The Single Server Queue With Semi-Markovian Arrivals and Services*

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Abstract

The single server queue with semi-Markovian arrivals and services is studied. The stability of the queue is proved under the usual assumption that the traffic intensity is less than 1.

Next, the waiting time and the type of the n^{th} customer are studied jointly. The transient behavior is studied using the matrix factorization theorem of Miller [9]. Asymptotic results are also obtained using a technique due to Smith [13].

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I. Introduction.

We consider a queueing model in which the n^{th} customer is of type J_{n-1} and arrives at a single server counter in the instant $T_n (0 \leq T_1 < T_2 < \ldots < T_n < \ldots)$. There are M customer types. The queue discipline is "first come, first served". Let S_n be the service time of C_n and W_n be the waiting time before being served. For $n \geq 1$, we write $t_n = T_{n+1} - T_n$, the interarrival time between the n^{th} and $n+1^{st}$ customers; we set $t_0 = s_0 = 0$ and put

(1) $$u_n = s_n - t_n \qquad (n \geq 0).$$

Let $\{(s_n, t_n, J_n), n \geq 0\}$ be defined on a complete probability space $(\Omega, \mathcal{G}, P)$ and having the following three properties:

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(P. 1) $P\{J_0 = k\} = P_k \quad (k=1,2,\ldots,M;\ M < \infty)$,

(P. 2) $$P\{S_n \leq x,\ t_n \leq y,\ J_n = j \mid (s_k,\ t_k,\ J_k,\ k \leq n);\ J_{n-1}=i\}$$

$$= P\{S_n \leq x,\ t_n \leq y,\ J_n=j \mid J_{n-1}=i\}$$

$$= K_{ij}(x,y) \qquad (n \geq 1;\ x \geq 0,\ y \geq 0;\ i,j=1,2,\ldots,M),$$

(P. 3) The imbedded Markov Chain $\{J_n,\ n \geq 0\}$ is irreducible and stationary such that

$$P\{J_n = j\} = P_j \qquad (n \geq 0;\ j = 1,2,\ldots,M).$$

The basic semi-Markov assumption (P. 2) states that the joint distribution of S_n, t_n depends only on the transition of the Markov Chain $\{J_n\}$ from the present state to the next. In other words, S_n, t_n depend on the types J_{n-1}, J_n only.

This general model is important and we are anticipating its application to concrete situations. A practical example occurs as follows. According to recent observations of traffic flow, especially during rush hours, trucks follow cars much closer than cars follow the trucks. Let vehicles, cars or trucks, arrive at a single server counter which offers certain facilities. If we count as the service time of a vehicle its time in the system together with the time required to adjust the server to the different type of vehicle, then interarrival times and service times depend only on the transitions in the Markov Chain $\{J_n,\ n \geq 0\}$.

We note that (P. 2) implies that the two double sequences $\{t_n, J_n, n \geq 0\}$ and $\{s_n, J_n, n \geq 0\}$ are proper semi-Markov ones (cf. [10]). For $-\infty < x < \infty$, let $Q_{ij}(x) = \int_0^\infty dK_{ij}(x+u,u)$. Then

(1) $Q_{ij}(x) = P\{u_n \leq x, J_n=j \mid J_{n-1}=i\} \qquad (i,j=1,2,\ldots,M),$

that is, $\{u_n, J_n, n > 0\}$ forms a (proper) semi-Markov sequence which implies that $Q_{ij}(+\infty)=P_{ij}$ and $\sum_{j=1}^{M} P_{ij}=1$ for all i. Where

(2) $$P_{ij} = P\{J_{n+1}=j \mid J_n=i\}.$$

In the subsequent sections, we shall discuss the stability, the transient behavior and the asymptotic behavior of the waiting time process W_n.

2. The Stability of the SM/SM/1 queue.

The definition of the stability of a queue is given in the following:

Definition 1. A sequence of a.e. finite random variables x_n with (honest) distribution functions F_n is said to be:

(i) stable, if F_n tends to some (honest) distribution function F at all its points of continuity;

(ii) substable, if each subsequence contains a stable sub-subsequence;

(iii) unstable, if it is not substable.

Definition 2. A queue is called stable, substable, or unstable according to the behavior of the waiting time sequence W_n.

If we introduce:

(3) $$\zeta_i = \sum_{j=1}^{M} \int_{-\infty}^{\infty} x \, d\, Q_{ij}(x) ;$$

(4) $$\zeta_i^* = \sum_{j=1}^{M} \int_{-\infty}^{\infty} |x| dQ_{ij}(x)$$

for $i = 1,2,\ldots M$, then we have the following stability theorem:

Theorem 1.

Suppose that $\zeta_i^* < \infty$ $(i=1,2,\ldots,M)$, then the SM/SM/1 queue is stable if $\sum_{i=1}^{M} P_i \zeta_i < 0$ and unstable if $\sum_{i=1}^{M} P_i \zeta_i > 0$. Where $\{P_i\}$ were given by (P. 3).

In order to prove this theorem, we need a lemma:

Lemma 1.

For all $j = 1,2,\ldots,M$, we have:

(5) $$\frac{{}_1P^*_{1j}}{m_{11}} = \frac{{}_2P^*_{2j}}{m_{22}} = \ldots = \frac{{}_MP^*_{Mj}}{m_{MM}} = P_j .$$

Where ${}_iP^*_{ij}$ is the expected number of times the Markov chain $\{J_n\}$ is in the state j before it is in the state i, starting from i; m_{ii} is the mean recurrence time of the state i. (These notations are the same as in Chung [2]).

Proof:

Since the Markov chain $\{J_n\}$ is irreducible, positive and such that $P_j = \sum_{k=1}^{M} P_k P_{kj}$, from a theorem of Chung ([2], p. 35), we have:

(6) $$P_j = c\, \pi_j \qquad (j=1,2,\ldots,M) ,$$

where

(7) $$\pi_j = \frac{1}{m_{jj}}$$

and c is a constant. Moreover,

(8) $$\sum_{j=1}^{M} P_j = \sum_{j=1}^{M} \Pi_j = 1 .$$

Thus we have $c = 1$ which implies that:

(9) $$P_j = \Pi_j \qquad (j=1,2,\ldots,M).$$

Next, let us denote by f^*_{ij} the probability that the Markov chain will be in the state j at least once, given that it starts from the state i. Then we get from Chung [2]:

(10) $$\Pi_{ij} = \frac{f^*_{ij}}{m_{jj}} \qquad (i,j=1,2,\ldots,M) ,$$

where

(11) $$\Pi_{ij} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^{(k)}_{ij} \quad \text{and}$$

$$P^{(k)}_{ij} = P\{J_k=j \mid J_0=i\}.$$

Furthermore, $f^*_{ij} = 1$ because of the recurrence of the Markov chain, whence

(12) $$\Pi_{ij} = \frac{1}{m_{jj}} \qquad (i,j=1,2,\ldots,M) .$$

If we introduce taboo probabilities, then a formula in Chung [2] yields that:

$$(13) \qquad {}_{i,H}P^{*}_{ij} = \lim_{N\to\infty} \frac{\sum_{n=1}^{N} {}_{H}P^{(n)}_{ij}}{1 + \sum_{n=1}^{N} {}_{H}P^{(n)}_{ii}} ,$$

where

$$(14) \qquad {}_{i,H}P^{*}_{ij} = \sum_{n=1}^{\infty} {}_{i,H}P^{(n)}_{ij} ;$$

$${}_{i,H}P^{(n)}_{ij} = P\{J_n=j; J_k \neq i, J_k \notin H,\ 0 < k < n \mid J_0=i\} .$$

Therefore, setting $H = \phi$, with the help of (11), (12), we conclude:

$$(15) \qquad \frac{\frac{1}{m_{jj}}}{\frac{1}{m_{ii}}} = {}_{i}P^{*}_{ij} \qquad (i,j=1,2,\ldots,M),$$

which is the stated result (5) by (7) and (9). Thus the lemma is proved.

Proof of theorem 1:

Let $U_n = \sum_{k=1}^{n} u_k$. We fix an arbitrary state i and apply the method of the dissection principle (Chung [2]). For a sample point ω, let $\tau_1(\omega) < \tau_2(\omega) < \tau_3(\omega) < \ldots\ldots < \tau_k(\omega) < \ldots\ldots$ be the increasing infinite sequence of those $n \geq 0$ for which $J_n(\omega)=i$. Then:

$$(16) \qquad U_n = y'(n) + \sum_{k=1}^{\ell(n)-1} y_k + y''(n) \qquad (n \geq 1) ,$$

where

(17) $$y'(n) = \sum_{s=1}^{\tau_1 - 1} U_s(\omega) \quad \text{independent of } n;$$

(18) $$y_k = \sum_{s=\tau_k}^{\tau_{k+1}-1} U_s(\omega) \ ;$$

(19) $$y'(n) = \sum_{s=\tau_{\ell(n)}}^{n} U_s(\omega) \ ;$$

and for given n, ω, $\ell(n)$ is the unique integer satisfying that:

(20) $$\tau_{\ell(n)}(\omega) \leq n < \tau_{\ell(n)+1}(\omega)$$

Since τ_1 is finite a.e., we have:

(21) $$\frac{y'(n)}{n} \to 0 \text{ a.e. as } n \to \infty.$$

Further, a theorem of Chung yields that:

(22) $$\frac{y''(n)}{n} \overset{P}{\to} 0 \text{ as } n \to \infty.$$

To estimate $\sum_{k=1}^{\ell(n)-1} y_k$, by using Chung's theorems, we obtain that:

(23) $$\frac{1}{n} \sum_{k=1}^{\ell(n)-1} y_k = \frac{\ell(n)-1}{n} \cdot \frac{1}{\ell(n)-1} \sum_{k=1}^{\ell(n)-1} y_k \to \Pi_i \, Ey_1 \text{ a.e.}$$

$$= \frac{1}{m_{ii}} Ey_1 \text{ a.e. as } n \to \infty.$$

Now, lemma 4.1 of Pyke and Schaufele ([12], p. 1756) gives:

$$(24) \qquad Ey_1 = \sum_{j=1}^{M} {}_iP^*_{ij} \ \zeta_j \ ,$$

whence

$$(25) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\ell(n)-1} y_k = \sum_{j=1}^{M} \frac{{}_iP^*_{ij}}{m_{ii}} \ \zeta_j \quad \text{a.e.} \ .$$

Therefore, applying lemma 1, we have from (16), (21), (22) and (25):

$$(26) \qquad \frac{U_n}{n} \xrightarrow{P} \sum_{j=1}^{M} P_j \zeta_j \quad \text{as} \quad n \to \infty.$$

On the other hand, it can be shown from the assumptions (P. 2) and (P. 3) that $\{u_n, n \geq 1\}$ is a strictly stationary sequence such that:

$$(27) \qquad Eu_1 = \sum_{j=1}^{M} P_j \zeta_j .$$

Thus, by Doob's theorem for strictly stationary processes ([4], p. 465), we deduce that:

$$(28) \qquad \frac{U_n}{n} \to E[u_1|\varphi] \quad \text{a.e.} \quad \text{as} \quad n \to \infty ,$$

where φ is the Borel field of invariant ω sets. Consequently:

(29) $$E[u_1|\zeta] = E\, u_1 = \sum_{j=1}^{M} P_j\, \zeta_j \quad \text{a.e.}\ ,$$

which proves theorem 1 according to a stability theorem due to Loynes [7].

It remains to discuss the critical case $\sum_{j=1}^{M} P_j \zeta_j = 0$. According to Doob ([4], p. 456), the strictly stationary process $\{u_n, n \geq 1\}$ can be extended to form a strictly stationary process $\{u_n, -\infty < n < \infty\}$. Combining (29) and a stability theorem of Loynes [7], we have the following:

Theorem 1'.

Suppose that $\zeta_i^* < \infty$ $(i=1,2,\ldots,M)$. Then the $_M$SM/SM/1 queue may be either stable, properly substable, or unstable, if $\sum_{j=1}^{M} P_j \zeta_j = 0$:

(i) If w is dishonest, the queue is unstable for all initial conditions;

(ii) If w is honest, the queue can be either stable or properly substable. Initial conditions will affect the asymptotic distributions. Where

(30) $$w = \left[\sup_{r \geq 1} \sum_{k=1}^{r} u_{-k} \right]^+ .$$

Remark: Suppose $\zeta_i^* < \infty$ $(i=1,2,\ldots,M)$, then the queue has a unique stationary waiting time distribution if and only if $\sum_{j=1}^{M} P_j \zeta_j < 0$.

This remark follows from theorems 1 and 1'.

3. Transient behavior of the joint process $\{w_n, J_{n-1}\}$.

In this section, we shall express the probabilities

(31) $G_{ij}^{(n)}(x) = P\{w_n \leq x,\ J_{n-1}=j \mid w_1=a, J_0=i\}$ $(n \geq 1;\ i,j=1,2,\ldots,M)$

in terms of the given distributions of $\{s_n\}$, $\{t_n\}$ or $\{u_n\}$.

Since $G_{ij}^{(n)}(x)$ is a mass function, it induces a measure also denoted by $G_{ij}^{(n)}(.)$. The same convention applies to other mass functions in the context.

It is known that

(32) $$w_{n+1} = (w_n + u_n)^+ .$$

Next, from (P. 2), we find that $\{u_n,\ n \geq 1\}$ are conditionally independent given the Markov chain $\{J_n\}$. (See (3.6) of Pyke [11]). Hence W_n and u_n are conditionally independent given $J_0, J_1, J_2, \ldots, J_n$. By this property and (32), we get that:

(33) $$G_{ij}^{(n+1)} = \pi\left(\sum_{k=1}^{M} G_{ik}^{(n)} * Q_{kj}\right) \qquad (n \geq 1;\ i,j=1,2,\ldots,M) ;$$

(34) $$G_{ij}^{(1)}(.) = \epsilon_{ij}(a,.) .$$

Where $*$ is convolution of measures and:

(35) $$\epsilon_{ij}(a,B) = \begin{cases} \delta_{ij} & \text{if } a \in B \\ 0 & \text{if } a \notin B \end{cases} \quad (B \text{ is any Borel set});$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} ;$$

(36) $$P\{W_1 = a\} = 1 \quad (a \geq 0) ;$$

and π is a Wendel projection defined by:

(37) $$(\pi\,\mu)\,(B) = \mu\,\{x:\ x^+ \in B\} \qquad (B \text{ is any Borel set}).$$

(See (10) of Kingman [6]).

Now we introduce a matrix algebra. Let

(38) $$M^* = \left\{ f:\ f = \begin{pmatrix} f_{11} & \cdots & f_{1M} \\ \cdot & \cdots & \cdot \\ f_{M1} & \cdots & f_{MM} \end{pmatrix} \equiv (f_{ij}),\ f_{ij} \text{ is a finite signed measure on Borel sets of } R \text{ for each } i,j. \right\}$$

Define addition and multiplications of elements in M^* by:

$$f + g = (f_{ij} + g_{ij})\ ;$$

$$f\,g = \left(\sum_{k=1}^{M} f_{ik} * g_{kj} \right);$$

$$c\,f = (cf_{ij})\ .\ (c \text{ is real}).$$

Then M^* is an algebra with an identity $\epsilon^* = \begin{pmatrix} \epsilon & & 0 \\ & \ddots & \\ 0 & & \epsilon \end{pmatrix}$, where

(39) $$\epsilon(B) = \begin{cases} 1 & \text{if } 0 \in B \\ 0 & \text{if } 0 \notin B \end{cases} \qquad (B \text{ is any Borel set}).$$

<u>Remark</u>: Since M^* is not commutative, the arguments of the GI/G/1 queue in Kingman [6] cannot be applied to the SM/SM/1 queue.

Furthermore, define $\pi^* : M^* \to M^*$ by $\pi^* f = (\pi\, f_{ij})$, then π^* is a Wendel projection on M^*, that is,

(40) $$M^* = M^*_+ \oplus M^*_- ,$$

and M_+^*, M_-^* are subalgebras, where

(41) $$M_+^* = \{\pi^* f: \ f \in M^*\} ;$$

(42) $$M_-^* = \{f \in M^*: \ \pi^* f = 0\} .$$

In other words, $M_+^* \cap M_-^* = \{0\}$ and any f in M^* can be written as $f=f_1+f_2$ with $f_1 \in M_+^*$, $f_2 \in M_-^*$.

Introduce the following:

(43) $$M^*[x] = \left\{ \Psi: \ \Psi = \sum_{n=0}^{\infty} f^{(n)} x^n \text{ with } f^{(n)} \in M^* \right\} ,$$

and define addition, multiplications in $M^*[x]$ by:

$$\sum_{n=0}^{\infty} f^{(n)} x^n + \sum_{n=0}^{\infty} g^{(n)} x^n = \sum_{n=0}^{\infty} (f^{(n)} + g^{(n)}) x^n ;$$

$$\left[\sum_{n=0}^{\infty} f^{(n)} x^n \right]\left[\sum_{n=0}^{\infty} g^{(n)} x^n \right] = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} f^{(k)} g^{(n-k)} \right] x^n ;$$

$$c \left[\sum_{n=0}^{\infty} f^{(n)} x^n \right] = \sum_{n=0}^{\infty} [c f^{(n)}] x^n .$$

Identify $M^* \subset M^*[x]$ by $f = f + 0 . x + 0 . x^2 + \ldots$, then ϵ^* is an identity in $M^*[x]$. Upon extending π^* to $M^*[x]$ by:

$$\pi^* \left[\sum_{n=0}^{\infty} f^{(n)} x^n \right] = \sum_{n=0}^{\infty} [\pi^* f^{(n)}] x^n ,$$

then we conclude that $M^*[x] = M_+^*[x] \oplus M_-^*[x]$ and $M_+^*[x]$, $M_-^*[x]$ are subalgebras.

In order to use the Miller's matrix factorization theorem [9] to obtain the transient behavior of the waiting time process w_n, we make two additional assumptions:

(A_1): $\{J_n, n \geq 0\}$ is aperiodic ,

(A_2): There exists c', $-\infty \leq c' < 0$ such that for each pair i,j, $\hat{Q}_{ij}(t) \equiv \int_{-\infty}^{\infty} e^{-tx}\, dQ_{ij}(x)$ is an analytic function of t for $c' < \text{Re } t \leq 0$ and for some pair i_0, j_0, $\hat{Q}_{i_0 j_0}$ (Re t) tends to infinity as Re $t \to c'+$. Besides, we assume that $P\{u_1 > 0\} > 0$.

First, we note that $\hat{Q}(0) = P$ is a primitive, irreducible, non-negative matrix with Perron eigenvalue $\lambda(0) = 1$ (See Miller [8] [9]). Also, for real t, $\hat{Q}(t)$ has the Perron eigenvalue $\lambda(t)$, (See [3]).

Let

$$\hat{f}_{ij}(t) = \int_{-\infty}^{\infty} e^{-tx}\, f_{ij}(dx) \tag{44}$$

be the Laplace-Stieltjes transform (L.S.T.) of f_{ij} if it exists. Set $\hat{f} = (\hat{f}_{ij})$ and:

$$C = \{\hat{f}: \hat{f}(t) \text{ exists for } c' < \text{Re } t \leq 0\}.$$

Therefore (33) becomes:

(34) $$G^{(n+1)} = \pi^* [G^{(n)} Q] \text{ and:}$$

(35) $$\hat{G}^{(n+1)}(t) = \hat{\pi}^* [\hat{G}^{(n)} \hat{Q}] \quad (\text{Re } t = 0) .$$

Where

(36) $$\hat{\pi}^* \hat{\mu} = \widehat{\pi^* \mu} .$$

Note that $\widehat{\mu\nu} = \hat{\mu} \hat{\nu}$ if all exist.

Applying the L.S.T. on $M^*[x]$ for those elements in C, we obtain that $\hat{M}^*[x] = \hat{M}^*_+[x] \oplus \hat{M}^*_-[x]$ with subalgebras $\hat{M}^*_+[x]$ and $\hat{M}^*_-[x]$. (At least, Fourier transforms exist).

Forming the generating functions:

(37) $$\Psi = G^{(1)} + G^{(2)}x + G^{(3)}x^2 + \ldots\ldots\ldots \in M^*_+[x] \; ;$$

(38) $$\hat{\Psi} = \hat{G}^{(1)} + \hat{G}^{(2)}x + \hat{G}^{(3)}x^2 + \ldots\ldots \in \hat{M}^*_+[x] \; ,$$

We have from (35) that $\hat{\Psi} = \hat{G}^{(1)} + x \, \hat{\Pi}^*(\hat{\Psi} \hat{Q})$ which yields:

(39) $$\hat{\Pi}^* \{\hat{\Psi}(I-x \hat{Q}) - \hat{G}^{(1)}\} = 0; \text{ and}$$

(40) $$\tau = \hat{\Psi}(I - x \hat{Q}) - \hat{G}^{(1)} \in \hat{M}^*_-[x].$$

Under the assumption $EU_1 < 0$, theorem 3 of Miller [8] with the help of assumptions (A_1), (A_2) implies that $\lambda(t)$ attains the unique minimum at $\tau^* \in (c', 0)$. Thus, for each $0 < x < [\lambda(t^*)]^{-1}$, there are two real roots of

(41) $$x\,\lambda(t) = 1$$

in $[\tilde{t},0]$, say $\tau_0(x) < \tau_1(x)$, where $\lambda(\tilde{t}) = \lambda(0)=1$.

Next, we note that (40) implies that $\hat{\Psi}(I-x\,Q)-\hat{G}^{(1)}$ can be continued analytically into $\text{Re } t < 0$. Therefore, the matrix factorization theorem of Miller ([9], p.277) gives:

(42) $$I-x\,Q = B_+(x,t)B_-(x,t) \quad (0 < x < [\lambda(t^*)]^{-1}).$$

Where

(43) $B_+(x,t)$, $B_+^{-1}(x,t)$ are analytic and bounded in $\text{Re } t \geq \tau_0(x)+\epsilon$;
$B_-(x,t)$, $B_-^{-1}(x,t)$ are analytic and bounded in $\text{Re } t \leq \tau_1(x)-\epsilon$;
$\epsilon > 0$ is arbitrarily small such that $\tau_0+\epsilon < \tau_1 - \epsilon$.

We are ready to prove the main theorem concerning the generating function of the L.S.T. of the distribution function of the joint process $\{w_n, J_{n-1}\}$.

<u>Theorem 2</u>.

Under the assumption $EU_1 < 0$, for each $0 < x < [\lambda(t^*)]^{-1}$,

we have that:

(44) $$\hat{\Psi} = [\hat{\pi}^* (\hat{G}^{(1)} B_-^{-1})]\, B_+^{-1} .$$

Where B_+, B_- were given by (42).

Proof

By using (39), (40) and (42), we find that:

$$(45)\qquad \tau = \hat{\Psi}\, B_+\, B_- - \hat{G}^{(1)} \in \hat{M}^*_-[x] \;;$$

$$(46)\qquad \hat{\Psi}\, B_+ - \hat{G}^{(1)}\, B_-^{-1} \in \hat{M}^*_-[x].$$

(For details, see [1] or P.273 of [9]). Moreover,

$$(47)\qquad \hat{\Psi}\, B_+ \in \hat{M}^*_+[x] \;,$$

hence (46) yields that

$$(48)\qquad \hat{\Psi}\, B_+ = \hat{\pi}^* \left(\hat{G}^{(1)}\, B_-^{-1}\right)$$

which agrees with (44). Thus theorem 2 is proved.

4. On the solution of the stationary waiting time distributions.

The purpose of this section is to obtain the L.S.T.'s of $\left\{ \lim_{n\to\infty} G_{ij}^{(n)}(x)\right\}$ (if they exist) by the technique of Wiener-Hopf factorization due to Smith [13]. In order to carry out our objective, we make four additional assumptions which are somehow weaker than those assumed in [13].

(A_1'): The same as (A_1).

(A_2'): The same as (A_2).

(A_3'): $\hat{Q}(t) = O(|t_2'|^{-1})$ as $|t_2'|$ is sufficiently large, where $t = t_1' + i\, t_2'$ and $c' < t_1' \leq 0$.

We remark that a sufficient condition to guarantee (A_3') follows from lemma 1 of [13].

(A_4'): For each $i=1,2,\ldots,M$, $Q_{ij}(x)$ is absolutely continuous for some j. More generally, it will suffice to suppose that for each i, either

$$P\{s_n \leq x,\ J_n=j \mid J_{n-1}=i\} \text{ or } P\{t_n \leq x,\ J_n=j \mid J_{n-1}=i\}$$

is absolutely continuous for some j.

First of all, we consider the problem of existences of $\left\{ \lim_{n\to\infty} G_{ij}^{(n)}(x) \right\}$. Let, for $i,j=1,2,\ldots,M$,

(49) $$\limsup_{n\to\infty} G_{ij}^{(n)}(x) = \lambda_{ij}(x) ;$$

(50) $$\liminf_{n\to\infty} G_{ij}^{(n)}(x) = \mu_{ij}(x) .$$

Under the usual assumption:

(51) $$EU_1 = \sum_{j=1}^{M} P_j \zeta_j < 0 ,$$

We come to the following:

Theorem 3.

The condition (52) below is a necessary and sufficient condition for the existence of $\left\{ \lim_{n\to\infty} G_{ij}^{(n)}(x) \right\}$.

Condition (52): There exist two subsequences $n_k = n_k(x)$, $m_k = m_k(x)$ such that $\lim_{k\to\infty} G_{ij}^{(n_k)}(x) = \lambda_{ij}(x)$ and $\lim_{k\to\infty} G_{ij}^{(m_k)}(x) = \mu_{ij}(x)$ for all $i,j=1,2,\ldots,M$.

Proof:

That the existences of $\{\lim_{n\to\infty} G_{ij}^{(n)}(x)\}$ imply (52) is obvious. Now, we assume that (52) is satisfied. The assumption (51) ensures that W_n tends to an a.e. finite random variable w in distribution regardless of the initial condition on w_1. Thus:

$$(52') \sum_{j=1}^{M} \sum_{i=1}^{M} P_i \lambda_{ij}(x) = \lim_{k\to\infty} P\{W_{n_k} \leq x | w_1 = a\} = P\{w \leq x\}$$

$$= \lim_{k\to\infty} P\{W_{m_k} \leq x | w_1 = a\} = \sum_{j=1}^{M} \sum_{i=1}^{M} P_i \mu_{ij}(x).$$

Since $\lambda_{ij}(x) \geq \mu_{ij}(x)$ and $P_i > 0$ for all i, we deduce from (52') that $\lambda_{ij}(x) = \mu_{ij}(x)$ for all i,j. Whence $\lim_{n\to\infty} G_{ij}^{(n)}(x) = G_{ij}(x)$ exists for all i,j.

From now on, we suppose that $\lim_{n\to\infty} G_{ij}^{(n)}(x) = G_{ij}(x)$ exists for all i,j. Then, as $n \to \infty$, (33) can be written as:

$$(53) \qquad G_{ij} = \sum_{k=1}^{M} \pi(G_{ik} * Q_{kj}) \qquad (i,j=1,2,\ldots,M).$$

In matrix notation, we find that:

$$(54) \qquad G = \pi^{*}(G * Q) \; ;$$

$$(55) \qquad \hat{G}(t) = \hat{\Pi}^{*}(\hat{G}^{(t)} \hat{Q}(t)) \qquad (\text{Re } t = 0),$$

which yields:

(56) $\hat{\Pi}^{*}[\hat{G}(I-\hat{Q})] = 0$;

(57) $\hat{G}(I-\hat{Q}) \in \hat{M}_{-}^{*}$ with $\hat{M}_{-}^{*} \equiv \hat{M}_{-}^{*}$ [1].

Next, we note that (57) implies that $\hat{G}(I-\hat{Q})$ can be continued analytically into $\mathrm{Re}\ t < 0$.

With the support of (51), (A_1') and (A_2'), the matrix factorization theorem of Miller gives:

(58) $$I - \hat{Q}(t) = B_{+}(1,t)\, B_{-}(1,t) ,$$

Where $B_{+}(1,t)$ and $B_{-}(1,t)$ have the properties described in (43), that is, $B_{+}(1,t)$ and $B_{+}^{-1}(1,t)$ are analytic and bounded in $\mathrm{Re}\ t \geq \tilde{t} + \epsilon$; $B_{-}(1,t)$ and $B_{-}^{-1}(1,t)$ are analytic in $\mathrm{Re}\ t < 0$ and bounded in $\mathrm{Re}\ t \leq -\epsilon$; $\epsilon > 0$ is arbitrarily small so that $\tilde{t} + \epsilon < -\epsilon$.

Theorem 4.

The matrix of the L.S.T.'s of the stationary waiting time distributions of the joint process $\{W_n, J_{n-1}\}$ is independent of the initial condition on (W_1, J_0) and is given by:

(59) $$\hat{G}(t) = P^{*}\, B_{+}(1,0)\, B_{+}^{-1}(1,t).$$

Moreover, $B_{+}(1,0)B_{+}^{-1}(1,t)$ is independent of the factorization of (58) in the sense that if $I - \hat{Q}(t) = C_{+}(1,t)\, C_{-}(1,t)$, then $B_{+}(1,0)B_{+}^{-1}(1,t) = C_{+}(1,0)\, C_{+}^{-1}(1,t)$. Where P^{*} is the matrix of stationary probabilities corresponding to the stochastic matrix P, that is,

(60) $$P^* = \begin{pmatrix} P_1 & P_2 & \cdots & P_M \\ P_1 & P_2 & \cdots & P_M \\ \vdots & \vdots & & \vdots \\ P_1 & P_2 & \cdots & P_M \end{pmatrix}.$$

__Proof:__

Evidently (A_3') implies that $I-\hat{Q}(t)$ is non-singular for sufficiently large $|t_2'|$ and $\tilde{t} < t_1' \leq 0$, where $t = t_1' + i\, t_2'$.

By the condition (A_4') and Wintner ([14], p.14), we have that for each $i=1,2,\ldots,M$, there exists some j such that:

(61) $$|\hat{Q}_{ij}(t)| < P_{ij} \quad \text{for} \quad \text{Re}\, t = 0 \quad \text{and} \quad t \neq 0.$$

Consequently,

(62) $$||\hat{Q}(t)|| < I \quad (\text{for} \quad \text{Re}\, t = 0 \quad \text{and} \quad t \neq 0),$$

where the matrix norm $|| \ \ ||$ is defined by

(63) $$||A|| = \max_{1 \leq i \leq M} \left\{ \sum_{j=1}^{M} |A_{ij}| \right\}$$

for any $M \times M$ matrix $A = (A_{ij})$.

In view of the inequality (62) just proved, theorems 2 and 4 of Faddeeva ([5], p.61) yield that the inverse of $I - \hat{Q}(t)$ exists for $\text{Re}\, t = 0$ and $t \neq 0$. Whence $B_-^{-1}(1,t)$ exists for $\text{Re}\, t = 0$ and $t \neq 0$. So we conclude that

(64) $$B_-^{-1}(1,t) \text{ exists for } \text{Re}\, t \leq 0,\ t \neq 0.$$

Next, we define

$$K(t) = \hat{G}(t)\, B_+(1,t) \quad \text{for} \quad \mathrm{Re}\, t = 0. \tag{65}$$

Clearly $K(t)$ has a bounded analytic continuation into the half-plane $\mathrm{Re}\, t > 0$ and $K(t)$ is continuous in $\mathrm{Re}\, t \geq 0$. Also define another function:

$$(66) \qquad H(t) = \hat{G}(t)\, [I-\hat{Q}(t)]\, B_-^{-1}(1,t) \qquad \text{if} \quad \mathrm{Re}\, t = 0, t \neq 0$$

$$= \hat{G}(0)\, B_+\,(1,0) \qquad \text{if } t = 0.$$

Since $\hat{G}(I-\hat{Q}) \in \hat{M}_-^*$, $H(t)$ has an analytic continuation in the plane $\mathrm{Re}\, t < 0$ and bounded in $\mathrm{Re}\, t \leq -\epsilon$ for any $\epsilon > 0$ (cf. [6], p.303). Furthermore, by means of (A_3') and (64), $H(t)$ is continuous in $\mathrm{Re}\, t \leq 0$ (In fact, $t = 0$ is a removable singularity of $H(t)$).

Taking into consideration that $H(t) = K(t)$ for $\mathrm{Re}\, t = 0$, by the principle of analytic continuation, we find that $H(t)$ or $K(t)$ can be continued to be an analytic function over the whole complex plane.

Moreover, (A_3') implies that $H(t)$ is bounded for large $|t_2'|$ and $\tilde{t} < t_1' \leq 0$. Applying the maximum principle, we conclude that $H(t)$ is bounded in $\mathrm{Re}\, t \leq 0$.

Finally, Liouville's theorem gives:

$$H(t) = H(0) = K(0) = \text{constant}. \tag{67}$$

Whence

(68) $$\hat{G}(t)\, B_+(1,t) = K(0) = \hat{G}(0)\, B_+\,(1,0) = P^*\, B_+\,(1,0)\ ,$$

which is the stated result (59).

Next, if $I - \hat{Q}(t) = C_+(1,t)\, C_-(1,t)$ and $C_+(1,t)$, $C_-(1,t)$ have those properties enunciated in (43), then from [9], we have:

(69) $$B_+(1,t) = C_+(1,t)\, D\ ,$$

where D is a non-singular matrix independent of t. Hence:

(70) $$B_+(1,0)B_+^{-1}(1,t) = C_+(1,0)\, D\, D^{-1}\, C_+^{-1}(1,t) = C_+(1,0)\, C_+^{-1}(1,t)$$

which completes the proof of theorem 4.

<u>Remark</u>: Theorem 4 provides the unique solution of the matrix equation $f = \hat{\Pi}^*(f\, \hat{Q})$, if $\hat{Q}$ satisfies (A_2'), (A_3') and (A_4').

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