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The Infinite Server Queue With Semi-Markovian Arrivals and Negative Exponential Services

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Abstract

The queue with an infinite number of servers with a semi-Markovian arrival process and with negative exponential service times is studied. The queuelength process and the type of the last customer to join the queue before time t are studied jointly, both in continuous and in discrete time.

Asymptotic results are also obtained.

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1. Introduction.

The infinite server queue with semi-Markovian arrivals has potential applicability in modelling a system with many servers in which there are significant variations in the interarrival times between successive customers. In many cases of practical interest such variations in the interarrival times can be expressed in terms of a semi-Markov process with a sufficiently large number of states.

Furthermore as the natural generalization to "the matrix case" of the $GI|M|\infty$ queue, this model is also of independent theoretical interest.

We consider a queueing model in which the n^{th} customer C_n is of type J_{n-1} and arrives at a service counter in the instant T_n $(0 = T_1 < T_2 < \ldots < T_n < \ldots)$; $t = 0$ is taken as an arrival instant. There are M customer types. There are infinitely servers, which is equivalent to saying that each customer starts being served as soon as he arrives. S_n is the service time of C_n. It is assumed that $S_1, S_2, \ldots, S_n, \ldots$ are independent, identically distributed positive random variables with common distribution:

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 , \end{cases} \tag{1}$$

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and that the sequence $\{S_n\}$ is independent of the arrival process $\{t_n, J_n\}$, where $t_n = T_{n+1} - T_n$ is the interarrival time between C_n and C_{n+1}. Our basic assumption for the arrival process is that the pairs $\{ (t_n, J_n), n \geq 0 \}$ form a semi-Markov sequence:

$$
\begin{aligned}
(2) \qquad & P\{t_n \leq x, J_n = j \mid (t_k, J_k, k \leq n-2); t_{n-1}, J_{n-1} = i \} \\
& = P\{t_n \leq x, J_n = j \mid J_{n-1} = i \} \\
& = Q_{ij}(x)
\end{aligned}
$$

for $n = 1, 2, \ldots ; i, j = 1, 2, \ldots, M ; M < \infty$.

We further assume that the underlying Markov chain $\{J_n, n \geq 0\}$ is irreducible For the standard definitions and properties of Markov renewal and semi-Markov processes, we refer to Pyke [1, 2]. Suppose that at time $t = 0$, there are i_0-1 ($i_0 \geq 1$) initial customers. Denoting by $\xi(t)$ the queue length at time $t-0$, we let $\xi_n = \xi(T_n-0)$ be the queuelength immediately preceding the arrival of the customer C_n. In particular, $\xi_1 = i_0-1$. The customer arriving at $t = 0$ is therefore not counted among the i_0-1.

The transient and asymptotic behavior of the processes ξ_n and $\xi(t)$ is discussed in the subsequent sections.

2. The transient behavior of the discrete-time queuelength process ξ_n.

It is clear from the assumptions that the sequence $(\xi_n, J_{n-1}, n \geq 1)$ is a homogeneous bivariate Markov chain with stationary transition probabilities:

$$
\begin{aligned}
(3) \qquad & P(k,\ell; j,h) = P \{ \xi_{n+1} = k, J_n = \ell \mid \xi_n = j, J_{n-1} = h \} \\
& = \int_0^\infty \Pi_{jk}(x) \, dQ_{h\ell}(x) \qquad (h, \ell = 1, 2, \ldots, M; \; j, k = 0, 1, 2, \ldots)
\end{aligned}
$$

where

(4) $$\Pi_{jk}(x) = \binom{j+1}{k} e^{-k\mu x} (1 - e^{-\mu x})^{j+1-k}, \quad \text{for } j + 1 \geq k$$
$$= 0, \qquad \text{elsewhere}$$

The derivation of these expressions is completely analogous to that given in Takàcs [3] p. 164.

We now introduce the binomial moments:

(5) $$B_{ij}^{(n-1)}(r) = E \left\{ \binom{\xi_n}{r} I_{\{J_{n-1}=j\}} \mid J_0 = i \right\},$$

for $i, j = 1, 2, \ldots, M$; $n \geq 2$; $r \geq 0$. I_A is the indicator function of the event A. Evidently, for $r \geq 1$, $n \geq 1$, we have:

(6) $$B_{ij}^{(n)}(r) = E \left\{ \binom{\xi_{n+1}}{r} I_{\{J_n=j\}} \mid J_0 = i \right\}$$

$$= \sum_{h=1}^{M} \sum_{k=0}^{\infty} E \left\{ \binom{\xi_{n+1}}{r} I_{\{J_n=j\}} \mid \xi_n = k, J_{n-1} = h, J_0 = i \right\} \cdot$$
$$\cdot \, P \{\xi_n = k, J_{n-1} = h \mid J_0 = i\}$$

$$= \sum_{h=1}^{M} \sum_{k=0}^{\infty} q_{hj}(r\mu) \binom{k+1}{r} P \{ \xi_n = k, J_{n-1} = h \mid J_0 = i \}$$

$$= \sum_{h=1}^{M} \sum_{k=0}^{\infty} q_{hj}(r\mu) \left[\binom{k}{r} + \binom{k}{r-1} \right] P \{ \xi_n = k, J_{n-1} = h \mid J_0 = i\}$$

$$= \sum_{h=1}^{M} q_{hj}(r\mu) \left\{ E \left[\binom{\xi_n}{r} I_{\{J_{n-1} = h\}} \mid J_0 = i \right] + \right.$$
$$\left. + E \left[\binom{\xi_n}{r-1} I_{\{J_{n-1} = h\}} \mid J_0 = i \right] \right\}$$

$$= \sum_{h=1}^{M} q_{hj}(r\mu) \left[B_{ih}^{(n-1)}(r) + B_{ih}^{(n)}(r-1) \right],$$

where:

(7) $$q_{hj}(s) = \int_0^\infty e^{-sx}\, dQ_{hj}(x) \qquad (\mathrm{Re}\, s \geq 0) .$$

Further, from the definition of $B_{ij}^{(n)}(r)$, we find that

(8) $$B_{ij}^{(n)}(0) = P\{ J_n = j \mid J_0 = i \} = P_{ij}^{(n)} ,$$

the n-step transition probability of the Markov chain $\{J_n\}$ with:

$$P_{ij} = P\{ J_n = j \mid J_{n-1} = i \} = q_{ij}(0+) .$$

In matrix notation with $P = (P_{ij})$, $B^{(n)}(r) = (B_{ij}^{(n)}(r))$, etc., we have

(9) $$B^{(n)}(r) = [B^{(n-1)}(r) + B^{(n-1)}(r-1)]\, q(r\mu) \; ; \text{ for } r \geq 1$$

and:

(10) $$B^{(n)}(0) = P^n .$$

Moreover, $\xi_1 = i_0$ implies that:

$$B_{ij}^{(1)}(r) = E\{ \binom{\xi_2}{r} I_{\{J_1=j\}} \mid J_0 = i \} = \binom{i_0}{r} q_{ij}(r\mu) .$$

and therefore:

(11) $$B^{(1)}(r) = \binom{i_0}{r} q(r\mu), \text{ for } r \geq 0.$$

The matrices $B^{(n)}(r)$ can be uniquely determined by formulae (9), (10), (11), but it is first convenient to introduce the matrix generating functions:

$$\phi_r(w) = \sum_{n=1}^{\infty} B^{(n)}(r)\, w^n \text{ for } r \geq 0 \text{ and } |w| < 1 .$$

and we set:

$$\phi_{-1}(w) \equiv 0 .$$

Then (9), (10) and (11) yield:

$$\phi_r(w)\,[\,I - w\,q(r\mu)\,] = [\,\phi_{r-1}(w) + \binom{i_0}{r}\,]\,w\,q(r\mu) \qquad (r \geq 0)\;,$$

which implies by successive substitutions that:

$$\phi_r(w) = \sum_{j=0}^{r} \binom{i_0}{j} \prod_{k=j}^{r} [w\,q(k\mu)]\,[I - w\,q(k\mu)]^{-1}, \qquad \text{for } r \geq 0. \tag{12}$$

where I is M x M identity matrix.

Let now:

$$P_{ij}^{(n)}(k) = P\,\{\,\xi_{n+1} = k,\ J_n = j \mid J_0 = i\,\}\;. \tag{13}$$

then we note that:

$$B_{ij}^{(n)}(r) = E\,\{\,\binom{\xi_{n+1}}{r}\,I_{\{J_n=j\}} \mid J_0 = i\,\} = \sum_{k=r}^{\infty} \binom{k}{r}\,P_{ij}^{(n)}(k)\;,$$

and hence that:

$$P_{ij}^{(n)}(k) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k}\,B_{ij}^{(n)}(r)\;;$$

or in matrix notation, that:

$$P^{(n)}(k) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k}\,B^{(n)}(r)\;. \tag{14}$$

whence

$$\sum_{n=1}^{\infty} P^{(n)}(k)\,w^n = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k}\,\phi_r(w) \qquad (|w| < 1)\;. \tag{15}$$

Consequently, we have proved the following theorem about the transient behavior of the joint process (ξ_n, J_{n-1}).

<u>Theorem 1.</u>

Suppose that $\xi_1 = i_0$ is fixed, then the matrix of probabilities $P^{(n)}(k) \equiv (P_{ij}^{(n)}(k))$ is uniquely determined by the matrix generating function:

(16) $$\sum_{n=1}^{\infty} P^{(n)}(k)\, w^n = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \phi_r(w), \qquad (k \geq 0) .$$

where $\phi_r(w)$ and $\{ P_{ij}^{(n)}(k) \}$ were given in formulae (12) and (13) respectively.

3. The asymptotic behavior of the queuelength in discrete time.

The asymptotic behavior of the Markov chain (J_{n-1}, ξ_n) follows readily from the general theory of irreducible Markov chains.

If the Markov chain (J_n) is aperiodic so is (J_{n-1}, ξ_n) and if every state of (J_n) has period d, the same holds for the bivariate chain.

In either case, it is known that the Abel limits:

(17) $$\lim_{w \to 1-} (1-w) \sum_{n=1}^{\infty} P^{(n)}(k)\, w^n, \qquad k \geq 0$$

exist and are equal to the Cesaro limits:

(18) $$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} P^{(n)}(k), \qquad k \geq 0$$

Moreover in the aperiodic case these limits also equal:

(19) $$\lim_{n \to \infty} P^{(n)}(k), \qquad k \geq 0$$

the stationary probability matrices for the bivariate chain. In the periodic case, the classical modification must be made in the latter limit [3].

Theorem 2.

The limit matrices:

(20) $$P(k) = \lim_{w \to 1-} (1-w) \sum_{n=1}^{\infty} P^{(n)}(k)\, w^n, \qquad k \geq 0$$

are given by:

(21) $$P(k) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B(r), \qquad k \geq 0$$

where the matrices $B(r)$ are the binomial moment matrices of $P(k)$, $k \geq 0$ and are themselves given by:

(22) $$B(r) = P^{*} \prod_{k=1}^{r} q(k\mu) \, [I - q(k\mu)]^{-1}, \qquad r \geq 1$$

$$B(0) = P^{*},$$

where P^{*} is the matrix whose entries are the stationary probabilities corresponding to the stochastic matrix P.

The matrix P^{*} has constant columns and so do the matrices $P(k)$, $k \geq 0$. This expresses the fact that the limit probabilities of the bivariate Markov chain do not depend on the initial queuelength i_0 and on the initial customer type J_0.

Proof: By theorem 1, we have:

(23) $$P(k) = \lim_{w \to 1-} (1-w) \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \Big\{ \sum_{j=0}^{r} \binom{i_0}{j} \prod_{k=j}^{r} w\, q(k\mu) \, [I - w\, q(k\mu)]^{-1} \Big\}$$

$$= \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B(r),$$

where:

(24) $$B(r) = \lim_{w \to 1-} (1-w) \sum_{j=0}^{r} \binom{i_0}{j} \prod_{k=j}^{r} w\, q(k\mu) \, [I - w\, q(k\mu)]^{-1}, \; r \geq 1.$$

The interchange of the limit and the summation on r is justified by the Lebesgue dominated convergence theorem.

$B(r)$ may be further simplified. Except for the first term, all other terms in the sum on j vanish in the limit as $w \to 1-$.

It follows that:

$$(25) \qquad B(r) = P \lim_{w \to 1-} (1-w)\,[I - w\,P]^{-1}.\ \prod_{k=1}^{r} q(k\mu)\,[I - q(k\mu)]^{-1}$$

but:

$$(26) \qquad \lim_{w \to 1-} (1-w)\,[I - w\,P]^{-1} = P^{*},$$

and:

$$(27) \qquad P\,P^{*} = P^{*},$$

so:

$$B(r) = P^{*} \prod_{k=1}^{r} q(k\mu)\,[I - q(k\mu)]^{-1}, \qquad r \geq 1.$$

and clearly:

$$B(0) = P^{*}.$$

Clearly $B(r)$ and hence $P(k)$, $r \geq 0$, $k \geq 0$ have constant columns. Thus the theorem is proved.

4. The transient behavior in continuous time.

In this section we wish to study the queuelength $\xi(t)$ in continuous time. We also define the process $J(t)$, where for every $t > 0$, $J(t)$ is the type of the last customer to arrive in $(0, t]$ and $J(0) = i$. We shall determine the probabilities $P_{ij}(k;\, t)$ defined by:

(28) $$P\{\xi(t) = k,\ J(t) = j \mid J_0 = i\} = P_{ij}(k;\ t),$$

for $k \geq 0$, $i, j = 1,\ldots,M$.

The process $\{\xi(t), J(t);\ t \geq 0\}$ will be studied in relation to the imbedded Markov renewal sequence $\{J_n, \xi_{n+1}, T_{n+1} - T_n\}$ on the state space $\{1,\ldots,M\} \times \{0, 1,\ldots\} \times \{[0, \infty)\}$.

Let us define for $\xi_0 = i_0$ fixed, the transforms:

(29) $$A_{ij}^{(n)}(r;\ s) = E\{e^{-sT_{n+1}} \binom{\xi_{n+1}}{r} I_{\{J_n = j\}} \mid J_0 = i\},$$

Re $s \geq 0$, as well as:

(30) $$\phi_{ij}(r;\ s,w) = \sum_{n=1}^{\infty} A_{ij}^{(n)}(r;\ s)\, w^n, \qquad |w| < 1.$$

and the matrices:

(31) $$A^{(n)}(r;\ s) = (A_{ij}^{(n)}(r;\ s)),\ \phi(r;\ s,w) = (\phi_{ij}(r;\ s,w))$$

<u>Lemma 1</u>

For Re $s > 0$, $|w| \leq 1$ or Re $s \geq 0$, $|w| < 1$ we have

(32) $$\phi(r;\ s,w) = \sum_{j=0}^{r} \binom{i_0}{j} \prod_{k=j}^{r} w\, q(s + k\mu)\, [I - w\, q(s + k\mu)]^{-1}, \qquad r \geq 0$$

<u>Proof</u>:

Under the condition $\xi_n = j$, $t_n = x$, ξ_{n+1} has a binomial distribution with parameters $j + 1$ and $e^{-\mu x}$. Applying the law of total probability and after an easy calculation we obtain:

$$(33)\quad A_{ij}^{(n)}(r;\, s) = \sum_{h=1}^{M} \sum_{k=0}^{\infty} E\{ e^{-s\, T_{n+1}} \binom{\xi_{n+1}}{r} I_{\{J_n=j\}} \mid \xi_n = k,\, J_{n-1} = h,\, J_0 = i\}$$

$$P\{ \xi_n = k,\, J_{n-1} = h \mid J_0 = i$$

$$= \sum_{h=1}^{M} q_{hj}\,(s+r\mu)\,\{ E\,[e^{-st_n} \binom{\xi_n}{r} I_{\{J_{n-1}=h\}} \mid J_0 = i]$$

$$+ E\,[e^{-st_n} \binom{\xi_n}{r-1} I_{\{J_{n-1}=h\}} \mid J_0 = i]\,\}$$

$$= \sum_{h=1}^{M} q_{hj}\,(s + r\mu)\,[A_{ih}^{(n-1)}(r;\, s) + A_{ih}^{(n-1)}(r-1;\, s)\,]\,,$$

or in matrix notation:

$$(34)\quad A^{(n)}(r;\, s) = [A^{(n-1)}(r;\, s) + A^{(n-1)}(r-1;\, s)]\, q(s + r\mu),$$

for $n \geq 2$. Also:

$$(35)\qquad A^{(1)}(r;\, s) = \binom{i_0}{r} q(s + r\mu).$$

Substituting in the definition of $\phi(r;\, s,w)$, we obtain:

$$(36)\quad \phi(r;\, s,w) - w \binom{i_0}{r} q(s+r\mu) = w\,[\,\phi(r;\, s,w) + \phi(r-1;\, s,w)]\, q(s+r\mu),$$

for $r \geq 0$.

Successive substitution leads to:

$$(37)\quad \phi(r;\, s,w) = \sum_{j=0}^{r} \binom{i_0}{j} \prod_{k=j}^{r} w\, q(s+k\mu)\, [I - w\, q(s+k\mu)]^{-1}\,,$$

which is the stated formula (32).

Let now $M_{ij}^{(h)}(t)$ be the expected number of visits to the state (j,h) in the imbedded semi-Markov process in $(0,t]$, given that the initial state was (i,i_0) - then:

$$(38) \qquad M_{ij}^{(h)}(t) = \sum_{n=1}^{\infty} P\{T_{n+1} \leq t,\ \xi_{n+1} = h,\ J_n = j \mid J_0 = i\},$$

If we take Laplace-Stieltjes transforms $\mu_{ij}^{(h)}(s)$, defined by:

$$(39) \qquad \mu_{ij}^{(h)}(s) = \int_0^{\infty} e^{-st}\, d\, M_{ij}^{(h)}(t), \qquad \text{Re } s > 0.$$

then:

$$(40) \qquad \sum_{h=r}^{\infty} \binom{h}{r} \mu_{ij}^{(h)}(s) = \sum_{n=1}^{\infty} E\left\{ e^{-s T_{n+1}} \binom{\xi_{n+1}}{r} I_{\{J_n = j\}} \mid J_0 = i\right\} = \phi_{ij}(r; s,1).$$

So that the transform (40) of the renewal functions $M_{ij}^{(h)}(t)$ is known in view of formula (32).

We now return to the probabilities $P_{ij}(k; t)$ and prove the following theorem.

<u>Theorem 3</u>.

Let $\xi(0) = i_0$ fixed, then the Laplace transform of $P_{ij}(k; t)$ is given by:

$$(41) \qquad \int_0^{\infty} e^{-st} P_{ij}(k; t)\, dt = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \beta_{ij}(r; s),$$

for Re $s > 0$, where the $\beta_{ij}(s)$ are given by:

$$(42) \qquad \beta_{ij}(r; s) = \frac{1}{s+r\mu} \left\{ \binom{i_0}{r} \left[1 - \sum_{h=1}^{M} q_{ih}(s+r\mu)\right] + \right.$$

$$\left. \left[1 - \sum_{h=1}^{M} q_{jh}(s+r\mu)\right] \left[\phi_{ij}(r; s,1) + \phi_{ij}(r-1; s,1)\right] \right\}$$

and the functions $\phi_{ij}(r; s,1)$ are given by (32).

Proof:

It is known that if $B_{ij}(k;\ t)$ is the binomial moment:

(43)
$$B_{ij}(k;\ t) = E\ \{\ \binom{\xi(t)}{k}\ I_{\{J(t)=j\}}\ \mid J_0 = i\ \} = \sum_{r=k}^{\infty} \binom{r}{k} P_{ij}\ (r;\ t),$$

then conversely:

(44)
$$P_{ij}(k;\ t) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_{ij}\ (r;\ t).$$

or, upon taking transforms:

(45)
$$\int_0^\infty e^{-st}\ P_{ij}(k;\ t)\ dt = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \beta_{ij}(r;\ s)\ ,$$

where:

(46)
$$\beta_{ij}\ (r;\ s) = \int_0^\infty e^{-st}\ B_{ij}(r;\ t)\ dt, \qquad \text{Re } s > 0$$

Now, using the theorem of total expectation [3] we can write:

(47)
$$B_{ij}\ (r;\ t) = \binom{i_0}{r} e^{-r\mu t}\ [\ 1 - \sum_{h=1}^{M} Q_{ih}(t)\ ] + \sum_{h'=r-1}^{\infty} \binom{h'+1}{r} \int_0^t e^{-r\mu(t-u)}\ [1 - \sum_{h=1}^{M} Q_{jh}(t-u)]\ d\ M_{ij}^{(h')}(u)\ ,$$

and upon taking transforms, we obtain:

(48)
$$\beta_{ij}\ (r;\ s) = \frac{1}{s+r\mu}\ \{\ \binom{i_0}{r}\ [\ 1 - \sum_{h=1}^{M} q_{ih}\ (s+r\mu)] + \sum_{h'=r-1}^{\infty} \binom{h'+1}{r} [1 - \sum_{h=1}^{M} q_{jh}\ (s+r\mu)]\ \mu_{ij}^{(h')}(s)\ \}$$

By appealing to formula (40) and after some obvious manipulations we obtain the stated result.

5. The asymptotic behavior of the continuous time process.

We shall assume throughout this section that the Markov renewal process with transition matrix $Q(x) = (Q_{ij}(x))$ is positive recurrent or equivalently that the mean row sums

$$\sum_{j=1}^{M} \int_0^{\infty} x \, d\, Q_{ij}(x)$$

are finite.

We now assume that the semi-Markov sequence of interarrival times and customer types is non-lattice i.e. we exclude the case where all the distributions $Q_{ij}(\cdot)$ are defined on a common lattice.

We may then apply the key renewal theorem for Markov renewal processes to the expressions in formula (47) and we obtain:

(49)
$$\lim_{t\to\infty} B_{ij}(r,t) = B_{ij}(r) =$$

$$\sum_{h'=r-1}^{\infty} \binom{h'+1}{r} \frac{1}{\mu_j^{(h')}} \int_0^{\infty} e^{-r\mu t} \left[1 - \sum_{h=1}^{M} Q_{jh}(t)\right] dt \,,$$

where $\mu_j^{(h')}$ is the mean recurrence time of the state (j, h') in the imbedded semi-Markov process.

Formula (49) further leads to:

$$B_{ij}(r) = \sum_{h'=r-1}^{\infty} \binom{h'+1}{r} \frac{1}{r\,\mu\,\mu_j^{(h')}} \left[1 - \sum_{h=1}^{M} q_{jh}(r\mu)\right] \quad (50)$$

for $r \geq 1$ and:

(51) $$B_{ij}(0) = \sum_{h'=r-1}^{\infty} \binom{h'+1}{r} \frac{1}{\mu_j^{(h')}} \cdot \left[\sum_{h=1}^{M} \int_0^{\infty} x \, d\, Q_{jh}(x) \right],$$

However, by formula (40) and the ordinary renewal theorem, we have:

(52) $$\sum_{h=r}^{\infty} \binom{h}{r} [\mu_j^{(h)}]^{-1} = \lim_{s\to 0+} s \sum_{h=r}^{\infty} \binom{h}{r} \mu_{ij}^{(h)}(s)$$

$$= \lim_{s\to 0+} s\, \phi_{ij}(r; s,1). \quad ([3], \text{ p. } 234).$$

The latter limits are identified in the following theorem.

Theorem 4.

The limit of the matrix $s\,\phi(r; s,1)$ with entries $s\,\phi_{ij}(r; s,1)$ is given by:

(53) $$\lim_{s\to 0+} s\,\phi(r; s,1) =$$

$$\lim_{s\to 0+} s \prod_{k=0}^{r} q(s+k\mu)\, [I - q(s+k\mu)]^{-1}$$

$$= M \prod_{k=1}^{r} q(k\mu)\, [I - q(k\mu)]^{-1}, \qquad r \geq 1$$

and $\lim_{s\to 0+} s\,\phi(0; s,1) = M$, where:

(54) $$M = \lim_{s\to 0+} s\,[I - q(s)]^{-1},$$

is the matrix whose entries $M_{ij} = \frac{1}{\theta_j}$, where θ_j is the mean recurrence time of the state j in the Markov renewal process with transition matrix $Q(x)$.

Proof:

Using formula (32) to express $\phi(r; s,1)$, we note that the limits of all terms but the one corresponding to $j = 0$ vanish. This leads to the equality of the first and the second limits.

The identification of the limit and the validity of (54) follow from well-known results for finite state Markov renewal processes [2].

We now obtain from (50) and (51) or alternately by taking the limits:

(55) $$\lim_{s \to 0+} s\, \beta_{ij}(r; s)$$

in formula (42), that:

(56) $B_{ij}(r) =$

$$\frac{1}{r\mu}\left[1 - \sum_{h=1}^{M} q_{jh}(r\mu)\right] \left\{ M \prod_{k=1}^{r-1} q(k\mu)[I - q(k\mu)]^{-1} . [I - q(r\mu)]^{-1} \right\}_{ij} \qquad r \geq 1$$

It is easy to check that, since M has constant columns, $B_{ij}(r)$ does not depend on i.

Also:

(57) $$B_{ij}(0) = \lim_{s \to 0+} s\, \beta_{ij}(0; s) =$$

$$\lim_{s \to 0+} \frac{1 - \sum_{h=1}^{M} q_{jh}(s)}{s} \lim_{s \to 0+} s\, \phi_{ij}(0; s,1)$$

$$= \left[\sum_{h=1}^{M} \int_0^{\infty} x \, d\, Q_{jh}(x) \right] . \frac{1}{\theta_j},$$

We can now identify the limits of the probabilities $P_{ij}(k; t)$ as t tends to infinity.

Theorem 4

If the Markov renewal process with matrix $Q(\cdot)$ is nonlattice and positive recurrent, then the limits

$$\lim_{t\to\infty} P_{ij}(k;\, t) \qquad k \geq 0$$

exist and are given by:

$$\lim_{t\to\infty} P_{ij}(k;\, t) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_{ij}(r), \qquad k \geq 0 \tag{58}$$

where the $B_{ij}(r)$ are given by (56) and (57). These limits do not depend on the initial conditions.

Proof:

Evaluation of $\lim_{s\to 0+} s \int_0^\infty e^{-st} P_{ij}(k;\, t)\, dt$, using formula (41).

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13. ABSTRACT

The queue with an infinite number of servers with a semi-Markovian arrival process and with negative exponential service times is studied. The queuelength process and the type of the last customer to join the queue before time t are studied jointly, both in continuous and in discrete time.

Asymptotic results are also obtained.

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infinite server queue						
semi-Markov arrivals						
negative exponential service times						
transient behavior						
limiting behavior						
Markov renewal theory						

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