Distribution of Occupation Time and Virtual Waiting Time of a General Class of Bulk Queues*

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Abstract

Bulk queues have been studied by several authors. Neuts(1967) discusses a general class of bulk queues and studies the queue length and busy periods. Because of the computational complexity the distribution of the occupation time and the virtual waiting time has not been studied so far. In this paper we closely follow the notation and terminology of [6]. We study the occupation time and virtual waiting time with the help of a simple lemma proved in the Appendix.

1. Introduction

Let us recall the definition of the bulk queues studied in [6]. Customers arrive at a counter according to a Poisson process of parameter λ and are served in groups according to the following policy: If at the time of a departure, K or more customers are waiting, a group of K customers is served and the others must wait. If the number waiting does not exceed K but is greater than or equal to L all are served together. If the number of customers is less than L, the server waits until L customers are present. Practical applications of this model are discussed in [6]. Here we study the distributions of the occupation time and the virtual waiting time (and their limiting moments) for this queue.

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We assume that the successive service times are conditionally independent, given the batch sizes, but their distributions may depend on the batch size. The different batches are formed according to first-come, first served rule.

2. Distribution of Occupation time

As defined by Takács [7] $\eta(t)$ denotes the occupation time of the server at the instant t. $\eta(t)$ is the time until the server becomes idle if no customers join the queue after t. Let $\xi(t)$ be the queue length at time t. We assume that $\xi(0)=i \geq L$. If $i < L$ the process starts with an idle period having an Erlang distribution of order L-i.

Denoting:

(1) $W_i(t,x) = P\{\eta(t) \leq x | \xi(0)=i\}$

and

(2) $\Lambda_i(t,x) = P\{0 < \eta(t) \leq x, \eta(\tau) \neq 0 \text{ for all } \tau \varepsilon (0,t] | \xi(0)=i\}$

we obtain by a standard renewal argument:

$$W_i(t,x) = \Lambda_i(t,x) + \int_0^t \Lambda_L(t-\tau,x)dM_1(\tau) \tag{3}$$

$$+ P\{\eta(t)=0|\xi(0)=i\}U(x)$$

where $M_1(\cdot)$ is the renewal function of the general renewal process formed by the beginnings of busy periods and $U(\cdot)$ is the distribution degenerate at zero. Let $m_1(\xi)$ be the Laplace-Stieltjes Transform (L.S.T.) of $M_1(\cdot)$ and:

$$W_i^{**}(\xi,s) = \int_0^\infty e^{-\xi t}\int_0^\infty e^{-sx} dW_i(t,x)dt \tag{4}$$

$$\Lambda_i^{**}(\xi,s) = \int_0^\infty e^{-\xi t}\int_0^\infty e^{-sx} d\Lambda_i(t,x)dt \tag{5}$$

Let G(j,n,x) be the probability that a busy period consists of atleast n services which lasts for atmost time x and that at the end of the nth service there are j customers waiting. Let $H_j(\cdot)$ denote the distribution function of service time for a batch of j customers, j=L, L+1,...,K. For the sake of easy notation we define:

$$H_\nu(\cdot) \equiv U(\cdot)\ ,\ 0 \le \nu \le L-1 \tag{6}$$

Further let $h_\nu(s)$ be the L.S.T. of $H_\nu(x)$ and $\Gamma(j,n,s)$ the L.S.T. of $G(j,n,x)$ and:

$$E_j(1,\xi) = \sum_{n=1}^{\infty} \Gamma(j,n,\xi),\ j=0,1,\dots,K-1 \tag{7}$$

<u>Lemma 1</u>. The transform $\Lambda_i^{**}(\xi,s)$ is given by

$$\Lambda_i^{**}(\xi,s) = \frac{1}{K}\sum_{\rho=0}^{K-1} \frac{\sum_{\nu=0}^{K-1}(\omega_\rho h_K^{\frac{1}{K}}(s)\ )^{-\nu} h_\nu(s)}{\xi-s+\lambda-\lambda\omega_\rho h_K^{\frac{1}{K}}(s)} \{(\omega_\rho h_K^{\frac{1}{K}}(s))^i$$

$$+ \sum_{j=L}^{K-1}[h_j(s) - (\omega_\rho h_K^{\frac{1}{K}}(s))^j][\delta_{ij} + E_j(1,\xi)]$$

$$- \sum_{j=0}^{L-1} E_j(1,\xi)(\omega_\rho h_K^{\frac{1}{K}}(s))^j\}$$

where $1=\omega_0,\omega_1,\dots,\omega_{K-1}$ are K-th roots of unity.

<u>Proof</u>

In terms of $G(\cdot,\cdot,\cdot)$ we have:

$$\Lambda_i(t,x) = \int_{0\,(u)}^{t} \int_{t\,(v)}^{t+x} \int_{v\,(v_1)}^{t+x} \sum_{n=0}^{\infty}\sum_{j=L}^{K-1} dG(j,n,u)\ dH_j(v-u) \tag{9}$$

$$\cdot\ e^{-\lambda(t-u)} \sum_{m=0}^{\infty}\sum_{\nu=0}^{K-1} \frac{[\lambda(t-u)]^{mK+\nu}}{(mK+\nu)!}\ dH_K^{(m)} * H_\nu(v_1-v)$$

$$+ \int_0^t \int_t^{t+x} \int_v^{t+x} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{j=0}^{K-1} dG(j+(r+1)K,n,u)$$
$$(u)\,(v)\quad (v_1)$$

$$\cdot\ dH_K(v-u)e^{-\lambda(t-u)} \sum_{m=0}^{\infty} \sum_{\nu=0}^{K-1} \frac{[\lambda(t-u)]^{mK-j+\nu}}{(mK-j+\nu)!} dH_K^{(m+r)} * H_\nu(v_1-v)$$

where $H_K^{(m)}(\cdot)$ is the m-fold convolution of $H_K(\cdot)$.

The probabilistic argument to get the first term is the following: If the server has never become idle in $(0,t]$, let the last service completion before time t occur between u and u+du and let $L \leq j \leq K-1$ be the number of customers left with at this time. Let the service completion of these j customers occur between v and v+dv. In the interval (u,t), $mK+\nu(m \geq 0, 0 \leq \nu \leq K-1)$ customers arrive, and the distribution of service time of these $mK+\nu$ customers is the convolution $H_K^{(m)} * H_\nu(\cdot)$.

To obtain the second term we assume that at the time u of the completion of last service before t, there are $j+(r+1)K(r \geq 0,\ 0 \leq j \leq K-1)$ customers left with. Out of these, K customers have service completion between v and v+dv. The number of arrivals in (u,t) is $mK+\nu-j$, so that the number waiting at t is $mK+rK+\nu$ whose service time distribution is $H_K^{(m+r)} * H_\nu(\cdot)$.

Taking the L.S.T. of (9) we obtain:

$$(10)\qquad \Lambda_i^{**}(\xi,s) = \sum_{n=0}^{\infty} \sum_{j=L}^{K-1} \Gamma(j,n,\xi) \int_0^{\infty} e^{-sv} dH_j(v) \int_0^v e^{-(\xi-s+\lambda)t}$$

$$\cdot \sum_{m=0}^{\infty} \sum_{\nu=0}^{K-1} \frac{(\lambda t)^{mK+\nu}}{(mK+\nu)!} h_K^m(s) h_\nu(s)dt$$

$$+ \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{j=0}^{K-1} \Gamma(j+(r+1)K,n,\xi) \int_0^{\infty} e^{-sv} dH_K(v)$$

$$\cdot \int_0^v e^{-(\xi-s+\lambda)t} \sum_{m=0}^{\infty} \sum_{\nu=0}^{K-1} \frac{(\lambda t)^{mK-j+\nu}}{(mK-j+\nu)!} h_K^{m+r}(s) h_\nu(s)dt$$

To sum the series inside the integrals we use the lemma in Appendix, which gives:

(11) $$\Lambda_i^{**}(\xi,s) = \sum_{n=0}^{\infty}\sum_{j=L}^{K-1}\Gamma(j,n,\xi)\int_0^{\infty}e^{-sv}dH_j(v)\int_0^{v}e^{-(\xi-s+\lambda)t} \cdot \frac{1}{K}\sum_{\rho=0}^{K-1}e^{\omega_\rho\lambda h_K^{\frac{1}{K}}(s)t}\sum_{\nu=0}^{K-1}(\omega_\rho h_K^{\frac{1}{K}}(s))^{-\nu}h_\nu(s)dt$$

$$+\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{K-1}\Gamma(j+(r+1)K,n,\xi)\int_0^{\infty}e^{-sv}dH_K(v)\int_0^{v}e^{-(\xi-s+\lambda)t} \cdot \frac{h_K^r(s)}{K}\sum_{\rho=0}^{K-1}e^{\omega_\rho\lambda h_K^{\frac{1}{K}}(s)t}\sum_{\nu=0}^{K-1}(\omega_\rho h_K^{\frac{1}{K}}(s))^{-(\nu-j)}h_\nu(s)dt$$

$$=\frac{1}{K}\sum_{\rho=0}^{K-1}\frac{\sum_{\nu=0}^{K-1}(\omega_\rho h_K^{\frac{1}{K}}(s))^{-\nu}h_\nu(s)}{\xi-s+\lambda-\lambda\omega_\rho h_K^{\frac{1}{K}}(s)}\{\sum_{n=0}^{\infty}\sum_{j=L}^{K-1}\Gamma(j,n,\xi)\,[h_j(s)-h_j(\xi+\lambda-\lambda\omega_\rho h_K^{\frac{1}{K}}(s))]$$

$$+\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{K-1}(\omega_\rho h_K^{\frac{1}{K}}(s))^{j+rK}\Gamma(j+(r+1)K,n,\xi)[h_K(s)-h_K(\xi+\lambda-\lambda\omega_\rho h_K^{\frac{1}{K}}(s))]\}$$

using equation (22) of Neuts [6] and noting that $\Gamma(j,0,\xi) = \delta_{ij}$, (11) simplifies to:

(12) $$\Lambda_i^{**}(\xi,s) = \frac{1}{K}\sum_{\rho=0}^{K-1}\frac{\sum_{\nu=0}^{K-1}(\omega_\rho h_K^{\frac{1}{K}}(s))^{-\nu}h_\nu(s)}{\xi-s+\lambda-\lambda\omega_\rho h_K^{\frac{1}{K}}(s)}\{\sum_{j=L}^{K-1}\delta_{ij}h_j(s)$$

$$+\sum_{j=K}^{\infty}\delta_{ij}(\omega_\rho h_K^{\frac{1}{K}}(s))^j+\sum_{j=L}^{K-1}E_j(1,\xi)h_j(s)$$

$$-\sum_{j=0}^{K-1}E_j(1,\xi)(\omega_\rho h_K^{\frac{1}{K}}(s))^j\}$$

which proves the lemma.

If we consider the general renewal process formed by the beginnings of busy periods and $f_1(\xi)$ the L.S.T. of the distribution function of the initial renewal, and $f(\xi)$ the L.S.T of the common distribution function of other renewals, then:

(13) $$m_1(\xi) = \frac{f_1(\xi)}{1-f(\xi)}$$

If I_n is the number of customers in the queue at the end of nth busy period and Y_n the length of the nth busy period and:

(14) $$G_j(x) = P\{I_n=j, Y_n \leq x\}, \ j=0,1,\ldots,L-1$$

then:

(15) $$f(\xi) = \sum_{j=0}^{L-1} E_j(1,\xi)\left(\frac{\lambda}{\lambda+\xi}\right)^{L-j}$$

This is so because if there are j customers left at the end of a busy period $(0 \leq j \leq L-1)$ then the ensuing idle period will have an Erlang distribution of order L-j.

(16) $$\int_0^\infty e^{-\xi t} P\{\eta(t)=0|\xi(0)=i\}dt$$

$$=\int_0^\infty e^{-\xi t}\int_0^t \sum_{j=0}^{L-1} d(U+M_1)*G_j(v) \sum_{r=0}^{L-j-1} e^{-\lambda(t-u)}\frac{[\lambda(t-u)]^r}{r!}\, dt$$

$$=\frac{1}{\xi}[1+m_1(\xi)] \sum_{j=0}^{L-1} E_j(1,\xi)\left[1-\left(\frac{\lambda}{\lambda+\xi}\right)^{L-j}\right]$$

Hence the transform of (3) gives:

<u>Theorem 1</u>. The transform $W_i^{**}(\xi,s)$ of the distribution function of occupation time is given by:

(17) $$W_i^{**}(\xi,s) = \Lambda_i^{**}(\xi,s) + m_1(\xi)\Lambda_L^{**}(\xi,s)$$

$$+ \frac{1}{\xi}[1+m_1(\xi)] \sum_{j=0}^{L-1} E_j(1,\xi)\left[1-\left(\frac{\lambda}{\lambda+\xi}\right)^{L-j}\right]$$

where $\Lambda_i^{**}(\xi,s), i\geq L$ is given by lemma 1.

Limiting Distribution

Let $W(x)$ be the limiting distribution function of $W_i(t,x)$ as $t\to\infty$ and $W^*(s)$ be its L.S.T., then:

Theorem 2. The L.S.T. $W^*(s)$ of the limiting distribution of occupation time is given by:

$$(18)\qquad W^*(s) = 1+\lambda[L+\lambda\mu-\sum_{j=0}^{L-1} jE_j(1,0)]^{-1}\{-\mu+\Lambda_L^{**}(0,s)\}$$

where $\Lambda_L^{**}(0,s)$ is given from lemma 1 and:

$$(19)\qquad \mu = -\sum_{j=0}^{L-1} E_j'(1,0)$$

The proof is immediate from Theorem 1 and: $W^*(s) = \lim_{\xi\to 0} \xi W_i^{**}(\xi,s)$.

From equation (19) of Neuts [6] we obtain:

$$(20)\qquad \mu = [K-\lambda\alpha_K]^{-1}\{K\alpha_L+K\sum_{j=L}^{K-1} E_j(1,0)\alpha_j-\alpha_K\sum_{j=0}^{K-1} jE_j(1,0)\}$$

where α_j is the mean of the distribution $H_j(\cdot)$. From Theorem 2 it can be verified that $W^*(0)=1$.

If $M_\eta(L,K)$ denotes the steady state expected occupation time, then

$$(21)\qquad M_\eta(L,K) = -\frac{\partial}{\partial s} W^*(s)]_{s=0+}$$

$$= \frac{\lambda}{2(K-\lambda\alpha_K)}[L+\lambda\mu-\sum_{j=0}^{L-1} jE_j(1,0)]^{-1}\{K\beta_L+K\sum_{j=L}^{K-1} E_j(1,0)\beta_j$$

$$-(\beta_K+\frac{K-\lambda\alpha_K}{\lambda K}\sum_{j=L}^{K-1}\alpha_j)\sum_{j=0}^{K-1} jE_j(1,0)$$

$$-\frac{\alpha_K}{\lambda}\sum_{j=0}^{K-1} j(j-K)E_j(1,0)$$

$$-((K-1)\alpha_K-\lambda\beta_K-2(\frac{K-\lambda\alpha_K}{K})\sum_{j=L}^{K-1}\alpha_j)\mu\}$$

Special cases

(i) L=K is the case of batches of fixed sizes [Takàcs(1962)].

(ii) L=1 is the case where the service commences when there is atleast one customer in the system[Miller (1959)].

(iii) L=0 is the case of transportation process [Bailey (1954), Downtown (1955)].

3. Virtual Waiting time

Let $\tilde{\eta}(t)$ denote the waiting time of a (virtual) customer arriving at time t.

Defining:

(22) $$\tilde{W}_i(t,x) = P\{\tilde{\eta}(t) \leq x \mid \xi(0)=i\}$$

and

(23) $$\tilde{\Lambda}_i(t,x) = P\{\tilde{\eta}(t) \leq x, \tilde{\eta}(\tau) \neq 0 \text{ for all } \tau \varepsilon (0,t] \mid \xi(0)=i\}$$

we have as in (3):

(24) $$\tilde{W}_i(t,x) = \tilde{\Lambda}_i(t,x) + \int_0^t \tilde{\Lambda}_L(t-\tau,x)\,dM_1(\tau) + P\{\tilde{\eta}(t)=0 \mid \xi(0)=i\}U(x)$$

Similar to (16):

$$\int_0^\infty e^{-\xi t} P\{\tilde{\eta}(t)=0 \mid \xi(0)=i\}dt$$

$$= \int_0^\infty e^{-\xi t} \int_0^t \sum_{j=0}^{L-1} d(U+M_1)*G_j(U)e^{-\lambda(t-u)} \frac{[\lambda(t-u)]^{L-j-1}}{(L-j-1)!}$$

(25) $$= \left[\frac{1+m_1(\xi)}{\lambda}\right] \sum_{j=0}^{L-1} E_j(1,\xi)\left(\frac{\lambda}{\lambda+\xi}\right)^{L-j}$$

Transform of (24) gives:

(26) $$\tilde{W}_i^{**}(\xi,s) = \tilde{\Lambda}_i^{**}(\xi,s) + m_1(\xi)\,\tilde{\Lambda}_L^{**}(\xi,s) + \left[\frac{1+m_1(\xi)}{\lambda}\right] f(\xi)$$

where $f(\xi)$ is defined in (15).

If $\tilde{W}^*(s)$ is the L.S.T. of the limiting distribution of $\tilde{W}_i(t,x)$ as $t\to\infty$, then from (26):

(27) $$\tilde{W}^*(s) = \frac{-1}{\lambda f'(0)}[1+\lambda\tilde{\lambda}_L^{**}(0,s)]$$

To find $\tilde{\lambda}_L^{**}(\cdot,\cdot)$ we proceed as follows:

Analogous ot the probabilistic argument given in (9) we obtain:

(28) $$\tilde{\lambda}_L(t,x) = \int_{0\,(u)}^{t}\int_{t\,(v)}^{t+x}\sum_{n=0}^{\infty}\sum_{j=0}^{L-1} dG(j,n,u)dE_{L-j-1}^{(v-u)}$$

$$+\int_{0\,(u)}^{t}\int_{t\,(v)}^{t+x}\int_{v\,(v_1)}^{t+x}\sum_{n=0}^{\infty}\sum_{j=L}^{K-1} dG(j,n,u)dH_j(v-u)e^{-\lambda(t-u)}$$

$$\cdot\sum_{m=0}^{\infty}\Big\{\sum_{\nu=L-1}^{K-1}\frac{[\lambda(t-u)]^{mK+\nu}}{(mK+\nu)!}dH_K^{(m)}(v_1-v)+\sum_{\nu=0}^{L-2}\frac{[\lambda(t-u)]^{mK+\nu}}{(mK+\nu)!}\sum_{r_1=L-\nu-1}^{\infty}e^{-\lambda(v-t)}\frac{[\lambda(v-t)]^{r_1}}{r_1!}dH_K^{(m)}(v_1-v)$$

$$+\sum_{\nu=0}^{L-2}\frac{[\lambda(t-u)]^{mK+\nu}}{(mK+\nu)!}\sum_{r_1=0}^{L-\nu-2}e^{-\lambda(v-t)}\frac{[\lambda(v-t)]^{r_1}}{r_1!}e^{-\lambda(v_1-v)}\sum_{r_2=L-r_1-\nu-1}^{\infty}\frac{[\lambda(v_1-v)]^{r_2}}{r_2!}dH_K^{(m)}(v_1-v)$$

$$+\sum_{\nu=0}^{L-2}\frac{[\lambda(t-u)]^{mK+\nu}}{(mK+\nu)!}\sum_{r_1=0}^{L-\nu-2}e^{-\lambda(v-t)}\frac{[\lambda(v-t)]^{r_1}}{r_1!}e^{-\lambda(v_1-v)}\sum_{r_2=0}^{L-r_1-\nu-2}\frac{[\lambda(v_1-v)]^{r_2}}{r_2!}dH_K^{(m)}(v_1-v)E_{L-r_1-r_2-\nu-1}(t+x-v_1)\Big\}$$

$$+\int_{0\,(u)}^{t}\int_{t\,(v)}^{t+x}\int_{v\,(v_1)}^{t+x}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{K-1} dG(j+K+rK,n,u)dH_K(v-u)e^{-\lambda(t-u)}$$

$$\cdot\sum_{m=0}^{\infty}\Big\{\sum_{\nu=L-1}^{K-1}\frac{[\lambda(t-u)]^{mK-j+\nu}}{(mK-j+\nu)!}dH_K^{(m+r)}(v_1-v)+\sum_{\nu=0}^{L-2}\frac{[\lambda(t-u)]^{mK-j+\nu}}{(mK-j+\nu)!}\sum_{r_1=L-\nu-1}^{\infty}e^{-\lambda(v-t)}\frac{[\lambda(v-t)]^{r_1}}{r_1!}dH_K^{(m+r)}(v_1-v)$$

$$+\sum_{\nu=0}^{L-2}\frac{[\lambda(t-u)]^{mK-j+\nu}}{(mK-j+\nu)!}\sum_{r_1=0}^{L-\nu-2}e^{-\lambda(v-t)}\frac{[\lambda(v-t)]^{r_1}}{r_1!}e^{-\lambda(v_1-v)}\sum_{r_2=L-r_1-\nu-1}^{\infty}\frac{[\lambda(v_1-v)]^{r_2}}{r_2!}dH_K^{(m+r)}(v_1-v)$$

$$+\sum_{\nu=0}^{L-2}\frac{[\lambda(t-u)]^{mK-j+\nu}}{(mK-j+\nu)!}\sum_{r_1=0}^{L-\nu-2}e^{-\lambda(v-t)}\frac{[\lambda(v-t)]^{r_1}}{r_1!}e^{-\lambda(v_1-v)}\sum_{r_2=0}^{L-r_1-\nu-2}\frac{[\lambda(v_1-v)]^{r_2}}{r_2!}dH_K^{(m+r)}(v_1-v)$$

$$\cdot E_{L-r_1-r_2-\nu-1}(t+x-v_1)\}$$

where $E_r(x)$ denotes the Erlang distribution of order r.

The transform of (28) gives:

$$\Lambda_L^{**}(\xi,s)=\sum_{n=0}^{\infty}\sum_{j=0}^{L-1}\Gamma(j,n,\xi)\frac{1}{(\xi-s)}[(\tfrac{\lambda}{\lambda+s})^{L-j-1}-(\tfrac{\lambda}{\lambda+\xi})^{L-j-1}] \tag{29}$$

$$+\sum_{n=0}^{\infty}\sum_{j=0}^{L-1}\Gamma(j,n,\xi)\int_0^\infty e^{-\xi t}dt\int_0^\infty e^{-sv}dH_j(v+t)e^{-\lambda t}$$

$$\cdot\sum_{m=0}^{\infty}\{\sum_{\nu=0}^{K-1}\frac{(\lambda t)^{mK+\nu}}{(mK+\nu)!}h_K^m(s)+\sum_{\nu=0}^{L-2}\frac{(\lambda t)^{mK+\nu}}{(mK+\nu)!}\sum_{r=0}^{L-\nu-2}e^{-\lambda v}\frac{(\lambda v)^r}{r!}$$

$$\cdot\int_0^\infty e^{-(\lambda+s)v_1}\sum_{r_1=0}^{L-r-\nu-2}\frac{(\lambda v_1)^{r_1}}{r_1!}dH_K^{(m)}(v_1)[(\tfrac{\lambda}{\lambda+s})^{L-r-r_1-\nu-1}-1]\}$$

$$+\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{K-1}\Gamma(j+K+rK,n,\xi)\int_0^\infty e^{-\xi t}dt\int_0^\infty e^{-sv}dH_K(v+t)e^{-\lambda t}$$

$$\cdot\sum_{m=0}^{\infty}\{\sum_{\nu=0}^{K-1}\frac{(\lambda t)^{mK-j+\nu}}{(mK-j+\nu)!}h_K^{m+r}(s)+\sum_{\nu=0}^{L-2}\frac{(\lambda t)^{mK-j+\nu}}{(mK-j+\nu)!}\sum_{r_1=0}^{L-\nu-2}e^{-\lambda v}\frac{(\lambda v)^{r_1}}{r_1!}$$

$$\cdot\int_0^\infty e^{-(\lambda+s)v_1}\sum_{r_2=0}^{L-r_1-\nu-2}\frac{(\lambda v_1)^{r_2}}{r_2!}dH_K^{(m+r)}(v_1)[(\tfrac{\lambda}{\lambda+s})^{L-r_1-r_2-\nu-1}-1]\}$$

$$=\sum_{n=0}^{\infty}\sum_{j=0}^{L-1}\Gamma(j,n,\xi)\frac{1}{(\xi-s)}[(\tfrac{\lambda}{\lambda+s})^{L-j-1}-(\tfrac{\lambda}{\lambda+\xi})^{L-j-1}]+\Lambda_L^{**}(\xi,s)]_{h_\nu=1}$$

$$+\sum_{\nu=0}^{L-2}\sum_{r_1=0}^{L-\nu-2}\sum_{r_2=0}^{L-r_1-\nu-2}\int_0^\infty e^{-(\xi+\lambda)t}dt\int_0^\infty e^{-(s+\lambda)v}\frac{(\lambda v)^{r_1}}{r_1!}\int_0^\infty e^{-(s+\lambda)v_1}\frac{(\lambda v_1)^{r_2}}{r_2!}$$

$$\cdot\left[\left(\frac{\lambda}{\lambda+s}\right)^{L-r_1-r_2-\nu-1}-1\right]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big\{\sum_{j=L}^{K-1}\Gamma(j,n,\xi)dH_j(v+t)\frac{(\lambda t)^{mK+\nu}}{(mK+\nu)!}dH_K^{(m)}(v_1)$$

$$+\sum_{r=0}^{\infty}\sum_{j=0}^{K-1}\Gamma(j+K+rK,n,\xi)dH_K(v+t)\frac{(\lambda t)^{mK+\nu-j}}{(mK+\nu-j)!}dH_K^{(m+r)}(v_1)\Big\}$$

where $\Lambda_L^{**}(\xi,s)$ is defined in (10).

Equation (29) does not simplify for the case of generat service time distribution. However, if we take:

$$H_j(x)=1-e^{-\mu_j x},\ j=L,L+1,\ldots,K,$$

then:

$$(30)\qquad \tilde{\Lambda}_L^{**}(\xi,s)=\Lambda_L^{**}(\xi,s)]_{h_\nu=1}+\frac{1}{(\xi-s)}\sum_{j=0}^{L-1}E_j(1,\xi)\left[\left(\frac{\lambda}{\lambda+s}\right)^{L-j-1}-\left(\frac{\lambda}{\lambda+\xi}\right)^{L-j-1}\right]$$

$$+\sum_{j=L}^{K-1}E_j(1,\xi)\frac{\mu_j}{\lambda^2}\sum_{\nu=0}^{L-2}\sum_{r_1=0}^{L-\nu-2}\left(\frac{\lambda}{\xi+\lambda+\mu_j}\right)^{\nu+1}\left(\frac{\lambda}{\lambda+\mu_j+s}\right)^{r_1+1}$$

$$\Big\{\sum_{r_2=0}^{L-r_1-\nu-2}\left[\left(\frac{\lambda}{\lambda+s}\right)^{L-r_1-\nu-1}\left(\frac{\lambda+s}{\lambda+s+\mu_K}\right)^{r_2}-\left(\frac{\lambda}{\lambda+s+\mu_K}\right)^{r_2}\right]\sum_{m=1}^{\infty}\binom{r_1+m-1}{m-1}\left(\frac{\lambda}{\xi+\lambda+\mu_j}\right)^{mK}$$

$$\cdot\left(\frac{\mu_K}{\lambda+s+\mu_K}\right)^{m}+\left(\frac{\lambda}{\lambda+s}\right)^{L-r_1-\nu-1}-1\Big\}$$

$$+\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{j=0}^{K-1}\Gamma(j+K+rK,n,\xi)\frac{\mu_K}{\lambda^2}\left(\frac{\mu_k}{\lambda+s+\mu_K}\right)^{r}$$

$$\Big\{\sum_{\nu=0}^{L-2}\sum_{r_1=0}^{L-\nu-2}\sum_{r_2=0}^{L-r_1-\nu-2}\sum_{m=1}^{\infty}\binom{r_2+m+r-1}{m+r-1}\left(\frac{\lambda}{\xi+\lambda+\mu_K}\right)^{mK}\left(\frac{\mu_K}{\lambda+s+\mu_K}\right)^{m}$$

$$+\sum_{\nu=j}^{L-2}\sum_{r_1=0}^{L-\nu-2}\sum_{r_2=0}^{L-r_1-\nu-2}\binom{r_2+r-1}{r-1}\Big\}\left(\frac{\lambda}{\xi+\lambda+\mu_K}\right)^{\nu-j+1}\left(\frac{\lambda}{\lambda+s+\mu_K}\right)^{r_1+1}\left[\left(\frac{\lambda}{\lambda+s}\right)^{L-r_1-\nu-1}\left(\frac{\lambda+s}{\lambda+s+\mu_K}\right)^{r_2}-\left(\frac{\lambda}{\lambda+s+\mu_K}\right)^{r_2}\right]$$

In the case L=1,

$$(31)\qquad \tilde{\lambda}_L^{**}(0,s)=\frac{1}{K}\ \frac{1-h_K(s)}{h_K(s)}\sum_{\rho=0}^{K-1}\frac{\omega_\rho h_K^{\frac{1}{K}}(s)}{1-\omega_\rho h_K^{\frac{1}{K}}(s)}\left[s-\lambda+\lambda\omega_\rho h_K^{\frac{1}{K}}(s)\right]^{-1}$$

$$\cdot\ \{1-h_1(s)+\sum_{j=0}^{K-1}\ [(\omega_\rho h_K^{\frac{1}{K}}(s))^j-h_j(s)]E_j(1,0)\}$$

and

$$(32)\qquad M_{\tilde{\eta}}(1,K)=\frac{\lambda}{2(K-\lambda\alpha_K)}(1+\lambda\mu)^{-1}\{K\beta_1-((K-1)\alpha_K-\lambda\beta_K)\mu$$

$$+\sum_{j=1}^{K-1}\ [K\beta_j-j\beta_K+\frac{\alpha_K}{\lambda}j(K-j)]E_j(1,0)\}$$

APPENDIX

Lemma

For all x the sum of the infinite series:

$$(A\cdot 1)\qquad \sum_{n=0}^{\infty} \frac{x^{nK+\nu}}{(nK+\nu)!} = \frac{1}{K}\sum_{m=0}^{K-1} \omega_m^{-\nu} e^{\omega_m x}, \text{ for } \nu \leq K-1$$

$$= -1+\frac{1}{K}\sum_{m=0}^{K-1} \omega_m^{-\nu} e^{\omega_m x} \text{ for } \nu=K$$

where $\omega_0,\omega_1,\ldots,\omega_{K-1}$ are K-th roots of unity

Proof

Let:

$$(A\cdot 2)\qquad f(x)= \sum_{n=0}^{\infty} \frac{x^{nK+\nu}}{(nK+\nu)!}$$

and

$$(A\cdot 3)\qquad \hat{f}(s)= \int_0^{\infty} e^{-sx} f(x)dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{s^{nK+\nu+1}}$$

$$= \frac{s^{K-\nu-1}}{s^K-1}$$

To find the inverse transform we use Bateman(1954), Tables of Integral Transforms (p. 232)

That is, if

$$\hat{f}(s) = \frac{Q(s)}{P(s)},$$

where $P(s)=(s-\alpha_1)\ldots(s-\alpha_n), \alpha_i \neq \alpha_j$ for $i \neq j$

and Q(s) is a polynomial of degree $\leq$ n-1, then the inverse transform of $\hat{f}(s)$ is given by

(A·4) $$f(x)=\sum_{m=1}^{n} \frac{Q(\alpha_m)}{P_m(\alpha_m)} e^{\alpha_m x},$$

where $$P_m(s)=\frac{P(s)}{s-\alpha_m}.$$

Comparing this with (A·3), we have:

$$P(s)=s^K-1$$
$$=(s-\omega_0)(s-\omega_1)\ldots.(s-\omega_{K-1})$$

so that $\alpha_m=\omega_m$ (m=0,...K-1), where $\omega_0,\omega_1,\ldots\omega_{K-1}$ are the roots of $z^K-1=0$.

In order to apply form (A·4) we calculate the various factors on the r.h.s.:

$$Q(s)=s^{K-\nu-1}$$

$$\left.\begin{aligned} P_m(\alpha_m)&=K\omega_m^{K-1} \\ Q(\alpha_m)&=\omega_m^{K-\nu-1} \end{aligned}\right\} m=0,1,\ldots,K-1$$

Hence lemma follows from (A·4).

The proof of the last part follows from:

$$\sum_{n=0}^{\infty} \frac{x^{nK+K}}{(nK+K)!} = -1+\sum_{n=0}^{\infty} \frac{x^{nK}}{(nK)!} .$$

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Bulk queues have been studied by several authors. Neuts (1967) discusses a general class of bulk queues and studies the queue length and busy periods. Because of the computational complexity the distribution of the occupation time and the virtual waiting time has not been studied so far. In this paper we closely follow the notation and terminology of [6]. We study the occupation time and virtual waiting time with the help of a simple lemma proved in the Appendix.