The Infinite Server Queue With Poisson Arrivals and Semi-Markovian Services*

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Abstract

The queue with an infinite number of servers with a Poisson arrival process and with semi-Markovian service times is considered. The queue length process and the type of the first customer to join the queue after t are studied jointly and we obtain the transient and asymptotic results which are of matrix extensions of the corresponding results of the $M/G/\infty$ queue. In particular, we prove that the limiting distribution of the queue length process is Poisson.

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I. Introduction

The infinite server queue with semi-Markovian service times has potential applicability in modelling a system with a sufficiently large number of servers in which service times depend on the types of customers themselves.

Moreover, as the natural generalization to "the matrix case" of the M/G/∞ queue, this model is also of independent theoretical interest.

We consider a queueing model in which the n^{th} customer C_n is of type J_{n-1} and arrives at a service counter in the instant $T_n (0 = T_1 < T_2 < T_3 < \dots < T_n < \dots)$. $t = 0$ is taken as an arrival instant. There are M customer types. There are infinitely servers, which is equivalent to saying that each customer starts being served as soon as he arrives. For $n \geq 1$, let $t_n = T_{n+1} - T_n$ be the interarrival time between C_n and C_{n+1}. It is assumed that $t_1, t_2, \dots, t_n, \dots$ are independent, identically distributed positive random variables with common distribution:

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$$(1) \qquad P\{x_n \le x\} = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases} ;$$

and that the sequence $\{t_n\}$ is independent of the service process $\{S_n, J_n\}$, where S_n is the service time of C_n. Our basic assumption for the service process is that the pairs $\{(S_n, J_n), n \ge 0\}$ form a semi-Markov sequence:

$$(2) \qquad P\{S_n \le x, J_n = j \,|\, (S_k, J_k, k \le n-2); S_{n-1}, J_{n-1} = i\}$$

$$= P\{S_n \le x, J_n = j \,|\, J_{n-1} = i\}$$

$$= Q_{ij}(x)$$

for $n = 1,2,\ldots;\ i,j = 1,2,\ldots, M;\ M < \infty$; and $S_0 = 0$.

We further assume that the underlying Markov chain $\{J_n, n \ge 0\}$ is irreducible. For the standard definitions and properties of Markov renewal and semi-Markov processes, we refer to Pyke [4, 5]. Suppose that at time $t = 0$, there are k $(k \ge 1)$ initial customers. Denoting by $\xi(t)$ the queue length at time $t + 0$. In particular, $\xi(0) = k$. The customer arriving at $t = 0$ is therefore counted among the k.

Let us order the k initial customers according to their arrivals and let $J(0)$ be the type of the first customer among k; $J(t)$ is the type of the first customer to join the queue after time t $(t > 0)$.

The transient and asymptotic behavior of the joint process $(\xi(t), J(t))$ is discussed in the subsequent sections.

2. The transient behavior of the queuelength process $\xi(t)$.

For brevity, set:

$$(3) \quad P_{ij}(k; m, t) = P\{\xi(t) = m, J(t) = j \mid \xi(0) = k, J(0) = i\} \quad (m \geq 0),$$

for $i, j = 1,2,\ldots, M$. It is more convenient to introduce the generating function:

$$(4) \quad \tilde{P}_{ij}(k; Z,t) = \sum_{m=0}^{\infty} P_{ij}(k; m,t)\, Z^m \quad |Z| \leq 1 .$$

In matrix notation with $\tilde{P}(k; Z,t) = (\tilde{P}_{ij}(k; Z,t))$, $Q(x) = (Q_{ij}(x))$, etc., by using the method of collective marks (Runnenberg [6]). We have the following theorem about the transient behavior of the joint process $(\xi(t), J(t))$.

Theorem 1.

The matrix generating function in (4) is given by

$$(5) \quad \tilde{P}(k;Z,t)=[Q(t)+Z(P-Q(t)]^k \exp\left\{-\lambda \int_0^t [I-Q(x)-Z(P-Q(x)]dx\right\} .$$

where I is the M x M indentity matrix and P is the transition matrix of the Markov chain $\{J_n\ n \geq 0\}$, that is,

$$P_{ij} = P\{J_n = j \mid J_{n-1} = i\} \ ; \ P = (P_{ij}).$$

Proof.

First of all, assume that $0 \leq Z \leq 1$. Let each customer be independently marked with probability $1 - Z$. Then the generating function (4) means the probability that there are no marked customers present at time t and the type of the first customer to join the queue after time t is j, given that $\xi(0) = k$ and $J(0) = i$.

Suppose that $\ell \geq 0$ customers arrive in the time interval $(0\ t]$, by the Markov property of Poisson input, the ℓ arrival points are independently uniformly distributed in the interval $(0\ t]$.

Let us denote the successive types of k initial customers by $i, i_1, i_2, \ldots, i_{k-1}$ and the successive types of the ℓ new arrivals by $i_k, i_{k+1}, \ldots, i_{k+\ell-1}$. Then we get:

(5')
$$\tilde{P}_{ij}(k;Z,t) = \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \cdots \sum_{i_{k-1}=1}^{M} \sum_{i_k=1}^{M} \cdots \sum_{i_{k+\ell-1}=1}^{M} \sum_{\ell=0}^{\infty} [Q_{ii_1}^{(t)} + Z(P_{ii_1} - Q_{ii_1}^{(t)})] \cdot$$
$$\cdot [Q_{i_1 i_2}^{(t)} + Z(P_{i_1 i_2} - Q_{i_1 i_2}^{(t)})] \cdots\cdots [Q_{i_{k-1} i_k}^{(t)} + Z(P_{i_{k-1} i_k} - Q_{i_{k-1} i_k}^{(t)})] \cdot$$
$$\cdot e^{-\lambda t} \frac{(\lambda t)^\ell}{\ell!} \Big\{ \ell! \int_0^t \frac{du_1}{t} [Q_{i_k i_{k+1}}^{(t-u_1)} + Z(P_{i_k i_{k+1}} - Q_{i_k i_{k+1}}^{(t-u_1)})] \cdot$$
$$\cdot \int_{u_1}^t \frac{du_2}{t} [Q_{i_{k+1} i_{k+2}}^{(t-u_2)} + Z(P_{i_{k+1} i_{k+2}} - Q_{i_{k+1} i_{k+2}}^{(t-u_2)})] \cdots\cdots\cdots$$
$$\cdot \int_{u_{\ell-1}}^t \frac{du_\ell}{t} [Q_{i_{k+\ell-1}}^{(t-u_\ell)} + Z(P_{i_{k+\ell-1} j} - Q_{i_{k+\ell-1} j}^{(t-u_\ell)})] \Big\} .$$

In matrix notation, we can write that

$$\tilde{P}(k;\ Z,t) = [Q(t) + Z(P-Q(t))]^k \sum_{\ell=0}^{\infty} e^{-\lambda t}\,\lambda^{\ell} \int_0^t \int_{u_1}^t \cdots \int_{u_{\ell-1}}^t$$

$$[Q(t-u_1)+Z(P-Q(t-u_1))] \,.\, [Q(t-u_2) + Z(P-Q(t-u_2))] \,.\, \cdots\cdots$$

$$.[Q(t-u_\ell) + Z(P-Q(t-u_\ell))]\ du_\ell\ du_{\ell-1} \cdots du_1.$$

We now prove by induction that

$$(6)\quad \int_0^t \int_{u_1}^t \cdots \int_{u_{\ell-1}}^t [Q(t-u_1)+Z(P-Q(t-u_1))] \cdots\cdots [Q(t-u_\ell)+Z(P-Q(t-u_\ell))]$$

$$du_\ell\ du_{\ell-1} \cdots du_1$$

$$= \frac{1}{\ell!} \left\{ \int_0^t [Q(x) + Z(P-Q(x))]\ dx \right\}^{\ell} \quad (\ell \geq 0)$$

First, it is clear that the equality holds for $\ell = 0,1$ (the left hand side is defined as I for $\ell = 0$). Next, assume that the equality is true for $\ell = n-1$. Let us fix the order of $0 < u_1 < u_2 < \ldots < u_{n-1}$, next, we vary the position of u_n from the last to the first, then we have:

$$\int_0^t \int_{u_1}^t \cdots \int_{u_{n-1}}^t F(t-u_1)\ F(t-u_2) \ldots F(t-u_n) du_n\ du_{n-1} \ldots du_1 =$$

$$= \int_0^t \int_{u_1}^t \cdots \int_{u_{n-2}}^t \int_{u_{n-2}}^{u_{n-1}} F(t-u_1)\ F(t-u_2) \ldots F(t-u_n) du_n\ du_{n-1} \ldots du_1$$

$$= \int_0^t \int_{u_1}^t \cdots\cdots \int_{u_{n-2}}^t \int_{u_{n-3}}^{u_{n-2}} F(t-u_1)\, F(t-u_2)\, \ldots\, F(t-u_n)\, du_n\, du_{n-1} \ldots du_1$$

$$= \ . \ . \ .$$

$$= \int_0^t \int_{u_1}^t \cdots\cdots \int_{u_{n-2}}^t \int_{u_1}^{u_2} F(t-u_1)\, F(t-u_2)\, \ldots\, F(t-u_n)\, du_n\, dy_{n-1} \ldots du ,$$

$$= \int_0^t \int_{u_1}^t \cdots \int_{u_{n-2}}^t \int_0^{u_1} F(t-u_1)\, F(t-u_2) \ldots F(t-u_n)\, du_n\, du_{n-1} \ldots du,$$

where

$$(6') \qquad F(t-x) = Q(t-x) + Z(P-Q(t-x)).$$

Therefore,

$$\int_0^t \int_{u_1}^t \cdots\cdots \int_{u_{n-1}}^t F(t-u_1)\, F(t-u_2)\, \ldots\ldots\, F(t-u_n)\, du_n\, du_{n-1} \ldots du,$$

$$= \frac{1}{n} \int_0^t \int_{u_1}^t \cdots\cdots \int_{u_{n-2}}^t \int_0^t F(t-u_1) \ldots\ldots F(t-u_{n-1})\, F(t-u_n)\, du_n\, du_{n-1} \ldots du_1$$

$$= \frac{1}{n} \left[\int_0^t \int_{u_1}^t \cdots \int_{u_{n-2}}^t F(t-u_1) \ldots\ldots F(t-u_{n-1})\, du_{n-1} \ldots du_1 \right] \cdot \int_0^t F(x)\, dx$$

$$= \frac{1}{n} \cdot \frac{1}{(n-1)!} \left\{ \int_0^t F(x)dx \right\}^{n-1} \int_0^t F(x)dx$$

$$= \frac{1}{n!} \left\{ \int_0^t F(x)dx \right\}^n .$$

Thus (6) follows. Therefor (5') becomes:

$$\tilde{P}(k;\, Z,t) = [Q(t) + Z(P-Q(t))]^k \sum_{\ell=0}^{\infty} e^{-\lambda t} \lambda^{\ell} \frac{1}{\ell!} \left\{ \int_0^t [Q(x)+Z(P-Q(x))]dx \right\}^{\ell}$$

$$= [Q(t)+Z(P-Q(t))]^k \, e^{-\lambda t} \exp\left\{\lambda \int_0^t [Q(x)+Z(P-Q(x))]dx\right\}$$

$$= [Q(t) + Z(P-Q(t))]^k \exp\left\{-\lambda \int_0^t [I-Q(x)-Z(P-Q(x))]dx\right\} ,$$

which implies (5) for the case $0 \leq Z \leq 1$. By the principle of analytic continuation, (5) is valid for $|Z| \leq 1$. Thus we have derived the desired result.

3. The asymptotic behavior of the queuelength process $\xi(t)$.

In this section, we study the limiting behavior of the joint process $(\xi(t), J(t))$, that is, we want to find the limit $\lim_{t\to\infty} \tilde{P}(k;\, Z,t)$, particularly, we find that the limiting distribution of $\xi(t)$ is Poisson.

For simplicity, we set:

$$(7) \quad F(Z;t) = \exp\left\{- \lambda \int_0^t [I-Q(x)-Z(P-Q(x))]\, dx \right\} .$$

So (5) can be written as:

$$(8) \qquad \tilde{P}(k;\, Z,t) = [Q(t) + Z(P-Q(t))]^k \, F(Z;\, t).$$

Let the matrix of the stationary probabilities (or the Cesaro limit) corresponding to the stochastic matrix P be

$$(9)\qquad P^* = \begin{pmatrix} P_1 & P_2 & \cdots & P_M \\ P_1 & P_2 & \cdots & P_M \\ \vdots & \vdots & & \vdots \\ P_1 & P_2 & \cdots & P_M \end{pmatrix};$$

that is,

$$(10)\qquad P^* P = P^* = PP^*. \quad \text{(See [2], p. 34)}$$

Assume that the mean service time of each type is finite, that is,

$$(11)\qquad \mu_i = \sum_{j=1}^{M} \mu_{ij} \equiv \sum_{j=1}^{M} \int_0^\infty x \, d\, Q_{ij}(x) < \infty.$$

We now state the main theorem and prove it after the following three lemmas.

<u>Theorem 2.</u>

The limit $\lim_{t\to\infty} \tilde{P}(k; Z,t) = \tilde{P}(Z)$ exists and is independent of the initial conditions on $(\xi(0), J(0))$ and we have:

$$(12)\qquad \tilde{P}(Z) = e^{-(1-Z)\lambda \sum_{i=1}^{M} P_i \mu_i} P^* .$$

Where P^* and $\{\mu_i\}$ are given by (10) and (11) respectively.

<u>Lemma 1</u>.

Let the eigen values of the stochastic matrix P be $1, \lambda_2, \lambda_3, \ldots, \lambda_M$, then the eigen value 1 is simple and $|\lambda_i| \leq 1$ for $i = 2,3,\ldots M$. Therefore, all eigen values of $P-I$ are $0, \rho_2, \rho_3, \ldots, \rho_M$ such that 0 is a simple eigen value and $\operatorname{Re} \rho_i < 0$ for $i = 2,3,\ldots,M$. Where $\rho_i = \lambda_i - 1$.

Proof.

See Debreu, G. and Herstein, I. N. [3].

Let $i_1, i_2, \ldots, i_\ell$ be the sequence of non-negative integers such that:

(13) $$1 = \lambda_{i_1}; \ \lambda_2 = \lambda_{i_1+1} = \lambda_{i_1+2} = \ldots = \lambda_{i_1+i_2}; \ \lambda_{i_1+i_2+1} = \ldots$$
$$= \lambda_{i_1+i_2+i_3}; \ \ldots\ldots ; \ \lambda_{i_1+i_2+\ldots+i_{\ell-1}+1} = \ldots\ldots = \lambda_{i_1+i_2+\ldots+i_\ell} = \lambda_M .$$

Lemma 2.

Let P be similar to its Jordan canonical form:

(14) $$P = U\,J\,U^{-1} ,$$

where

$$J = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & J_2 \end{matrix} & O \\ O & J_\ell \end{pmatrix}$$

and

$$J_h = \begin{pmatrix} \lambda_{i_h} & & \epsilon_{12}^{(h)} & & O \\ & \lambda_{i_h} & & \epsilon_{23}^{(h)} & \\ O & & \ddots \lambda_{i_h} & & \ddots \epsilon_{i_{h-1} i_h}^{(h)} \end{pmatrix} \qquad (h=2,3,\ldots,\ell)$$

is a $(i_h \times i_h)$ matrix with the property that:

(15) $$\epsilon_{i\ i+1}^{(h)} = 0 \text{ or } 1 \qquad (i = 1,2,\ldots,i_h-1;\ h = 2,3,\ldots,\ell).$$

Then we have, for any complex number x,

(16) $$e^{(P - I)x} = U J^{*} U^{-1} ,$$

where

(17) $$J^{*} = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & J_2^{*} & & \\ & & \ddots & \\ 0 & & & J_\ell^{*} \end{pmatrix}$$

and

(18) $$J_h^{*} = \begin{pmatrix} e^{\rho_{i_h} x} & x\,\epsilon_{12}^{(h)} e^{\rho_{i_h} x} & & & 0 \\ & e^{\rho_{i_h} x} & x\epsilon_{23}^{(h)} e^{\rho_{i_h} x} & & \\ & & \ddots & \ddots & \\ & & & \ddots & x\epsilon_{i_h-1\, i_h}^{(h)} e^{\rho_{i_h} x} \\ 0 & & & & e^{\rho_{i_h} x} \end{pmatrix}$$

$(h = 2,3,\ldots,\ell)$.

where $\{\rho_{i_h}\}$ are eigen values of $P - I$. Moreover,

(19) $$\lim_{x\to\infty} e^{(P-I)x} = U I^{*} U^{-1} ,$$

where

(20) $I^{*} = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}$ is the matrix whose components are zeros except the 1^{st} row, 1^{st} column component whose value is 1.

<u>Proof:</u>

First note that:

(21) $$P - I = U(J - I)\, U^{-1} \quad \text{and}$$

(22) $$e^{(P-I)x} = \sum_{n=0}^{\infty} \frac{1}{n!} (P - I)^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} U(J - I)^n U^{-1} x^n .$$

One can easily prove by induction that:

(23) $$(J - I)^n = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & J_2^{(n)} \end{matrix} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & J_\ell^{(n)} \end{pmatrix} \quad (n \geq 1)$$

and

(24) $$J_h^{(n)} = \begin{pmatrix} e^n_{i_h} & n\,\epsilon^{(h)}_{12}\,\rho^{n-1}_{i_h} & & & \mathbf{0} \\ & \rho^n_{i_h} & n\epsilon^{(h)}_{23}\;\rho^{n-1}_{i_h} & & \\ \mathbf{0} & & \ddots & \ddots & \\ & & & \ddots & h\epsilon^{(h)}_{i_h-1\; i_h}\;\rho^{n-1}_{i_h} \\ & & & & \rho^n_{i_h} \end{pmatrix} \quad (h=2,3,\ldots,\ell)$$

and

(25) $$(J - I)^0 = I.$$

Therefore, (22), (23), (24) and (25) imply (16), (17) and (18). Furthermore, noting that $\mathrm{Re}\ \rho_i < 0$ for $i = 2,3,\ldots,M$, we obtain (19) from (16), (17) and (18). Thus lemma 2 is proved.

Lemma 3.

We have:

$$P^* = U\, I^*\, U^{-1} , \tag{26}$$

Where U and I^* are given by (14) and (20) respectively.

Proof:

(14) implies that the first column of U is a right eigen vector of the eigen value 1 of the stochastic matrix P, whence all components $u_{11}, u_{21},\ldots,u_{M\,1}$ of the first column of U must be equal, so, without loss of generality, we may assume that $u_{11}=u_{21}=\ldots=u_{M\,1}=1$. On the other hand, the first row of U^{-1} is a left eigen vector corresponding to the eigen value 1 of P, from (10) and $U^{-1}U = I$ then the first row of U^{-1} must be $(P_1, P_2, \ldots, P_M)$. Combining the above two properties just proved, we obtain (26) which completes the proof of lemma 3.

Proof of theorem 2:

Upon differentiating $F(Z;t)$ in (7) with respect to Z, we find that:

$$(27)\quad F'(Z;t) = \lim_{\Delta Z\to 0} \frac{1}{\Delta Z} [\, F(z + \Delta Z;\, t) - F(Z;\, t)]$$

$$= \lim_{\Delta Z\to 0} \frac{1}{\Delta Z} \left\{ \exp\left\{-\lambda \int_0^t [I - Q(x) - Z(P-Q(x))]dx \; + \right.\right.$$

$$\left.\left. + \lambda\, \Delta\, Z \int_0^t [P-Q(x)]dx\right\} - \exp\left\{-\lambda \int_0^t [I - Q(x)-Z(P-Q(x))]dx\right\}\right\}$$

$$= \lim_{\Delta Z\to 0} \frac{1}{\Delta Z} \{\, e^{A+\Delta Z B} - e^{A}\} ,$$

where

(28) $$A = -\lambda \int_0^t [I - Q(x) - Z(P-Q(x))]dx \; ;$$

(29) $$B = \lambda \int_0^t [P-Q(x)]dx.$$

Applying Perturbation theory (Bellman [1], p. 171) we can write that:

(30) $$F'(Z;t) = \lim_{\Delta Z \to 0} \frac{1}{\Delta Z} \left\{ e^{A} + \Delta Z \int_0^1 e^{A(1-s)} B e^{AS} ds + O((\Delta Z)^2) - e^{A} \right\}$$

$$= \int_0^1 e^{A(1-s)} B e^{As} ds.$$

Setting $Z = 1$ and noting that $A = \lambda(P-I)t$ and $B = \int_0^t (P-Q(x))dx$, then we obtain from (8) and (30) the asymptotic expected queuelength:

(31) $$\lim_{t\to\infty} \tilde{P}'(k;1,t) \equiv \tilde{P}'(k;1)$$

$$= \lim_{t\to\infty} (E(\xi(t), J(t) = j \,|\, \xi(0)=k, J(0)=i))$$

$$= P^k \lim_{t\to\infty} \int_0^1 e^{\lambda(P-I)t(1-s)} \lambda \int_0^\infty [P-Q(x)]dx \; e^{\lambda(P-I)ts} ds \; .$$

Now lemmas 2 and 3 imply that:

(32) $$\lim_{t\to\infty} e^{\lambda(P-I)t(1-s)} = U I^* U^{-1} = P^* = \lim_{t\to\infty} e^{\lambda(P-I)ts}.$$

Therefore, we have from (31) and (10) that:

$$(33)\qquad \tilde{P}'(k;1) = P^k P^* \lambda \mu P^* = P^* \lambda \mu P^*$$

$$= \lambda \begin{pmatrix} P_1 & P_2 & \cdots & P_M \\ P_1 & P_2 & \cdots & P_M \\ P_1 & P_2 & \cdots & P_M \end{pmatrix} \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1M} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2M} \\ - & - & - & - \\ \mu_{M1} & \mu_{M2} & \cdots & \mu_{MM} \end{pmatrix} \begin{pmatrix} P_1 & P_2 & \cdots & P_M \\ P_1 & P_2 & \cdots & P_M \\ \vdots & \vdots & & \vdots \\ P_1 & P_2 & \cdots & P_M \end{pmatrix}$$

$$= \lambda \begin{pmatrix} P_1 & P_2 & \cdots & P_M \\ \vdots & \vdots & & \vdots \\ P_1 & P_2 & & P_M \end{pmatrix} \begin{pmatrix} \mu_1 P_1 & \mu_1 P_2 & \cdots & \mu_1 P_M \\ \mu_2 P_1 & \mu_2 P_2 & \cdots & \mu_2 P_M \\ \vdots & \vdots & & \vdots \\ \mu_M P_1 & \mu_M P_2 & \cdots & \mu_M P_M \end{pmatrix}$$

$$= \lambda \left(\sum_{i=1}^{M} P_i \mu_i \right) P^* .$$

Next, we want to determine the second moment by the same technique. Upon differentiating (30) with respect to z, we find that:

$$(34)\qquad F''(z;t) = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left\{ F'(z+\Delta z;t) - F'(z;t) \right\}$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left\{ \int_0^1 \left\{ e^{(1-s)A+\Delta z(1-s)B}\, B\, e^{sA+\Delta z sB} - \right.\right.$$

$$\left.\left. - e^{(1-s)A}\, B\, e^{sA} \right\} ds \right\}$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left\{ \int_0^1 \left\{ \left[e^{(1-s)A} + \Delta z \int_0^1 e^{(1-s)A(1-\nu)} (1-s)B\, e^{(1-s)A\nu}\, d\nu \right.\right.\right. +$$

$$\left. + O(\Delta z)^2) \right] . B . \left[e^{sZ} + \Delta z \int_0^1 e^{sA(1-\nu)} sBe^{sA\nu}\, d\nu + O((\Delta z)^2) \right]$$

$$\left.\left. - e^{(1-s)A}\, B\, e^{sA} \right\} ds \right\}$$

$$= \int_0^1 \left\{ \int_0^1 e^{(1-s)A(1-\nu)} (1-s)\ B\ e^{(1-s)A\nu}\ d\nu\ .\ B\ .\ e^{sA} + \right.$$

$$\left. + e^{(1-s)A} .\ B\ .\ \int_0^1 e^{sA(1-\nu)}\ s\ B\ e^{sA\nu}\ d\nu \right\} ds.$$

Setting $z = 1$ and letting $t \to +\infty$ and applying lemmas 2 and 3, we obtain that:

$$(35)\quad \tilde{P}''(k;1) \equiv \lim_{t\to+\infty} \tilde{P}''(k;1,t)$$

$$= P^k \lim_{t\to+\infty} F''(1;t)$$

$$= P^k \int_0^1 \left\{ \int_0^1 P^*(1-s)\ \lambda\ \mu\ P^*\ d\nu\ .\ \lambda\mu\ .\ P^* + \right.$$

$$\left. + P^*\ \lambda\ \mu \int_0^1 P^*\ s\ \lambda\ \mu\ P^*\ d\nu \right\} ds$$

$$= P^k\ P^*\ \lambda\ \mu\ P^*\ \lambda\ \mu\ P^*$$

$$= P^*(\lambda\ \mu\ P^*)^2\ .$$

From (33), we note that:

$$(36)\qquad P^*\ \lambda\ \mu\ P^* = \left(\lambda \sum_{i=1}^{M} P_i\ \mu_i \right) P^*.$$

So (35) implies that

$$(37)\qquad \tilde{P}''(k;1) = \left(\lambda \sum_{i=1}^{M} P_i\ \mu_i \right)^2 P^*\ .$$

In the same manner, by differentiating (34) with respect to z and then setting z = 1, it is not difficult to establish the following:

$$\lim_{t\to\infty} F^{(n)}(1;t) = P^*(\lambda\ \mu\ P^*)^n \quad (n \geq 1)\ ;\ \text{and} \tag{38}$$

$$\lim_{t\to\infty} \tilde{P}^{(n)}(k;1,t) = P^k\, P^*(\lambda\ \mu\ P^*)^n \tag{39}$$

$$= (\lambda \sum_{i=1}^{M} P_i\, \mu_i)^n\, P^* \qquad (n \geq 1)\ .$$

Moreover, from (5) and lemmas 2,3, we have:

$$\lim_{t\to\infty} \tilde{P}^{(0)}(k;1,t) \equiv \lim_{t\to\infty} \tilde{P}\,(k;1,t) \tag{40}$$

$$= P^k \lim_{t\to\infty} e^{\lambda(P-I)t}$$

$$= P^k\, P^*$$

$$= P^*\ .$$

From (39) and (40), we obtain the limiting behavior of the joint process $(\xi(t),\ J(t))$ in terms of the generating function:

$$\lim_{t\to\infty} \tilde{P}\,(k;z,t) = \lim_{t\to\infty} \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}\, \tilde{P}^{(n)}(k;1,t) \tag{41}$$

$$= \sum_{n=0}^{\infty} \frac{\left[\lambda \sum_{i=1}^{M} P_i\, \mu_i\ (z-1)\right]^n}{n!}\, P^*$$

$$= e^{(z-1)\,\lambda \sum_{i=1}^{M} P_i\, \mu_i}\, P^*\ ,$$

which is the stated result (12). Thus theorem 2 is proved.

From theorem 2, we immediately have the following corollaries:

Corollary 1.

We have:

(42) $$\lim_{t\to\infty} P\{\xi(t)=m, J(t)=j \mid \xi(0)=k, J(0)=i\} = e^{-\lambda \sum_{h=1}^{M} P_h \mu_h} \frac{\left(\lambda \sum_{h=1}^{M} P_h \mu_h\right)^m}{m!} P_j$$

for $m = 0,1,2,\ldots;\ j=1, 2,\ldots,M$. So the limiting distribution of the queuelength process $\xi(t)$ is Poisson:

(43) $$\lim_{t\to\infty} P\{\xi(t)=m\} = e^{-\lambda \sum_{i=1}^{M} P_i \mu_i} \frac{\left(\lambda \sum_{i=1}^{M} P_i \mu_i\right)^m}{m!} \qquad (m \geq 0).$$

Corollary 2.

The asymptotic expected queuelength is given by:

(44) $$\lim_{t\to\infty} E(\xi(t)) = \lambda \sum_{i=1}^{M} P_i \mu_i ,$$

and the asymptotic variance of queuelength is given by:

(45) $$\lim_{t\to\infty} \mathrm{Var.}\ \xi(t) = \lambda \sum_{i=1}^{M} P_i \mu_i .$$

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13. ABSTRACT

The queue with an infinite number of servers with a Poisson arrival process and with semi-Markovian service times is considered. The queue length process and the type of the first customer to join the queue after time t are studied jointly and we obtain the transient and asymptotic results which are of matrix extensions of the corresponding results of the M/G/∞ queue. In particular, we prove that the limiting distribution of the queue length process is Poisson.

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