilon, \quad \text{where} \quad t_i = ch_i(\underset{\sim}{A}\,\underset{\sim}{R}),\ i = 1,\ldots,p \quad . $$

Proof. Let $f(\underset{\sim}{H}) = {}_{\mu}F_{\eta}(a_1,\ldots,a_{\mu},\, b_1,\ldots,b_{\eta},\, \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R})$. Then by lemma 2 of [3], $f(\underset{\sim}{H})$ is an increasing function in each of its characteristic roots. Thus varying $\underset{\sim}{H}\varepsilon N(\underset{\sim}{I})$, we note that first partial derivatives of $f(\underset{\sim}{H})$ with respect to each characteristic root exist, except possibly over a set of zero measure. Again as $f(\underset{\sim}{H})|_{\underset{\sim}{H}=\underset{\sim}{I}}$ exists, the mean value theorem applies and hence the lemma.

Now application of lemma 1 in conjunction with Kummer's formula (9) gives (10). Following Anderson [1], Chang [2], Li and Pillai [9],[10], and using (10) we get for large values of $(2s-t)$

$$E \doteq 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I}-\underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-(2s-t)}d(\underset{\sim}{H})\,{}_2F_1(t-s,t-s,t,\underset{\sim}{A}\,\underset{\sim}{R}) + 0(\varepsilon) \quad .$$

Further we consider

$$(11) \qquad F = 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-(2s-t)}d(\underset{\sim}{H}) \quad .$$

The integrand in (11) is quite similar to that of Chang [2] and hence what follows is essentially his technique as modified by Li and Pillai, [9],[10]. For the sake of continuity we write down the essential steps as applied in our case omitting the details to the above references with suitable modification.

Let us use the transformation

$$(12) \qquad \underset{\sim}{H} = \exp[\underset{\sim}{S}] \quad ,$$

where $\underset{\sim}{S}(p \times p)$ is a skew symmetric matrix. Then by Anderson [1]

$$(13) \qquad J(\underset{\sim}{S},\underset{\sim}{H}) = 1 + [(p-2)/4!]\ \mathrm{tr}\underset{\sim}{S}^2 + [(8-p)/(4.6!)](\mathrm{tr}\underset{\sim}{S}^4)$$

$$+ [(5p^2 - 20p + 14) / (8.6!)](\mathrm{tr}\underset{\sim}{S}^2)^2 + \ldots \qquad .$$

Under this transformation $N(\underset{\sim}{I}) \rightarrow N(\underset{\sim}{S} = \underset{\sim}{0})$. However as shown by Anderson [1] and Chang [2], for large $(2s-t)$ we can approximate F in (11) by integrating not exactly on $N(\underset{\sim}{S} = \underset{\sim}{0})$ but simply over intervals $-\infty < s_{ij} < \infty$ for each s_{ij}. Under the transformation (12) we have

$$|\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}| = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|\,|\underset{\sim}{I} + \{\underset{\sim}{S}\} + \{\underset{\sim}{S}^2\} + \{\underset{\sim}{S}^3\} + \ldots | \;,$$

i.e.

$$(14) \qquad |\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-(2s-t)} = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-(2s-t)}|\underset{\sim}{I} + \underset{\sim}{G}|^{-(2s-t)} \quad ,$$

where

$$\underset{\sim}{G} = \{\underset{\sim}{S}\} + \{\underset{\sim}{S}^2\} + \{\underset{\sim}{S}^3\} + \ldots \qquad .$$

Henceforth for notational ease we will write $2s-t = v$ i.e. (14) is rewritten as

$$|\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-v} = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-v}|\underset{\sim}{I} + \underset{\sim}{G}|^{-v} \quad .$$

Let $\underset{\sim}{T} = (\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R})^{-1}$. Since $\underset{\sim}{A}$ in our case is a fixed diagonal matrix and $\underset{\sim}{R}$ has random entries corresponding to sample canonical correlations, we neglect the set in which $\underset{\sim}{T}$ is undefined as at most it will contribute a set of measure zero. Thus without loss of generality we can write

(15)
$$T = (I - A\ R)^{-1}\ A = \begin{pmatrix} t_1 & 0 \ldots 0 \\ 0 & t_2 \ldots 0 \\ \vdots & \\ 0\ \ldots & \ldots\ t_p \end{pmatrix} ,$$

where

$$t_j = \ell_j(1-\ell_j r_j)^{-1}, \quad j = 1,\ldots, p.$$

Then

$$\{S\} = T\ (R\ S - S\ R),$$

$$\{S^2\} = \tfrac{1}{2}T\ (2S\ R\ S - S^2R - R\ S^2) \quad ,$$

and $\{S^3\}$ and other terms are obtainable with modification from Li and Pillai [9],[10]. Further we quote a lemma.

Lemma 2. Let b_j be the jth characteristic root of $B(p \times p)$ such that

$$\max_{1 \leq j \leq p} |b_j| < 1 \quad ,$$

then

$$|I + B|^v = \exp[v\ \mathrm{tr}(B - \tfrac{1}{2}B^2 + \tfrac{1}{3}B^3 \ldots)] \quad .$$

For proof see Chang [2].

Under transformation (12), $N(\underset{\sim}{I}) \rightarrow N(\underset{\sim}{S} = \underset{\sim}{0})$ and taking $\underset{\sim}{S}$ sufficiently close to $\underset{\sim}{0}$ we can take the maximum characteristic root of $\underset{\sim}{G}$ to be less than unity. Hence applying lemma 2 we get under (12)

$$|\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-v} = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-v}\,|\underset{\sim}{I} + \underset{\sim}{G}|^{-v}$$

$$= |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-v}\exp[-v\ \mathrm{tr}([\underset{\sim}{S}] + [\underset{\sim}{S}^2] + [\underset{\sim}{S}^3] + \dots)],$$

where

$$[\underset{\sim}{S}] = \{\underset{\sim}{S}\},$$

$$[\underset{\sim}{S}^2] = \{\underset{\sim}{S}^2\} - \tfrac{1}{2}\{\underset{\sim}{S}\}^2,$$

and $[\underset{\sim}{S}^3]$ and other terms are available from Li and Pillai, [9],[10], with obvious modification. Now putting $\underset{\sim}{S} = (s_{ij})$, and $\underset{\sim}{S}' = -\underset{\sim}{S}$ we have

$$\mathrm{tr}[\underset{\sim}{S}] = 0,$$

$$\mathrm{tr}[\underset{\sim}{S}^2] = \sum_{i<j} c_{ij}\, s_{ij}^2,$$

where

(16) $$c_{ij} = (t_{ji} - t_i t_j r_{ij}) r_{ij} = c_{ji},$$

$$t_{ij} = t_i - t_j,\ r_{ij} = r_i - r_j.$$

Thus we note that the above and other expressions follow from those of Li and Pillai [9], [10] changing $\underset{\sim}{R}$ to $-\underset{\sim}{R}$ and with accompanying change of notation. Hence following Li and Pillai [9], [10] we get after some lengthy algebra

$$F \approx 2^p |I - \underset{\sim}{A}\,\underset{\sim}{R}|^{-v} \prod_{i<j} \left(\frac{\Pi}{vc_{ij}}\right)^{\frac{1}{2}} \{1 + \frac{1}{4v} [\sum_{i<j} c_{ij}^{-1} + \alpha(p)] + \ldots\} \quad ,$$

where

$$(17) \qquad \alpha(p) = p(p-1)(2p+5)/12 \quad .$$

Thus substituting back this value in E we get the theorem:

<u>Theorem 1</u>. For large n, an asymptotic expansion of the distribution of $r_1, \ldots, r_p$ (the squares of the canonical correlation coefficients) where $1 > r_1 > \ldots > r_p > 0$ and the population parameters from (2) are such that $1 > \ell_1 > \ldots > \ell_p \geq 0$, is given by

$$(18) \qquad D_1 \prod_{i<j} \left(\frac{2\Pi}{(2n-q)c_{ij}}\right)^{\frac{1}{2}} |I - \underset{\sim}{A}\,\underset{\sim}{R}|^{-\frac{1}{2}(2n-q)} \{1 + \frac{1}{2(2n-q)} \Big[\sum_{i<j} c_{ij}^{-1} + \alpha(p)\Big] + \ldots\} \; {}_2F_1(\tfrac{1}{2}(q-n), \tfrac{1}{2}(q-n), \tfrac{1}{2}q, \underset{\sim}{A}\,\underset{\sim}{R}) + O(\varepsilon) \quad ,$$

where $\underset{\sim}{R}$, D_1, c_{ij} and $\alpha(p)$ are given by (3), (6), (16) and (17) respectively.

3. <u>The asymptotic expansion for canonical correlation-one extreme population multiple root</u>. James [7] has studied the distribution of smaller roots given the larger roots of a sample covariance matrix and has found a gamma type approximation with linkage factors between sample

roots corresponding to smaller and larger population roots. In their study of the two-sample case, Chang [2], Li and Pillai [9], [10], have found a beta type approximation in the same context. We obtain below a similar beta type approximation in the canonical correlation case. Let us assume

(19) $$\underset{\sim}{A} = \operatorname{diag}(\ell_1,\ldots,\ell_p),\ 1 > \ell_1 > \ell_2 > \ldots > \ell_k > \ell_{k+1} = \ldots = \ell_p = \ell \geq 0$$

and

$$\underset{\sim}{R} = \operatorname{diag}(r_1,\ldots, r_p), 1 > r_1 > \ldots > r_p > 0.$$

The joint distribution of $r_1,\ldots, r_p$ in this case is given by (4) with appropriate changes in definition of $\underset{\sim}{A}$ and as earlier we consider (8). Here we partition $\underset{\sim}{H}$ as follows

(20) $$\underset{\sim}{H} = \begin{array}{c c} p & \\ \begin{pmatrix} \underset{\sim}{H}_1 \\ \underset{\sim}{H}_2 \end{pmatrix} & \begin{array}{c} k \\ p-k \end{array} \end{array},$$

i.e. $\underset{\sim}{H}_1(k \times p)$ and $\underset{\sim}{H}_2((p-k) \times p)$. **Under** (19) we note that our integrand in (8) is invariant under choice of $\underset{\sim}{H}_2$ up to the restriction that the matrix $\underset{\sim}{H}$ is orthogonal [3]. Because of the above we can integrate out $\underset{\sim}{H}_2$ in (20) using the formula

(21) $$c_1 \int_{\underset{\sim}{H}_2} d(\underset{\sim}{H}) = c_2\, d(\underset{\sim}{H}_1) \quad ,$$

where

(22) $$c_1 = \Pi^{p^2/2}\{\Gamma_p(\tfrac{p}{2})\}^{-1} \quad \text{and} \quad c_2 = \Pi^{kp/2}\{\Gamma_k(\tfrac{p}{2})\}^{-1} \quad ,$$

where $d(\underset{\sim}{H}_1)$ denotes the invariant volume element of the Stiefel-manifold of orthonormal k-frames in p-space normalized to make its integral unity. Now by [3] and following Chang [2], Li and Pillai [9], [10], the integrand in (8) can be closely approximated for large ξ when $\underset{\sim}{H}$ has the following form

(23) $$\underset{\sim}{H} = \begin{pmatrix} \underset{\sim}{I}_o(k) & \underset{\sim}{0} \\ & \underset{\sim}{H}_2 \end{pmatrix} \quad ,$$

where $I_o(k) = \text{diag}(\pm 1, \ldots, \pm 1)$ and is of dimension k. Now restricting ourselves to orthogonal matrices we apply the following transformations

(24) $$\underset{\sim}{H} = \exp\ [\underset{\sim}{S}] \quad ,$$

where

(25) $$\underset{\sim}{S} = \begin{pmatrix} \underset{\sim}{S}_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}'_{12} & \underset{\sim}{0} \end{pmatrix} \quad ,$$

and $\underset{\sim}{S}_{11}(k \times k)$ is a skew symmetric matrix and $\underset{\sim}{S}_{12}(k \times (p-k))$, is a rectangular matrix. The jacobian of transformation (24) is given by (13). Also we have by analogy with Anderson [1], (James [7]),

(26) $$c_2 d(\underset{\sim}{H}_1) = c_3 d(\underset{\sim}{S}_{11})\ d(\underset{\sim}{S}_{12})(1+0(\text{squares of } s_{ij}\text{'s})) \quad ,$$

where

$$(27) \qquad c_3 = \Pi^{\frac{q^2}{2}} \{\Gamma_q(\tfrac{1}{2}q)\}^{-1} , \qquad q = p-k \quad ,$$

$d(\underset{\sim}{S}_{11})$ and $d(\underset{\sim}{S}_{12})$ stand for $\prod_{i<j=1}^{k} ds_{ij}$ and $\prod_{i=1}^{k} \prod_{j=k+1}^{p} ds_{ij}$ respectively. From equations (24) and (25) we get

$$h_{ii} = 1 - \frac{1}{2} \sum_{j=1}^{p} s_{ij}^2 , \qquad i \leq k \quad ,$$

and

$$h_{ij} = s_{ij} + \text{higher order terms}, \ (i \neq j), \quad s_{ij} = -s_{ji} \quad .$$

Now using the transformation (24) and following the technique used earlier and remembering that $q = p-k$ of the roots of $\underset{\sim}{A}$ are equal, we get

$$\mathrm{tr}[\underset{\sim}{S}^2] = \sum_{i<j} c_{ij} s_{ij}^2 + \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o} \, s_{ij}^2 \quad ,$$

where

$$(28) \qquad c_{ij} = (t_{ji} - t_i t_j r_{ij}) r_{ij} = c_{ji}, \ i,j=1,\ldots, k, \ \ i < j \quad ,$$

$$(29) \qquad c_{ij}^{o} = (t_{ji} - t_i t_j r_{ij}) r_{ij}, \qquad i = 1,\ldots,k, \ \ j = k+1,\ldots, p,$$

and

$$t_i = \begin{cases} \ell_i/(1-r_i \ell_i), & i = 1,\ldots, k \\ \ell/(1-r_i \ell), & i = k+1,\ldots, p, \end{cases}$$

$t_{ij} = t_i - t_j$ and $r_{ij} = r_i - r_j$.

Thus following Li and Pillai [9], [10], we get

$$|\underset{\sim}{I} - \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{R}|^{-(2s-t)} = |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-(2s-t)} \prod_{i<j=1}^{k} \exp[-(2s-t)c_{ij}s_{ij}^{2}]$$

$$\prod_{i=1}^{k} \prod_{j=k+1}^{p} \exp[-(2s-t)c_{ij}^{o}s_{ij}^{2}]\ \{1 + 0(s_{ij}^{2})\} \ .$$

Now for large $(2s-t)$, and remembering that in the present context the integrand (11) is invariant of the choice of H_2 in (20) and using (21) and (26) we get

$$(30) \qquad F \simeq 2^{k}\, c_3 c_1^{-1} |I - A\ R|^{-(2s-t)} \int_{\underset{\sim}{S}_{11}} \int_{\underset{\sim}{S}_{12}} \prod_{i<j=1}^{k} \exp[-(2s-t)c_{ij}s_{ij}^{2}]\ ds_{ij}$$

$$\prod_{i=1}^{k} \prod_{j=k+1}^{p} \exp[-(2s-t)c_{ij}^{o}s_{ij}^{2}]\ ds_{ij}\ \{1 + 0(\frac{1}{(2s-t)})\} \ .$$

Again when $(2s-t)$ is large and ℓ_i's and r_i's are well spaced $(i = 1,\ldots,p)$, most of the integral in (30) will be obtained from small values of the elements of $\underset{\sim}{S}_{11}$, $\underset{\sim}{S}_{12}$. Hence to obtain an asymptotic expansion, we can replace the range of elements of s_{ij} for all real values of them. With this stipulation, following Li and Pillai, [9],[10], we get after some lengthy algebra

$$(31) \qquad F \simeq 2^{k} c_3 c_1^{-1}\ |\underset{\sim}{I} - \underset{\sim}{A}\ \underset{\sim}{R}|^{-(2s-t)} \prod_{i<j=1}^{k} \left(\frac{\pi}{(2s-t)c_{ij}}\right)^{\frac{1}{2}} \prod_{i=1}^{k} \prod_{j=k+1}^{pp} \left(\frac{\pi}{(2s-t)c_{ij}^{o}}\right)^{\frac{1}{2}}$$

$$\{1 + \frac{1}{4(2s-t)}[\sum_{1<j=1}^{k} c_{ij}^{-1} + \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o\,-1} + \alpha(p,k)] + \ldots\},$$

where $\alpha(p,k)$ is defined below. Now using this value of F as in (31) and proceeding exactly as in the case of distinct roots in the matrix $\underset{\sim}{A}$ we get the following theorem.

<u>Theorem 1.1.</u> For large n, an asymptotic expansion of the distribution of $r_1, \ldots, r_p$, where $1 > r_1 > \ldots > r_p > 0$ and the parameters from (2) are such that $1 > \ell_1 > \ldots > \ell_k > \ell_{k+1} = \ldots = \ell_p \geq 0$, is given by

$$(32) \quad D_1 c_3 c_1^{-1} 2^k |\underset{\sim}{I} - \underset{\sim}{A}\,\underset{\sim}{R}|^{-\frac{1}{2}(2n-q)} \prod_{i<j=1}^{k} \left(\frac{2\Pi}{(2n-q)c_{ij}}\right)^{\frac{1}{2}} \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{2\Pi}{(2n-q)c_{ij}^{o}}\right)^{\frac{1}{2}}$$

$$\{1 + \frac{1}{2(2n-q)} [\sum_{i<j=1}^{k} c_{ij}^{-1} + \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o\,-1} + \alpha(p,k)] + \ldots\}$$

$$_2F_1(\tfrac{1}{2}(q-n), \tfrac{1}{2}(q-n), \tfrac{1}{2}q, \underset{\sim}{A}\,\underset{\sim}{R}) + 0(\varepsilon) \quad ,$$

where D_1, c_1, c_3, c_{ij} and c_{ij}^{o} are defined in (6), (22), (27), (28), (29) respectively and

$$\alpha(p,k) = \frac{k}{12}\{(k-1)(4k+1) + 6(p^2-k^2)\} \quad .$$

4. <u>Asymptotic expansion for MANOVA - population roots all distinct</u>. Let $\underset{\sim}{B}$ be the Between S.P. matrix and $\underset{\sim}{W}$ the Within S.P. matrix. Then $\underset{\sim}{B}(p \times p)$ has a non-central Wishart distribution with s d.f. and matrix of non-centrality parameter $\underset{\sim}{A}$, and $\underset{\sim}{W}$ has a central Wishart distribution on t d.f., the covariance matrix in each case being $\underset{\sim}{\Sigma}$, and

$$(33) \qquad \underset{\sim}{A} = \frac{1}{2}\,\underset{\sim}{\mu}\,\underset{\sim}{\mu}'\,\underset{\sim}{\Sigma}^{-1} \quad ,$$

where $\underset{\sim}{\mu}(p \times s)$ is the matrix of the mean vectors. Then the probability distribution function of the roots of the matrix

$$\underset{\sim}{R} = \underset{\sim}{B}(\underset{\sim}{B} + \underset{\sim}{W})^{-1} \quad , \tag{34}$$

is given by [4]

$$T_1 \int_{0(p)} {}_1F_1(\tfrac{1}{2}(s+t),\ \tfrac{1}{2}s,\ \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})\ d(\underset{\sim}{H}) \quad , \tag{35}$$

where

$$T_1 = \Pi^{\frac{p^2}{2}}\ \Gamma_p(\tfrac{1}{2}(s+t))\{\Gamma_p(\tfrac{1}{2}t)\Gamma_p(\tfrac{1}{2}s)\Gamma_p(\tfrac{1}{2}p)\}^{-1} \exp[-\mathrm{tr}\ \underset{\sim}{A}]$$
$$(\prod_{i=1}^{p} r_i)^{\frac{1}{2}(s-p-1)} \prod_{i=1}^{p} (1-r_i)^{\frac{1}{2}(t-p-1)} \prod_{i<j} (r_i-r_j) \quad .$$

Let

$$\underset{\sim}{R} = \mathrm{diag}(r_1,\ldots,r_p),\ 1 > r_1 > r_2 > \ldots > r_p > 0, \tag{36}$$
$$\underset{\sim}{A} = \mathrm{diag}(\ell_1,\ldots,\ell_p),\ \infty > \ell_1 > \ell_2 > \ldots > \ell_p \geq 0 \quad ,$$

where $\underset{\sim}{R}$ and $\underset{\sim}{A}$ are otherwise specified in (33) and (35), $0(p)$ and $d(\underset{\sim}{H})$ are as specified in the earlier problem.

As stated earlier, as in relation to the canonical correlation problem we consider the following:

$$E_1 = \int_{0(p)} {}_1F_1\ (\tfrac{1}{2}(s+t),\ \tfrac{1}{2}s,\ \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})\ d(\underset{\sim}{H}) \quad . \tag{37}$$

The integrand as it stands is not easy to work with, hence we apply the confluence relation (James [6]).

(38) $$\underset{c\to\infty}{\mathrm{Lt}}\ {}_2F_1(a,c,b,c^{-1}\underset{\sim}{S}) = {}_1F_1(a,b,\underset{\sim}{S}) \quad .$$

Applying the dominated convergence theorem, since the functions involved are well defined, we get using (38)

$$\underset{c\to\infty}{\mathrm{Lt}} \int_{O(p)} {}_2F_1(\tfrac{1}{2}(s+t),\ a,\ \tfrac{1}{2}s,\ a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})\ d(\underset{\sim}{H})$$

$$= \int_{O(p)} \underset{a\to\infty}{\mathrm{Lt}}\ {}_2F_1(\tfrac{1}{2}(s+t),\ a,\ \tfrac{1}{2}s,\ a^{-1}\ \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})\ d(\underset{\sim}{H})$$

$$= \int_{O(p)} {}_1F_1(\tfrac{1}{2}(s+t),\ \tfrac{1}{2}s,\ \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})\ d(\underset{\sim}{H}) = E_1 \quad .$$

Thus, for evaluating E_1, we consider, for large a,

(39) $$E_2 = \int_{O(p)} {}_2F_1(\tfrac{1}{2}(s+t),a,\tfrac{1}{2}s,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})d(\underset{\sim}{H}) \quad .$$

Thus we note that we can apply the earlier technique but with slight modification as would be noted in the process. For notational simplicity we use

(40) $$m = \tfrac{1}{2}(s+t) \quad \text{and} \quad n = \tfrac{1}{2}s \quad .$$

Now, using Kummer's relation given by James [6] we get

(41) $${}_2F_1(m,a,n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}) = |\underset{\sim}{I}-a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}|^{n-m-a}\ {}_2F_1(n-m,n-a,n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}) \ .$$

Again

$$|\underset{\sim}{I}-a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}|^{n-m-a} = |\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{D}||\underset{\sim}{I}-(\underset{\sim}{I}+\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{D}\ \underset{\sim}{H})^{-1}(\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{D} + a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})|$$

and

(42) $$\underset{\sim}{D} = \underset{\sim}{R}^{-1} \quad .$$

Thus we get from (41)

(43) $${}_2F_1(m,a,n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}) = |\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{D}|^{n-m}|\underset{\sim}{I}-a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}|^{-a}|\underset{\sim}{I}-(\underset{\sim}{I}+\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{D})^{-1}$$

$$(\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{D} + a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R})|^{n-m}\ {}_2F_1(n-m,n-a,n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}) \quad .$$

Also by [3], the integrand in (37) is maximized under the present set up when $\underset{\sim}{H}$ has the form (ii) in lemma 3 of [3]. Now if we expand the last three factors in (43) around $\underset{\sim}{H}\varepsilon N(\underset{\sim}{I})$, applying lemma 1, we get

(44) $${}_2F_1(m,a,n,a^{-1}\underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{R}) = |\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{D}|^{-(m-n)}\Phi(m,n,a,\underset{\sim}{A},\underset{\sim}{D},\underset{\sim}{R}) + 0(\delta) \quad ,$$

where

$$\Phi(m,n,a,\underset{\sim}{A},\underset{\sim}{D},\underset{\sim}{R}) = |\underset{\sim}{I}-a^{-1}\underset{\sim}{A}\ \underset{\sim}{R}|^{-a}|\underset{\sim}{I}-(\underset{\sim}{I}+\underset{\sim}{A}\ \underset{\sim}{D})^{-1}$$

$$(\underset{\sim}{A}\ \underset{\sim}{D} + a^{-1}\underset{\sim}{A}\ \underset{\sim}{R})|^{-(m-n)}\ {}_2F_1(n-m,n-a,n,a^{-1}\underset{\sim}{A}\ \underset{\sim}{R}) \quad .$$

Thus using (44) in (39) we get for large m

$$E_2 \simeq 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{D}|^{-(m-n)} \Phi(m,n,a,\underset{\sim}{A},\underset{\sim}{D},\underset{\sim}{R})d(\underset{\sim}{H}) + O(\epsilon) \quad .$$

Further we consider the following

$$E_3 = 2^p \int_{N(\underset{\sim}{I})} |\underset{\sim}{I} + \underset{\sim}{H}'\underset{\sim}{A}\,\underset{\sim}{H}\,\underset{\sim}{D}|^{-(m-n)} d(\underset{\sim}{H}) \quad ,$$

where

$$\underset{\sim}{D} = \text{diag}(d_1,\ldots,d_p),\ \infty > d_p > \ldots > d_1 > 1 \quad ,$$

and $\underset{\sim}{A}$ is as defined in (36). The integrand as it stands corresponds to that in Li and Pillai [9], [10], and hence following them as $m > n$ and for large m we get ,

$$E_3 \simeq 2^p|I + A\ D|^{-(m-n)} \prod_{i<j=1}^{p} \left(\frac{\Pi}{(m-n)c_{ij}}\right)^{\frac{1}{2}} \{1 + \frac{1}{4(m-n)} [\sum_{i<j} c_{ij}^{-1} + \alpha(p)] + \ldots\} \quad ,$$

where

$$(45) \qquad c_{ij} = \frac{(d_i-d_j)(\ell_j-\ell_i)}{(1+\ell_i d_i)(1+\ell_j d_j)} \ , \qquad i < j \quad ,$$

and $\alpha(p)$ given in (17) .

Thus putting all these results together we get the theorem:

Theorem 2. For large t (and hence for large sample size), an asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (34) with parameter matrix $\underset{\sim}{A}$ as in (33), where $\underset{\sim}{R}$ and $\underset{\sim}{A}$ satisfy (36) is given by

$$T_1 2^p \prod_{i < j=1}^{p} \left(\frac{2\Pi}{t\ c_{ij}}\right)^{\frac{1}{2}} \{1 + \frac{1}{2t} [\sum_{i < j} c_{ij}^{-1} + \alpha(p)] + \ldots \}$$

$$\exp[\operatorname{tr} \underset{\sim}{A}\ \underset{\sim}{R}]\ {}_1F_1(-\tfrac{1}{2}t, \tfrac{1}{2}s, -\underset{\sim}{A}\ \underset{\sim}{R}) + O(\varepsilon) \quad ,$$

where T_1 is given by (35) and c_{ij} by (45), $i < j$.

5. Asymptotic expansion for MANOVA —one extreme population multiple root.

The problem involved here is quite similar to the previous problem with the difference that the matrix $\underset{\sim}{A} = \operatorname{diag}(\ell_1, \ldots, \ell_p)$ defined in (33) now satisfies (46) instead of (36)

$$(46) \qquad \infty > \ell_1 > \ell_2 > \ldots > \ell_k > \ell_{k+1} = \ldots = \ell_p = \ell \geq 0 \quad .$$

Thus every step of the previous problem in canonical correlation Section 3 follows smoothly and we come to the consideration of (39). But now as in Section 3 we get by lemma 3.2 [3], the integrand in (33) is invariant of the choice of $\underset{\sim}{H}_2$ in (20). Thus, again, following the arguments and algebra as in Section 3, we get the following theorem (details of algebra are available from Li and Pillai [9], [10], with slight changes.).

Theorem 2.1. For large t (and hence for large sample size) an asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (34) with parameter matrix satisfying (46), is given by

$$T_1 c_3 c_1^{-1} 2^k \prod_{i<j=1}^{k} \left(\frac{2\Pi}{t\, c_{ij}}\right)^{\frac{1}{2}} \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{2\Pi}{t\, c_{ij}}\right)^{\frac{1}{2}} \exp[\mathrm{tr}\, \underset{\sim}{A}\, \underset{\sim}{R}]$$

$${}_1F_1(-\tfrac{1}{2}t, \tfrac{1}{2}s, -\underset{\sim}{A}\, \underset{\sim}{R})\ \{1 + \frac{1}{2t}[\mathop{}_{i<j=1}^{k} c_{ij}^{-1} + \sum_{i=1}^{t} \sum_{j=k+1}^{p} c_{ij}^{o\,-1}$$

$$+ \alpha(p,k)] + \ldots\} \quad + 0(\epsilon) \quad ,$$

where c_1, c_3 and T_1 are given by (22), (27) and (35) respectively, and c_{ij} and c_{ij}^{o} are defined as follows:

$$(47) \qquad c_{ij} = (t_{ji} - t_i t_j r_{ij}) r_{ij}, \quad i,j=1,\ldots,k,\ i < j\,(c_{ij} = c_{ji}) \quad ,$$

$$c_{ij}^{o} = (t_{ji} - t_i t_j r_{ij}) r_{ij}, \quad i=1,\ldots, k,\ j=k+1,\ldots,p,\ (c_{ij}^{o} = c_{ji}^{o}) \quad ,$$

where

$$t_{ij} = t_i - t_j \quad , \qquad r_{ij} = r_i - r_j \quad ,$$

$$t_i = \begin{cases} \ell_i/(1+\ell_i d_i), & i = 1,\ldots, k \\ \\ \ell/(1+\ell\, d_i), & i=k+1,\ldots, p. \end{cases}$$

$$\alpha(p,\kappa) = \frac{k}{12}\ \{(k-1)(4k+1) + 6(p^2-k^2)\}$$

and

$$\underset{\sim}{D} = \mathrm{diag}(d_1,\ldots,d_p) = \underset{\sim}{R}^{-1} \quad .$$

6. Asymptotic expansion for canonical correlation in the complex case — population roots all distinct. Let $x_1,\ldots, x_p$, $x_{p+1},\ldots, x_{p+q}$, $p \leq q$ be distributed complex normal $N_c(\underset{\sim}{0}, \underset{\sim}{\Sigma}_c)$, where

$$\underset{\sim}{\Sigma}_c = \begin{array}{c|cc} & & \\ p & \underset{\sim}{\Sigma}_{c11} & \underset{\sim}{\Sigma}_{c12} \\ q & \overline{\underset{\sim}{\Sigma}}'_{c12} & \underset{\sim}{\Sigma}_{c22} \\ & p & q \end{array} \quad .$$

Let $\underset{\sim}{P}^2_c = \mathrm{diag}(\rho^2_{c_1},\ldots, \rho^2_{cp})$, where ρ^2_{ci}, $i = 1,\ldots, p$ are the roots of

$$(48) \qquad |\underset{\sim}{\Sigma}_{c12}\, \underset{\sim}{\Sigma}^{-1}_{c22}\, \overline{\underset{\sim}{\Sigma}}'_{c12} - \rho^2_c\, \underset{\sim}{\Sigma}_{c11}| = 0 \quad ,$$

and let $\hat{\underset{\sim}{P}}^2_c = \mathrm{diag}(\hat{\rho}^2_{c1},\ldots, \hat{\rho}^2_{cp})$, where $\hat{\rho}^2_{ci}$, $i = 1,\ldots, p$, are the maximum likelihood estimators of ρ^2_{ci}, $i = 1,\ldots, p$, from a sample of size $n \geq p+q$, from the above population. Then the joint density of

$$(49) \qquad \hat{\underset{\sim}{P}}^2_c = \underset{\sim}{R} = \mathrm{diag}(r_1,\ldots, r_p),$$

is given by James [6], as

$$D_2 \int_{U(p)} {}_2\tilde{F}_1(n,n,q,\underset{\sim}{U}^* \underset{\sim}{A}\, \underset{\sim}{U}\, \underset{\sim}{R}) d(\underset{\sim}{U}) \quad ,$$

where

(50) $$\underset{\sim}{P}_c^2 = \underset{\sim}{A} = \mathrm{diag}(\ell_1,\ldots,\ell_p),\ 1 > \ell_1 > \ell_2 > \ldots > \ell_p \geq 0 \quad ,$$

$d(\underset{\sim}{U})$ is the invariant measure or the Haar measure defined on the group $U(p)$ of unitary matrices of order p,

(51) $$D_2 = [\pi^{p(p-1)}\tilde{\Gamma}_p(n)/\tilde{\Gamma}_p(n-q)\tilde{\Gamma}_p(q)\tilde{\Gamma}_p(p)]\,|\underset{\sim}{I}-\underset{\sim}{A}|^n|\underset{\sim}{R}|^{q-p}$$

$$|\underset{\sim}{I} - \underset{\sim}{R}|^{n-q-p} \prod_{i<j} (r_i - r_j)^2 \ ,$$

$$\tilde{\Gamma}_p(t) = \pi^{\frac{1}{2}p(p-1)} \prod_{j=1}^{p} \Gamma(t-j+1) \quad ,$$

where the hypergeometric function of the Hermitian matrix $\underset{\sim}{Z}$ is defined in [6] as

(52) $${}_\mu\tilde{F}_\eta(a_1,\ldots,a_\mu,b_1,\ldots,b_\eta,\ \underset{\sim}{Z}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]_\kappa \ldots [a_\mu]_\kappa}{[b_1]_\kappa \ldots [b_\eta]_\kappa} \frac{\tilde{C}_\kappa(\underset{\sim}{Z})}{k!} \quad ,$$

where

$$[a]_\kappa = \prod_{i=1}^{p} (a - i+1)_{k_i} \qquad ,$$

and $\kappa = (k_1,\ldots, k_p)$ is a partition of the integer k.

Let us now consider the elements of $\underset{\sim}{R}$ and $\underset{\sim}{A}$ as in (7) and (50) respectively. Then as in the real case we consider the following integral:

$$\int_{U(p)} {}_2\tilde{F}_1(n,n,a,\underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R})d(\underset{\sim}{U}) \quad .$$

To study the above integral let us consider a lemma analogous to lemma 1 in the real case.

<u>Lemma</u> <u>1.1</u>. If $\underset{\sim}{U}\epsilon U(p)$, and a_i's and b_j's are real $(i = 1,\dots,\mu,$ $j = 1,\dots,\eta\ )$, then

(i) $\quad {}_\mu\tilde{F}_\eta(a_1,\dots,a_\mu,b_1,\dots,b_\eta,\ \underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R})$ is real and if

$$\underset{\sim}{U}\epsilon N(\underset{\sim}{I}),\ a_i \geq (p-1),\ b_j \geq (p-1),\ (i = 1,\dots,\mu,j=1,\dots,\eta) \tag{53}$$

then

(ii) $$\quad {}_\mu\tilde{F}_\eta(a_1,\dots,a_\mu,b_1,\dots,b_\eta,\underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) = {}_\mu\tilde{F}_\eta(a_1,\dots,a_\mu,b_1,\dots,b_\eta,\underset{\sim}{A}\ \underset{\sim}{R}) + O(\varepsilon) \quad ,$$

provided

$$t_i - \varepsilon \leq ch_i(\underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) \leq t_i + \varepsilon,$$

where

$$t_i = ch_i(\underset{\sim}{A}\ \underset{\sim}{R}), \qquad i = 1,\dots, p \quad .$$

For proving (i), in view of (52), and a_i's and b_j's being real, it will suffice if we can show that $\hat{C}_\kappa(\underset{\sim}{U}^*\underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R})$ is real. This has been shown in

[3]. Now as $ch_i(\underset{\sim}{U}^* \underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R})$ is real $(i = 1,\ldots, p)$, and nonnegative in this case, under (53), we get (52) is an increasing function in each characteristic root. Result (ii) now follows by arguments similar to those in lemma 1.

Now as is done in the real case using Kummer's formula we get

$${}_2\tilde{F}_1(n,n,q,\underset{\sim}{U}^* \underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) = |\underset{\sim}{I} - \underset{\sim}{U}^* \underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}|^{-(2n-q)}\, {}_2\tilde{F}_1((q-n),(q-n),q,\underset{\sim}{U}^* \underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R})$$

and using lemma 1.1 and following Li and Pillai, [9], [10], we get the following theorem.

<u>Theorem</u> <u>1.2</u>. For large n, the asymptotic expansion for the distribution of $r_1,\ldots, r_p$, in (49) where $1 > r_1 > \ldots > r_p > 0$ and the parameters from (48) satisfy (50), is given by

$$D_2 \prod_{i<j=1}^{p} \left(\frac{\Pi}{(2n-q)c_{ij}}\right)\left\{1 + \frac{1}{3(2n-q)}\left[\sum_{i<j} c_{ij}^{-1} + \beta(p)\right] + \ldots\right\}$$

$$|\underset{\sim}{I} - \underset{\sim}{A}\ \underset{\sim}{R}|^{-(2n-q)}\ {}_2\tilde{F}_1((q-n),(q-n),q,\underset{\sim}{A}\ \underset{\sim}{R}) + O(\varepsilon) \quad ,$$

where $\beta(p) = p(p-1)(2p-1)/12$, $\underset{\sim}{R}$, $\underset{\sim}{A}$ and D_2 are given by (49),(50). and (51) respectively and c_{ij} are defined as in (16) with r_i and ℓ_j being replaced by corresponding elements of (49) and (50).

7. <u>Asymptotic expansion for canonical correlation in the complex case — one extreme multiple population root</u>. In this case we have the same model as in the distinct root case with the change that $\underset{\sim}{A}$ defined in (51) has the form

(54) $$\underset{\sim}{P}_c^2 = \underset{\sim}{A} = \text{diag}(\ell_1,\ldots,\ell_p),\ 1 > \ell_1 > \ell_2 > \ldots > \ell_k > \ell_{k+1} = \ldots$$
$$= \ell_p = \ell \geq 0 \quad ,$$

where $\underset{\sim}{P}_c^2$ is as defined in (48). Now, with necessary modifications in our procedure in the real case, and using lemma 1.1, and following Li and Pillai [11] we get the theorem:

Theorem 1.3. For large n, the asymptotic expansion for the distribution of $r_1,\ldots, r_p$, where $1 > r_1 > \ldots > r_p > 0$ and the parameters from (48) satisfy (54), is given by

$$D_2 D_3 \prod_{i<j} \left(\frac{\pi}{(2n-q)c_{ij}}\right) \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{\pi}{(2n-q)c_{ij}^{o}}\right) \{ 1 + \frac{1}{3(2n-q)} [\sum_{i<j=1}^{k} c_{ij}^{-1}$$

$$+ \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o\,-1} + \beta(p,k)] + \ldots \} \ |\underset{\sim}{I} - \underset{\sim}{A}\ \underset{\sim}{R}|^{-(2n-q)}$$

$${}_2\tilde{F}_1 \ ((q-n),(q-n),q,\underset{\sim}{A}\ \underset{\sim}{R}) + O(\varepsilon) \quad ,$$

where

$$D_3 = \pi^{q(q-1)} \{ \tilde{\Gamma}(q) \}^{-1}$$

and $q=p-k$, D_2 is as in (51), c_{ij} and c_{ij}^{o} are as in the real case with r_i's and ℓ_j's substituted from (49) and (54) and $\beta(p,k) = \frac{k}{2}\{(k-1)(2k-1) + 3(p-k)(p+k-1)\}$.

8. Asymptotic expansion for MANOVA in the complex case —population roots all distinct. Let $\underset{\sim}{B}_1$ be the Between S.P. matrix and $\underset{\sim}{W}_1$ the Within S.P. matrix in a complex multivariate normal case. Then $\underset{\sim}{B}_1$ $(p \times p)$ has a complex non-central Wishart distribution with s d.f. and matrix of noncentrality parameter $\underset{\sim}{A}$, and $\underset{\sim}{W}_1$ has a complex central Wishart distribution on t d.f., the covariance matrix in each case being $\underset{\sim}{\Sigma}_1$, and

$$\underset{\sim}{A} = \underset{\sim}{\mu}_1 \underset{\sim}{\mu}_1' \underset{\sim}{\Sigma}_1^{-1} \tag{55}$$

where $\underset{\sim}{\mu}_1(p \times s\ \ )$ is the matrix of the mean vectors. Then the density of the roots of the matrix

$$\underset{\sim}{R} = \underset{\sim}{B}_1(\underset{\sim}{W}_1 + \underset{\sim}{B}_1)^{-1} \tag{56}$$

is given by [6]

$$T_2 \int_{U(p)} {}_1\tilde{F}_1((s+t),s,\ \underset{\sim}{U}^* \underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) d(\underset{\sim}{U}) \quad ,$$

where

$$T_2 = [\pi^{p(p-1)} \tilde{\Gamma}_p(s+t)/\tilde{\Gamma}_p(p)\tilde{\Gamma}_p(s)\tilde{\Gamma}_p(t)]\,|\underset{\sim}{I}-\underset{\sim}{R}|^{t-p}$$

$$|\underset{\sim}{R}|^{(s-p)} \prod_{i<j} (r_i-r_j)^2 \quad \exp[-\mathrm{tr}\ \underset{\sim}{A}] \quad ,$$

where

(58) $$\underset{\sim}{R} = \mathrm{diag}(r_1, \dots, r_p),\ 1 > r_1 > \dots > r_p > 0$$

$$\underset{\sim}{A} = \mathrm{diag}\ (\ell_1, \dots, \ell_p),\ \infty > \ell_1 > \dots > \ell_p \geq 0$$

and $d(\underset{\sim}{U})$ is as defined earlier. Now as in real case we consider

$$\int_{U(p)} {}_1\tilde{F}_1((s+t),\ s,\ \underset{\sim}{U}^* \underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) d(\underset{\sim}{U}) \quad .$$

But as stated in the real case, the integrand as it stands is difficult to work with and as such we consider, instead, for large a

$$\int_{U(p)} {}_2\tilde{F}_1\ ((s+t),\ a,\ s,\ a^{-1}\underset{\sim}{U}^* \underset{\sim}{A}\ \underset{\sim}{U}\ \underset{\sim}{R}) d(\underset{\sim}{U}) \quad .$$

Thus proceeding as before and by Li and Pillai [9],[10], we get the theorem:

Theorem 2.2. For large t (and hence for large sample size), the asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (56) where $\underset{\sim}{R}$ and the parameter matrix $\underset{\sim}{A}$ in (55) satisfy (58), is given by

$$T_2 \prod_{i<j} \left(\frac{\pi}{t\ c_{ij}}\right) \{1 + \frac{1}{3t} [\sum_{i<j} c_{ij}^{-1} + \beta(p)] + \dots\} \ \exp\ [\mathrm{tr}\ \underset{\sim}{A}\ \underset{\sim}{R})$$
$$ {}_1\tilde{F}_1\ (-t,\ s,\ -\underset{\sim}{A}\ \underset{\sim}{R}) + 0(\varepsilon) \quad ,$$

where c_{ij} is given by (45) using $\underset{\sim}{A}$ and $\underset{\sim}{R}$ from (55) and (56) respectively, T_2 is given by (58) and $\beta(p) = p(p-1)(2p-1)/12$.

9. Asymptotic expansion for MANOVA in the complex case — one extreme multiple population root.

As in the canonical correlation case with one extreme multiple population root, here again the model is the same as in the distinct root case with the change that $\underset{\sim}{A}$ defined in (55) has the following form

$$\underset{\sim}{\mu}_1\underset{\sim}{\mu}_1'\underset{\sim}{\Sigma}_1^{-1} = \underset{\sim}{A} = \operatorname{diag}(\ell_1,\ldots,\ell_p); \quad \infty > \ell_1 > \ldots > \ell_k > \ell_{k+1} = \ldots = \ell_p \geq 0 \quad ,$$

where $\underset{\sim}{\mu}_1$, $\underset{\sim}{\Sigma}_1$ are defined as in the distinct root case. Now proceeding as in the earlier case with necessary changes and following Li and Pillai [11] we get the theorem:

<u>Theorem</u> <u>2.3</u>. For large t (and hence for large sample size), the asymptotic expansion for the distribution of the characteristic roots of $\underset{\sim}{R}$ in (56) where $1 > r_1 > \ldots > r_p > 0$ and parameter matrix $\underset{\sim}{A}$ in (55) satisfy (59), is given by

$$T_2D_3 \prod_{i<j=1}^{k} \left(\frac{\Pi}{t\ c_{ij}}\right) \prod_{i=1}^{k} \prod_{j=k+1}^{p} \left(\frac{\Pi}{t\ c_{ij}^{o}}\right) \{ 1 + \frac{1}{3} + [_{i} \sum_{j=1}^{k} c_{ij}^{-1}$$

$$+ \sum_{i=1}^{k} \sum_{j=k+1}^{p} c_{ij}^{o^{-1}} + \beta(p,k)] + \ldots \} \exp[\operatorname{tr} \underset{\sim}{A}\ \underset{\sim}{R}]$$

$$_1\tilde{F}_1\ (-t,\ s,\ -\underset{\sim}{A}\ \underset{\sim}{R}) + O(\varepsilon) \quad ,$$

where

$$D_3 = \Pi^{q(q-1)} \{\tilde{\Gamma}_q(q)\}^{-1}, \quad \beta(p,k) = \frac{k}{2}\{(k-1)(2k-1) + 3(p-k)(p+k-1)\}$$

and $q = p-k$, c_{ij} and c_{ij}^{o} are as defined in (47) but the r_i and ℓ_j's being taken from (56) and (57) respectively.

10. Remarks. As will be noted from the following there are some general remarks which apply to all cases discussed above and some others which pertain only to special cases.

1. The method as outlined above is a generalization of Anderson's result [1] and all his comments are applicable here also. Note especially the following one.

2. No proof has been given to show that we have an asymptotic expansion of the integrals involved on each case, but application of an extension of Laplace's method as given by Hsu [5] can be utilized to show that in each case the first term gives an asymptotic representation and has been explicitly shown by Chang [2] and hence we just refer to his result.

3. In approximating ${}_{\mu}F_{\eta}$ or ${}_{\mu}\tilde{F}_{\eta}$ by Kummer's formula we note that if we take $N(\underset{\sim}{I})$ involved in each case to be sufficiently close to $\underset{\sim}{I}$, which is possible for large enough sample size, we can neglect $O(\varepsilon)$ in each case for good enough approximation.

4. The direction of ordering of roots in each problem is immaterial as can be shown by results in [3], and as such the only restriction is that the roots of the sample and population matrices should be ordered in the same direction.

5. From remark 4 it may be seen that the expansion for one extreme multiple population root covers the largest root although the results given in the paper are for the smallest.

6. Each formula, as given, gives a considerable simplification in the ${}_{\mu}F_{\eta}$ function since each population root goes along with its sample counterpart

7. In the real case when a in $(a)_k$ is a negative integer the hypergeometric function involved reduces to a polynomial. In the complex situation a constant being negative ${}_{\mu}F_{\eta}$ always reduces to a polynomial expansion.

8. When all the population roots are equal we see that $O(\varepsilon)$ term in our expansion is identically zero. Here we have to take any empty product to be unity.

9. Though in [3] the limits of the elements of the matrices $\underset{\sim}{A}$ and $\underset{\sim}{R}$ are taken to be the whole real line, it does not matter even if we take it to be any interval [a,b], where a and b are two distinct real numbers.

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