Another Look at Variances of Variance Components

by

James N. Arvesen

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series #220

March 1970

Another Look at Variances of Variance Components

by

James N. Arvesen*

1. Introduction. Several years ago, some excellent work was done by Tukey [1956b, 1957] and Hooke [1956b] in obtaining the variance of estimates of variance components when the standard underlying assumption of normality is omitted. The method both authors employed was essentially the device of polykays (Tukey [1956a], and Hooke [1956a]. Tukey obtained results for one-way classification models, both of a balanced and unbalanced nature. Hooke was more interested in the balanced two-way classification. Unfortunately, the present author must agree with Scheffé [1959, p. 346] that the latter results "..look discouragingly complicated." It is hoped that the technique used in this paper, that of U-statistics (Hoeffding [1948]), is more illuminating.

Finally, it is hoped that the present technique may be profitably used in more complicated higher-way layouts, especially with unbalanced data. To this end, the technique is applied to the unbalanced two-way layout. The results are essentially in the same spirit as recent results of Harville [1969], however the weighting scheme in the present paper is different from those investigated by Harville. Also, the assumption of normality is dropped in the present paper.

Section 2 deals with the unbalanced one-way classification, section 3 treats the balanced two-way classification, and section 4 handles an unbalanced two-way classification. Some summary comments are made in section 5.

* This research was supported in part by the NIH Grant 5T01-GM-00024 at the Purdue University.

2. One-way classifications.

Consider the following model:

$Y_{ij} = \mu + a_i + e_{ij}$, $i = 1,\ldots, I$, $j = 1,\ldots, J_i$ where μ is some overall constant, and the following are the assumptions on the main random effects $\{a_i\}$, and the error random effects $\{e_{ij}\}$

	Means	Variances	Kurtoses
Main random effects	0	σ_A^2	γ_A
Error random effects	0	σ_e^2	γ_e

All terms in the table are finite. In addition, the $I + \sum_{i=1}^{I} J_i$ random variables are mutually independent. This is essentially the model considered in Tukey [1957].

Let $X_i = \begin{pmatrix} Y_{i.} \\ \sum_{j=1}^{J_i} (Y_{ij} - Y_{i.})^2 \end{pmatrix}$ $i = 1,\ldots,I$

be I independent (not necessarily indentically distributed) bivariate random vectors. In what follows, U-statistics (Hoeffding [1948]) will be constructed based on the vectors $\{X_i\}$.

Define $f^{*(1)}(X_{\alpha_1}, X_{\alpha_2}) = w_{\alpha_1} w_{\alpha_2} (Y_{\alpha_1 .} - Y_{\alpha_2 .})^2$

$$f^{*(2)}(X_{\alpha_1}) = u_{\alpha_1} \sum_{j=1}^{J_{\alpha_1}} (Y_{\alpha_{1j}} - Y_{\alpha_1 .})^2 ,$$

$$U^{(1)} = \sum_{\alpha_1 < \alpha_2} f^{*(1)}(X_{\alpha_1}, X_{\alpha_2}) = W \sum_{i=1}^{I} w_i (Y_{i.} - W^{-1} \sum_{j=1}^{I} w_j Y_{j.})^2 \qquad (2.1)$$

where $W = \sum_{i=1}^{I} w_i$,

$$U^{(2)} = \sum_{\alpha_1=1}^{I} f^{*(2)}(X_{\alpha_1}) = \sum_{i=1}^{I} \sum_{j=1}^{J_i} u_i (Y_{ij} - Y_{i.})^2 \quad .$$

Note that $U^{(1)}$ and $U^{(2)}$ are essentially U-statistics based on non-identically distributed random variables as outlined in Hoeffding [1948]. The only slight difference is that Hoeffding prefers to normalize each U-statistic differently.

Since $E[f^{*(1)}(X_{\alpha_1}, X_{\alpha_2})] = w_{\alpha_1} w_{\alpha_2} (2\sigma_A^2 + \sigma_e^2 (J_{\alpha_1}^{-1} + J_{\alpha_2}^{-1}))$,

and $E[f^{*(2)}(X_{\alpha_1})] = u_{\alpha_1} (J_{\alpha_1} - 1) \sigma_e^2$, one obtains

$$E(U^{(1)}) = (W^2 - \sum_{i=1}^{I} w_i^2) \sigma_A^2 + (\sum_{i=1}^{I} w_i (W - w_i) J_i^{-1}) \sigma_e^2 ,$$

$$E(U^{(2)}) = \sum_{i=1}^{I} u_i (J_i - 1) \sigma_e^2 \quad . \qquad (2.2)$$

Hence an unbiased estimate of σ_A^2 can be obtained from

$$\hat{\sigma}_A^2 = A^{-1}\ (U^{(1)} - U^{(2)}\ .\ B\ .\ C^{-1}) \text{ where}$$

$$A = (W^2 - \sum_{i=1}^{I} w_i^2) \tag{2.3}$$

$$B = \sum_{i=1}^{I} w_i\ (W-w_i)\ J_i^{-1}$$

$$C = \sum_{i=1}^{I} u_i\ (J_i-1)\ .$$

Thus to obtain var $(\hat{\sigma}_A^2)$, one only needs var $(U^{(1)})$, var $(U^{(2)})$, and cov $(U^{(1)},U^{(2)})$.

Using Hoeffding's notation,

$$\text{var}\ (U^{(1)}) = \binom{I}{2} \sum_{c=1}^{2} \binom{2}{c} \binom{I-2}{2-c} \binom{I}{c\quad (2-c)\quad (2-c)}^{-1} \tag{2.4}$$

$$\cdot \sum \zeta^{(1,1)}_{c(\alpha_1,\ldots,\alpha_c)\beta_1,\ldots,2-c;\gamma_1,\ldots,\gamma_{2-c}},$$

where $\sum$ is over all disjoint sets of integers $\{\alpha_1,\ldots,\alpha_c\}$, $\{\beta_1,\ldots,\beta_{2-c}\}$, $\{\gamma_1,\ldots,\gamma_{2-c}\}$ chosen from $(1,\ldots,I)$, and where

$$\zeta^{(1,1)}_{c(\alpha_1,\ldots,\ \alpha_c)\ \beta_1,\ldots,\ \beta_{2-c};\ \gamma_1,\ldots,\ \gamma_{2-c}} =$$

$$\text{cov}\ (f^{*(1)}_{c\ \beta_1,\ldots,\ \beta_{2-c}}(X_{\alpha_1},\ldots, X_{\alpha_c})\ ,\ f_{c\ \alpha_1,\ldots,\ \alpha_{2-c}}(X_{\alpha_1},\ldots, X_{\alpha_c}))$$

and $f^{*(1)}_{c\ \beta_1,\ldots,\ \beta_{2-c}}(X_{\alpha_1},\ldots, X_{\alpha_c}) =$

$$E\ \{\ f^{*(1)}(X_{\alpha_1},\ X_{\alpha_2},\ldots,X_{\alpha_c},\ X_{\beta_1},\ldots,X_{\beta_{2-c}}\ |\ X_{\alpha_1} = x_{\alpha_1},\ldots,X_{\alpha_c} = x_{\alpha_c})\ \}$$

Thus $$\zeta_{1\ (\alpha_1)\ \beta_1;\gamma_1}^{(1,1)} = \text{cov}\ (f_{1\ \beta_1}^{*(1)}\ (X_{\alpha_1}),\ f_{1\ \gamma_1}^{*(1)}\ (X_{\alpha_1}))$$

$$= w_{\alpha_1}^2\ w_{\beta_1}\ w_{\gamma_1}\ \text{var}\ (Y_{\alpha_1 .}^2) \tag{2.5}$$

$$= w_{\alpha_1}^2\ w_{\beta_1}\ w_{\gamma_1}\ (2\ (\sigma_A^2 + \sigma_e^2\ J_{\alpha_1}^{-1})^2 + \gamma_A \sigma_A^4 + J_{\alpha_1}^{-3} \gamma_e \sigma_e^4),$$

and $$\zeta_2^{(1,1)}\ (\alpha_1,\alpha_2) = \text{cov}\ (f_2^{*(1)}\ (X_{\alpha_1}, X_{\alpha_2})\ ,\ f_2^{*(1)}\ (X_{\alpha_1}, X_{\alpha_2}))$$

$$= w_{\alpha_1}^2\ w_{\alpha_2}^2\ \text{var}\ (Y_{\alpha_1 .} - Y_{\alpha_2 .})^2 \tag{2.6}$$

$$= w_{\alpha_1}^2\ w_{\alpha_2}^2\ (2(2\sigma_A^2 + (J_{\alpha_1}^{-1} + J_{\alpha_2}^{-1})\sigma_e^2)^2 + 2\gamma_A \sigma_A^4 + (J_{\alpha_1}^{-3} + J_{\alpha_2}^{-3})\ \gamma_e \sigma_e^4)\ .$$

Next, one obtains,

$$\text{var}\ (U^{(2)}) = \sum_{\alpha_1=1}^{I} \zeta_{1(\alpha_1)}^{(2,2)}\ , \tag{2.7}$$

where $$\zeta_{1\ (\alpha_1)}^{(2,2)} = \text{var}\ (\ f^{*(2)}\ (X_{\alpha_1}))$$

$$= u_{\alpha_1}^2\ J_{\alpha_1}^{-1}\ (J_{\alpha_1} - 1)^2\ (\gamma_e + 3 - (J_{\alpha_1} - 3)\ (J_{\alpha_1} - 1)^{-1})\ \sigma_e^4\ . \tag{2.8}$$

Finally,

$$\text{cov}\ (U^{(1)},\ U^{(2)}) = \sum \zeta_{1\ (\alpha_1)\ \beta_1}^{(1,2)}\ , \tag{2.9}$$

where $\sum$ is over all disjoint sets of integers $\{\alpha_1\}$, $\{\beta_1\}$ from $(1, \ldots, I)$,

and where
$$\zeta^{(1,2)}_{1(\alpha_1)\beta_1} = \operatorname{cov}\left(f^{*(1)}_{1\beta_1}(X_{\alpha_1}),\ f^{*(2)}_{1}(X_{\alpha_1})\right)$$
$$= w_{\alpha_1} w_{\beta_1} u_{\alpha_1} \operatorname{cov}\left(Y^2_{\alpha_1 .},\ \sum_{j=1}^{J_{\alpha_1}} (Y_{\alpha_{ij}} - Y_{\alpha_1 .})^2\right) \tag{2.10}$$
$$= w_{\alpha_1} w_{\beta_1} u_{\alpha_1} (J_{\alpha_1} - 1)\, J^{-2}_{\alpha_1} \gamma_e \sigma^4_e \, .$$

Next, one may readily verify that,

$$\sum_{\{\alpha_1\},\{\beta_1\},\{\gamma_1\}} h(J_{\alpha_1}) w^2_{\alpha_1} w_{\beta_1} w_{\gamma_1} = \sum_{i=1}^{I} h(J_i) w^2_i (W - w_i)^2$$
$$+ \sum_{i=1}^{I} h(J_i) w^4_i - \left(\sum_{i=1}^{I} w^2_i\right)\left(\sum_{j=1}^{I} h(J_j) w^2_j\right) , \tag{2.11}$$

where $h(J_i)$, $g(J_i)$ are any functions in J_i,

$$\sum_{\{\alpha_1\},\{\beta_1\}\{\gamma_1\}} h(J_{\beta_1}) w^2_{\alpha_1} w_{\beta_1} w_{\gamma_1} = W \left(\sum_{i=1}^{I} h(J_i) w_i\right)\left(\sum_{i=1}^{I} w^2_j\right)$$
$$- W \left(\sum_{i=1}^{I} w^3_i h(J_i) - \left(\sum_{i=1}^{I} w^3_i\right)\left(\sum_{j=1}^{I} w_j h(J_j)\right)\right) \tag*{$\cancel{(2.12)}$}$$
$$+ 2 \sum_{i=1}^{I} w^4_i h(J_i) - \left(\sum_{i=1}^{I} w^2_i\right)\left(\sum_{j=1}^{I} w^2_j h(J_j)\right) ,$$

$$\sum_{\{\alpha_1,\alpha_2\}} w^2_{\alpha_1} w^2_{\alpha_2} h(J_{\alpha_1}) = \left[\left(\sum_{i=1}^{I} w^2_i\right)\left(\sum_{j=1}^{I} w^2_j h(J_j)\right) - \left(\sum_{i=1}^{I} w^4_i h(J_i)\right)\right] / 2 \tag{2.12}$$

and $$\sum_{\{\alpha_1, \alpha_2\}} w_{\alpha_1}^2 w_{\alpha_2}^2 h(J_{\alpha_1}) g(J_{\alpha_2})$$

$$= \left[\left(\sum_{i=1}^{I} w_i^2 h(J_i)\right) \left(\sum_{j=1}^{I} w_j^2 g(J_j)\right) - \left(\sum_{i=1}^{I} w_i^4 h(J_i) g(J_i)\right) \right] / 2. \quad (2.13)$$

In each of the above summations on the left hand side, the sum is overall disjoint sets in the subscripts.

Recall that

$$W = \sum_{i=1}^{I} w_i ,$$

$$A = (W^2 - \sum_{i=1}^{I} w_i^2) ,$$

$$B = \sum_{i=1}^{I} w_i (W - w_i) J_i^{-1} ,$$

$$C = \sum_{i=1}^{I} u_i (J_i - 1) \quad \text{from (2.3).}$$

Let $$D = \sum_{i=1}^{I} w_i^2 (W - w_i)^2$$

$$E = \left(\sum_{i=1}^{I} w_i^2\right)^2 - \sum_{i=1}^{I} w_i^4 ,$$

$$F = \sum_{i=1}^{I} w_i^2 J_i^{-1} (W - w_i)^2 + \left(\sum_{i=1}^{I} w_i^2\right) \left(\sum_{j=1}^{I} w_j^2 J_j^{-1}\right) - \sum_{i=1}^{I} w_i^4 J_i^{-1} ,$$

$$G = \sum_{i=1}^{I} w_i^2 J_i^{-3} (W-w_i)^2$$

$$H = \sum_{i=1}^{I} w_i^2 J_i^{-2} (W-w_i)^2 - \sum_{i=1}^{I} w_i^4 J_i^{-2} + \left(\sum_{i=1}^{I} w_i^2 J_i^{-1}\right)^2 ,$$

2.14 ~~(2.15)~~

$$L = \sum_{i=1}^{I} u_i^2 (J_i-1)^2 J_i^{-1} ,$$

$$M = \sum_{i=1}^{I} u_i^2 (J_i-1) ,$$

and

$$N = \sum_{i=1}^{I} (W-w_i) w_i u_i (J_i-1) J_i^{-2} .$$

Thus, using (2.11)-(2.14) in (2.4), (2.7) and (2.9), one obtains ,

$$\text{var } (U^{(1)}) = (D(\gamma_A + 2) + 2 \, . \, E) \sigma_A^4 + 4F \sigma_A^2 \sigma_e^2$$

2.15 ~~(2.16)~~

$$+ (G \gamma_e + 2 H) \sigma_e^4 ,$$

$$\text{var } (U^{(2)}) = (L \gamma_e + 2M) \sigma_e^4 ,$$

2.16 ~~(2.17)~~

and

$$\text{cov } (U^{(1)}, U^{(2)}) = N \gamma_e \sigma_e^4 .$$

2.17 ~~(2.18)~~

Then, from (2.3), (2.14 ~~2.15~~) - (2.17 ~~2.18~~), one obtains

$$\text{var } (\hat{\sigma}_A^2) = A^{-2} \{ (D (\gamma_A + 2) + 2E) \sigma_A^4 + 4 F \sigma_A^2 \sigma_e^2$$

2.18 ~~(2.19)~~

$$+ ((G - 2 BC^{-1} N + B^2 C^{-2} L) \gamma_e + 2H + 2B^2 C^{-2} M) \sigma_e^4 .$$

If the data are normal, $\gamma_A = \gamma_e = 0$.

Note that ~~(2.19)~~ 2.18 agrees with Tukey [1957], even though the present approach and notation is substantially different (note that a factor of 2 was omitted from the coefficient of k_{22} in var {between} on page 52). The question of the selection of the weights $\{w_i\}$, $\{u_i\}$ remains a difficult one although it is discussed by Tukey. However, if one is also interested in making a test or confidence interval for σ_A^2, a choice has been suggested in Arvesen [1969]. That is, the jackknife technique can be profitably employed to obtain (asymptotically) a test or confidence interval for σ_A^2 if one chooses the weights $w_i = 1$, $u_i = (J_i-1)^{-1}$. This unconventional choice of weights is discussed in Tukey [1957], and they seem to perform well in the estimation problem, especially if γ_A, γ_e are positive.

3. Balanced two-way classifications.

Consider the model

$$Y_{ijk} = \mu + a(u_i) + b(v_j) + c(u_i, v_j) + e(u_i, v_j)_k ,$$

$$i = 1,\ldots,I,\ j = 1,\ldots,J,\ k = 1,\ldots,k ,$$

where the $a(\cdot)$, $b(\cdot)$, $c(\cdot,\cdot)$ represent the two main and interaction random effects, and the $\{e(u_i,v_j)\}_k$ form the error variance, while μ is the overall mean. In addition, the $\{a(u_i)\}$, $\{b(v_j)\}$, and $\{e(u_i,v_j\}_k$ are independent random variables. The $\{c(u_i, v_j)\}$ are uncorrelated random variables, are also uncorrelated with the other random variables, and are independent of the $\{e(u_i, v_j)\}_k$. The following are assumptions on the moments of these random variables

	Means	Variances	Kurtoses
$\{a(u_i)\}$ main random effects	0	σ_A^2	γ_A
$\{b(v_j)\}$ main random effects	0	σ_B^2	γ_B
$\{c(u_i,v_j)\}$ interaction random effects	0	σ_{AB}^2	γ_{AB}
$\{e(u_i,v_j)\}_k$ error random effects	0	σ_e^2	γ_e

All terms above are finite. This model is described in more detail in Cornfield and Tukey [1956], and Scheffé [1959].

As in section 2, let us now form the relevant U-statistics involved in estimation of σ_B^2. Define

$$f^{*(1)}(u_1,\ldots,u_I \,;\, v_{\beta_1}, v_{\beta_2}) = (Y_{.\beta_1.} - Y_{.\beta_2.})^2/2 \quad ,$$

$$f^{*(2)}(u_{\alpha_1}, u_{\alpha_2} \,;\, v_{\beta_1}, v_{\beta_2}) = (Y_{\alpha_1\beta_1.} - Y_{\alpha_1\beta_1} - Y_{\alpha_2\beta_1.} + Y_{\alpha_2\beta_2.})^2/4 \quad ,$$

$$U^{(1)} = \binom{J}{2}^{-1} \sum_{\beta_1<\beta_2} f^{*(1)}(u_1,\ldots,u_I \,;\, v_{\beta_1}, v_{\beta_2}) = (J-1)^{-1} \sum_{j=1}^{I} (Y_{.j.} - Y_{...})^2 \quad , \tag{3.1}$$

$$U^{(2)} = \binom{I}{2}^{-1} \binom{J}{2}^{-1} \sum_{\substack{\alpha_1<\alpha_2 \\ \beta_1<\beta_2}} f^{*(2)}(u_{\alpha_1}, u_{\alpha_2} \,;\, v_{\beta_1}, v_{\beta_2}) =$$

$$((I-1)(J-1))^{-1} \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij.} - Y_{i..} - Y_{.j.} + Y_{...})^2 \; .$$

An unbiased estimate of σ_B^2 is readily seen to be

$$\hat{\sigma}_\beta^2 = U^{(1)} - U^{(2)} / I \; . \tag{3.2}$$

To obtain var $(\hat{\sigma}_B^2)$, one only needs var $(U^{(1)})$, var $(U^{(2)})$, cov $(U^{(1)}, U^{(2)})$. One can also see that

$$\mathrm{var}\,(U^{(1)}) = \binom{J}{2}^{-1} \sum_{c=0}^{2} \binom{2}{c} \binom{J-2}{2-c} \zeta_{I;c}^{(1,1)} \tag{3.3}$$

where $\zeta_{I;c}^{(1,1)} = \mathrm{var}\,(f_{I;c}^{*(1)}(u_1,\ldots,u_I \,;\, v_{\beta_1},\ldots,v_{\beta_c}))$, and

$$f_{I;c}^{*(1)}(u_1,\ldots,u_I \,;\, v_{\beta_1},\ldots,v_{\beta_c}) =$$

$$E\,[\, f^{*(1)}(u_1,\ldots,u_I \,;\, v_{\beta_1} v_{\beta_2}) \mid u_1,\ldots,u_I \,;\, v_{\beta_1},\ldots,v_{\beta_c}] \; , \quad c = 0,1,2 \quad .$$

Let

$$\sigma_\tau^2 = \mathrm{var}\ (E\ [c^2(u_1,v_1)|u_1])\ ,$$

$$\sigma_\nu^2 = \mathrm{var}\ (E\ [c^2(u_1,v_1)|v_1])\ ,$$

$$\begin{aligned}\sigma_\omega^2 &= \mathrm{var}\ (E\ [c(u_1,v_1)\ .\ c(u_2,v_1)|u_1,u_2]) \\ &= \mathrm{var}\ (E\ [c(u_1,v_1)\ .\ c(u_1,v_2)|v_1,v_2])\ ,\end{aligned} \tag{3.4}$$

$$\sigma_\chi^2 = \mathrm{var}\ (E\ [b(v_1)\ .\ c(u_1,v_1)|u_1])\ ,$$

$$\sigma_\xi^2 = \mathrm{cov}\ (b^2(v_1),\ c^2(u_1,v_1))\ .$$

Then one can obtain,

$$\zeta_{I;0}^{(1,1)} = I^{-3}\ (\sigma_\tau^2 + 2(I-1)\ \sigma_\omega^2) + 4\ I^{-1}\ \sigma_\chi^2\ ,$$

$$\begin{aligned}\zeta_{I;1}^{(1,1)} = (1/4)\ (\ &(\gamma_B + 2)\ \sigma_B^4 + I^{-2}\ (I^{-1}\ \gamma_{AB} + 2)\ \sigma_{AB}^4\ , \\ &+ (KI)^{-2}\ (\ (KI)^{-1}\ \gamma_e + 2)\ \sigma_e^4 + 4I^{-1}\ \sigma_B^2\ \sigma_{AB}^2 \\ &+ 4I^{-2}\ K^{-1}\ \sigma_{AB}^2\ \sigma_e^2 + 4I^{-1}\ K^{-1}\ \sigma_B^2\ \sigma_e^2\ + 3I^{-3}\ \sigma_\tau^2 \\ &+ 3I^{-3}\ (I-1)\ \sigma_\nu^2 + 6I^{-3}\ (I-1)\ \sigma_\omega^2 + 12I^{-1}\ \sigma_\chi^2 + 6I^{-1}\ \sigma_\xi^2)\ ,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\zeta_{I;2}^{(1,1)} = (1/2)\ (\ &(\gamma_B + 4)\ \sigma_B^4 + I^{-2}\ (I^{-1}\ \gamma_{AB} + 4)\ \sigma_{AB}^4 + (KI)^{-2}\ (\ (KI)^{-1}\ \gamma_e + 4)\ \sigma_e^4 \\ &+ 8I^{-1}\ \sigma_B^2\ \sigma_{AB}^2 + 8I^{-2}\ K^{-1}\ \sigma_{AB}^2\ \sigma_e^2 + 8I^{-1}\ K^{-1}\ \sigma_B^2\ \sigma_e^2 + 3I^{-3}\ \sigma_\tau^2 \\ &+ 3I^{-3}\ (I-1)\ \sigma_\nu^2 + 6I^{-3}\ (I-1)\ \sigma_\omega^2 + 12\ I^{-1}\ \sigma_\chi^2 + 6I^{-1}\ \sigma_\xi^2)\ .\end{aligned}$$

Next, note that

$$\operatorname{var}(U^{(2)}) = \left(\binom{I}{2} \binom{J}{2} \right)^{-1} \sum_{c_1=0}^{2} \sum_{c_2=0}^{2} \binom{2}{c_1} \binom{I-2}{2-c_1} \binom{2}{c_2} \binom{J-2}{2-c_2} \zeta^{(2,2)}_{c_1;c_2} , \qquad (3.6)$$

where $\zeta^{(2,2)}_{c_1;c_2} = \operatorname{var}(f^{*(2)}_{c_1;c_2}(u_{\alpha_1},\ldots,u_{\alpha_{c_1}} ; v_{\beta_1},\ldots,v_{\beta_{c_2}}))$, and

$$f^{*(2)}_{c_1;c_2}(u_{\alpha_1},\ldots,u_{\alpha_{c_1}} ; v_{\beta_1},\ldots,v_{\beta_{c_2}})$$
$$= E\,[f^{*(2)}(u_{\alpha_1},u_{\alpha_2} ; v_{\beta_1},v_{\beta_2}) \mid u_{\alpha_1},\ldots,u_{\alpha_{c_1}} ; v_{\beta_1},\ldots,v_{\beta_1}]$$

One then obtains,

$$\xi^{(2,2)}_{1;0} = (1/4)\,\sigma^2_\tau$$

$$\zeta^{(2,2)}_{0;1} = (1/4)\,\sigma^2_\nu$$

$$\zeta^{(2,2)}_{1;1} = (1/16)\,((\gamma_{AB}+2)\,\sigma^4_{AB} + K^{-2}(K^{-1}\,\gamma_e+2)\,\sigma^4_e + 4K^{-1}\,\sigma^2_{AB}\,\sigma^2_e + 3\sigma^2_\tau + 3\sigma^2_\nu),$$

$$\zeta^{(2,2)}_{2;0} = (1/2)\,(\sigma^2_\tau + 2\sigma^2_\omega) ,$$

$$\zeta^{(2,2)}_{0;2} = (1/2)\,(\sigma^2_\nu + 2\,\sigma^2_\omega) , \qquad (3.7)$$

$$\zeta^{(2,2)}_{2;1} = (1/8)\,((\gamma_{AB}+4)\,\sigma^4_{AB} + K^{-2}\,(K^{-1}\,\gamma_e+4)\,\sigma^4_e + 8K^{-1}\,\sigma^2_{AB}\,\sigma^2_e + 3\sigma^2_\tau + 3\sigma^2_\nu + 6^2_{\sigma_\omega}) ,$$

$$\zeta^{(2,2)}_{1;2} = (1/8)\,((\gamma_{AB}+4)\,\sigma^4_{AB} + K^{-2}\,(K^{-1}\,\gamma_e+4)\,\sigma^4_e+8K^{-1}\,\sigma^2_{AB}\,\sigma^2_e + 3\sigma^2_\tau + 3\sigma^2_\nu +6\sigma^2_\omega) ,$$

$$\zeta^{(2,2)}_{2;2} = (1/4)\,((\gamma_{AB}+8)\,\sigma^4_{AB} + K^{-2}\,(K^{-1}\,\gamma_e+8)\,\sigma^4_e + 16K^{-1}\,\sigma^2_{AB}\,\sigma^2_e + 3\sigma^2_\tau + 3\sigma^2_\nu + 6\sigma^2_\omega) .$$

Finally, one obtains

$$\operatorname{cov}(U^{(1)}, U^{(2)}) = \binom{J}{2}^{-1} \sum_{c=0}^{2} \binom{2}{c} \binom{J-2}{2-c} \zeta_{2;c}^{(1,2)} , \tag{3.8}$$

where $\zeta_{2;c}^{(1,2)} = \operatorname{cov}\left(f_{I;c}^{*(1)}(u_1,\ldots,u_I,\ v_{\beta_1},\ldots,v_{\beta_c}),\ f_{2;c}^{*(2)}(u_{\alpha_1}, u_{\alpha_2}\ ;\ v_{\beta_1},\ldots,v_{\beta_c})\right)$.

From the definitions in (3.1), one obtains

$$\zeta_{2;0}^{(1,2)} = I^{-2}\ (\sigma_\tau^2 - 2\sigma_\omega^2) ,$$

$$\zeta_{2;1}^{(1,2)} = (1/4)\ I^{-2}\ (\gamma_{AB}\sigma_{AB}^4 + K^{-3}\ \gamma_e\ \sigma_e^4 + 3\sigma_\tau^2 + (I-3)\sigma_\nu^2 - 6\sigma_\omega^2 + I^2\ \sigma_\xi^2) , \tag{3.9}$$

$$\zeta_{2;2}^{(1,2)} = (1/2)\ I^{-2}\ (\gamma_{AB}\sigma_{AB}^4 + K^{-3}\ \gamma_e\sigma_e^4 + 3\sigma_\tau^2 + (I-3)\sigma_\nu^2 + 2(I-3)\sigma_\omega^2 + 2I^2\ \sigma_\chi^2 + I^2\ \sigma_\xi^2) .$$

Hence, combining (3.5), (3.7), (3.9) and (3.3), (3.6), (3.8), one obtains from (3.2) ,

$$\begin{aligned} \operatorname{var}(\hat\sigma_B^2) = {} & (J^{-1}\gamma_B + 2(J-1)^{-1})\sigma_B^4 + 2(I(I-1)(J-1))^{-1}\sigma_{AB}^4 + 2(K^2 I(I-1)(J-1)^{-1}\ \sigma_e^4 \\ & + 4\ (I(J-1))^{-1}\sigma_B^2\sigma_{AB}^2 + 4(KI\ (I-1)(J-1))^{-1}\sigma_{AB}^2\sigma_e^2 + 4(KI\ (J-1))^{-1}\ \sigma_B^2\ \sigma_e^2 \\ & + 2\ (I(I-1)J)^{-1}\ \sigma_\nu^2 + 2(I(I-1))^{-1}\ (J(J-1)^{-1} - 2\ J^{-1})\ \sigma_\omega^2 \\ & + 4(IJ)^{-1}\ ((J-1) + (J-1)^{-1})\ \sigma_\chi^2 + 4(IJ)^{-1}\ \sigma_\xi^2 . \end{aligned} \tag{3.10}$$

If the random effects are normal, $\gamma_B = \sigma_\nu^2 = \sigma_\omega^2 = \sigma_\chi^2 = \sigma_\xi^2 = 0$. This expression agrees with Hooke [1956b], except for the coefficient of σ_χ^2. The results were obtained by very different techniques.

The most striking aspect of the above expression for var $(\hat\sigma_B^2)$ is that unless <u>both</u> I, J $\to \infty$, one does not in general obtain a consistent estimate of σ_B^2. Thus if either σ_ω^2 or σ_χ^2 is non-zero, one needs both I, J $\to \infty$. Roughly speaking, σ_χ^2 is non-zero if interactions depend on the first main factor, and σ_ω^2 is non-zero if interactions depend on either main factor. This is in marked contrast to the situation where normality is assumed in which one needs only $J \to \infty$ for a consistent estimate of σ_B^2 (see Scheffé [1959]).

Finally, note that in this balanced model, (3.10) does not involve the kurtosis of the interaction or error random effects. This phenomenon was discussed earlier in Tukey [1956b].

4. Unbalanced two-way classifications.

Consider the model discussed in section 3, except omit the assumption of balance. That is

$$Y_{ijk} = \mu + a(u_i) + b(v_j) + c(u_i, v_j) + e(u_i, v_j)_k,$$

$$i = 1,\ldots,I \quad j = 1,\ldots,J, \; k = 1,\ldots,K_{ij} \; .$$

In what follows, assume $K_{ij} \geq 1$ for each pair (i,j), and that $K_{ij} \geq 2$ for some pair (i,j). Unfortunately, this assumption eliminates consideration of many interesting designs (e.g. BIBD, cells with no entries). It would be worthwhile to investigate how crucial this assumption is to what follows.

Several methods of obtaining estimates of the variance components in such unbalanced models are due to Henderson [1953] and are also discussed in Searle [1968]. Also, Harville [1969] has given expressions for the variance of estimates in a balanced incomplete block design. In an earlier result, Arvesen [1969] suggested an alternate method of obtaining estimates of variance components in unbalanced models. One advantage of the latter method is that asymptotically robust (against non-normality) tests or confidence intervals for the variance components can be made using the jackknife technique. It is this latter method that will be discussed in what follows. Expressions for the other weighting schemes discussed above are also possible using the U-statistic technique, but have been omitted because of their complex interpretation.

Let $\{w_i\}$ be a set of arbitrary weights, $W = \sum_{i=1}^{I} w_i$,

$$f^{*(1)}(u_1,\ldots,u_I \, ; \, v_{\beta_1}, v_{\beta_2}) = (W^{-1} \sum_{i=1}^{I} w_i \, (Y_{i\beta_1.} - Y_{i\beta_2.}))^2/2 \, ,$$

$$f^{*(2)}(u_{\alpha_1}, u_{\alpha_2} \, ; \, v_{\beta_1}, v_{\beta_2}) = (Y_{\alpha_1\beta_1.} - Y_{\alpha_1\beta_2.} - Y_{\alpha_2\beta_1.} + Y_{\alpha_2\beta_2.})^2/4 \qquad (4.1)$$

$$f^{*(3)}(u_{\alpha_1} \, ; \, v_{\beta_1}) = \sum_{k=1}^{K_{\alpha_1\beta_1}} (Y_{\alpha_1\beta_1 k} - Y_{\alpha_1\beta_1.})^2/(K_{\alpha_1\beta_1} - 1) \quad ,$$

where the last kernel is defined only for those cells where $K_{\alpha_1\beta_1} \geq 2$. Using (4.1), now define,

$$U^{(1)} = \binom{J}{2}^{-1} \sum_{\beta_1<\beta_2} f^{*(1)}(u_1,\ldots,u_I \, ; \, v_{\beta_1}, v_{\beta_2})$$

$$= (J-1)^{-1} \sum_{j=1}^{J} [W^{-1} \sum_{i=1}^{I} w_i \, (Y_{ij.} - J^{-1} \sum_{j'=1}^{J} Y_{ij'.})]^2 \, ,$$

$$U^{(2)} = [\binom{I}{2} \binom{J}{2}]^{-1} \sum_{\substack{\alpha_1<\alpha_2 \\ \beta_1<\beta_2}} f^{*(2)}(u_{\alpha_1}, u_{\alpha_2} \, ; \, v_{\beta_1}, v_{\beta_2}) \qquad (4.2)$$

$$= [\, (I-1)\,(J-1) \,]^{-1} \{ \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij.} - I^{-1} \sum_{i'=1}^{I} Y_{i'j.} - J^{-1} \sum_{j'=1}^{J} Y_{ij'} + (IJ)^{-1} \sum_{i'=1}^{I} \sum_{j'=1}^{J} Y_{i'j'.} \}^2 \, ,$$

$$U^{(3)} = c^{-1} \sum_{c} f^{*(3)}(u_{\alpha_1} ; v_{\beta_1}) = c^{-1} \sum (K_{\alpha_1\beta_1} - 1)^{-1} \sum_{k=1}^{K_{\alpha_1\beta_1}} (Y_{\alpha_1\beta_1 k} - Y_{\alpha_1\beta_1.})^2$$

where in the last expression (and in what follows), $\sum_c$ indicates summation over all cells where $K_{\alpha_1\beta_1} \geq 2$, and c is the total number of such cells.

Note that the practice of pooling mean squares as in $U^{(3)}$ may actually be preferred if γ_e is positive.

Next, note that

$$E[\, f^{*(1)}(u_1,\ldots,u_I\, ;\, v_{\beta_1}, v_{\beta_2})\,] = \sigma_B^2 + W^{-2} \sum_{i=1}^{I} w_i^2\, \sigma_{AB}^2 + \sum_{i=1}^{I} w_i^2\, (K_{i\beta_1}^{-1} + K_{i\beta_2}^{-1})\, \sigma_e^2/(2w^2), \tag{4.3}$$

$$E[\, f^{*(2)}(u_{\alpha_1}, u_{\alpha_2}\, ;\, v_{\beta_1}, v_{\beta_2})\,] = \sigma_{AB}^2 + (K_{\alpha_1\beta_1}^{-1} + K_{\alpha_1\beta_2}^{-1} + K_{\alpha_2\beta_1}^{-1} + K_{\alpha_2\beta_2}^{-1})\, \sigma_e^2/4\ ,$$

$$E[\, f^{*(3)}(u_{\alpha_1}\, ;\, v_{\beta_1})\,] = \sigma_e^2\ .$$

The leading term in each of the above three expressions has coefficient one. It can be checked that estimates based on such kernels satisfy theorems 10, 11, 12, 13, and 16 of Arvesen [1969], and hence one can obtain an asymptotically robust test or confidence interval from such estimates using the jackknife technique. The estimates obtained by Henderson's methods do not have this property.

From (4.3) one obtains,

$$E(U^{(1)}) = \sigma_B^2 + W^{-2} \sum_{i=1}^{I} w_i^2\, \sigma_{AB}^2 + W^{-2} J^{-1} \sum_{j=1}^{J} \sum_{i=1}^{I} w_i^2\, K_{ij}^{-1}\, \sigma_e^2\ ,$$

$$E(U^{(2)}) = \sigma_{AB}^2 + (IJ)^{-1} \sum_{i=1}^{I} \sum_{j=1}^{J} K_{ij}^{-1}\, \sigma_e^2\ ,$$

$$E(U^{(3)}) = \sigma_e^2\ .$$

Hence, an unbiased estimate of σ_B^2 is

$$\hat{\sigma}_B^2 = U^{(1)} - A\, U^{(2)} - B\, U^{(3)} \tag{4.4}$$

where $A = W^{-2} \sum_{i=1}^{I} w_i^2$,

$$B = W^{-2} J^{-1} \sum_{i=1}^{I} \sum_{j=1}^{J} (w_i^2 - I^{-1} \sum_{i'=1}^{I} w_{i'}^2)\, K_{ij}^{-1} .$$

Note that if $w_i \equiv 1$, the coefficient of $U^{(3)}$ in (4.4) vanishes. To illustrate the use of the U-statistic technique, the algebra becomes considerably simpler if we assume $w_i \equiv 1$. Hence, the estimate to be considered is

$$\tilde{\sigma}_B^2 = U^{(1)} - I^{-1} U(2) \tag{4.5}$$

where $U^{(1)} = (J-1)^{-1} \sum_{j=1}^{J} [\, I^{-1} \sum_{i=1}^{I} (Y_{ij'.} - J^{-1} \sum_{j'=1}^{J} Y_{ij'.})\,]^2$,

and $U^{(2)}$ as in (4.2). Note that with this choice for $\{w_i\}$, the assumption that $K_{ij} \geq 2$ for some (i,j) cell may be dropped.

To obtain var $(\tilde{\sigma}_B^2)$, one needs to obtain var $(U^{(1)})$, var $(U^{(2)})$ and cov $(U^{(1)}, U^{(2)})$. As in section 2, one is dealing with U-statistics based on independent, non-identically distributed random variables. Again, var $(U^{(1)})$ is as given in (3.3), except that

$$\zeta_{I;c}^{(1,1)} = \binom{J}{c}^{-1} \sum \zeta_{I;c(\beta_1,\ldots,\beta_c)} \tag{4.6}$$

where $\sum$ is over all combinations $(\beta_1,\dots,\beta_c)$ of c integers chosen from $(1,\dots,J)$, and

$$\zeta_{I;c(\beta_1,\dots,\beta_c)} = \text{var}\,(f^{*(1)}_{I;c(\beta_1,\dots,\beta_c)}(u_1,\dots,u_I\ ;\ v_{\beta_1},\dots,v_{\beta_c}))$$

and $f^{*(1)}_{I;c(\beta_1,\dots,\beta_c)}(u_1,\dots,u_I\ ;\ v_{\beta_1},\dots,v_{\beta_c}) =$

$$E\,\{f^{*(1)}(u_1,\dots,u_I\ ;\ v_{\beta_1,}v_{\beta_2})\ |\ u_1,\dots,u_I\ ;\ v_{\beta_1},\dots,v_{\beta_c}\},\ c = 0,1,2\ .$$

The computations that follow are essentially the same as in section 3, taking account of the unbalanced design. In an attempt to simplify notation, let

$$\begin{aligned}
C &= (IJ)^{-1} \sum_{i=1}^{I} \sum_{j=1}^{J} K_{ij}^{-3}\ , \\
D &= J^{-1} \sum_{j=1}^{J} (I^{-1} \sum_{i=1}^{I} K_{ij}^{-1})^2\ , \\
D' &= I^{-1} \sum_{i=1}^{I} (J^{-1} \sum_{j=1}^{J} K_{ij}^{-1})^2\ , \\
E &= (IJ)^{-1} \sum_{i=1}^{I} \sum_{j=1}^{J} K_{ij}^{-1}\ , \\
F &= (IJ)^{-1} \sum_{i=1}^{I} \sum_{j=1}^{J} K_{ij}^{-2}\ .
\end{aligned} \tag{4.7}$$

Then one obtains,

$$\zeta^{(1,1)}_{I;0} = I^{-3}(\sigma^2_\tau + 2(I-1)\sigma^2_\omega) + 4I^{-1}\,\sigma^2_\chi ,$$

$$\zeta^{(1,1)}_{I;1(\beta_1)} = (1/4)[(\gamma_B+2)\sigma^4_B + I^{-2}(I^{-1}\gamma_{AB}+2)\sigma^4_{AB} + I^{-4}(\sum_{i=1}^{I} K^{-3}_{i\beta_1}\gamma_e + 2(\sum_{i=1}^{I} K^{-1}_{i\beta_1})^2\,\sigma^4_e$$

$$+ 4I^{-1}\,\sigma^2_B\,\sigma^2_{AB} + 4I^{-3}\sum_{i=1}^{I} K^{-1}_{i\beta_1}\,\sigma^2_{AB}\,\sigma^2_e + 4I^{-2}\sum_{i=1}^{I} K^{-1}_{i\beta_1}\,\sigma^2_B\,\sigma^2_e$$

$$+ 3I^{-3}\,\sigma^2_\tau + 3I^{-3}(I-1)\sigma^2_\nu + 6I^{-3}(I-1)\sigma^2_\omega + 12I^{-1}\,\sigma^2_\chi + 6I^{-1}\,\sigma^2_\xi] ,$$

$$\zeta^{(1,1)}_{I;1} = (1/4)[(\gamma_B+2)\sigma^4_B + I^{-2}(I^{-1}\gamma_{AB}+2)\sigma^4_{AB} + I^{-3}(C\gamma_e+2ID)\sigma^4_e + 4I^{-1}\,\sigma^2_B\,\sigma^2_{AB}$$

$$+4I^{-2}\,E\,\sigma^2_{AB}\,\sigma^2_e + 4I^{-1}\,E\,\sigma^2_B\,\sigma^2_e + 3I^{-3}\,\sigma^2_\tau + 3I^{-3}\,(I-1)\,\sigma^2_\nu$$

$$+ 6I^{-3}\,(I-1)\,\sigma^2_\omega + 12I^{-1}\,\sigma^2_\chi + 6I^{-1}\,\sigma^2_\xi] , \tag{4.8}$$

$$\zeta^{(1,1)}_{I;2(\beta_1,\beta_2)} = (1/4)\,[2(\gamma_B+4)\sigma^4_B + 2I^{-2}\,(I^{-1}\gamma_{AB}+4)\,\sigma^4_{AB}$$

$$+ I^{-4}\,(\sum_{i=1}^{I}(K^{-3}_{i\beta_1} + K^{-3}_{i\beta_2})\,\gamma_e + 2\,(\sum_{i=1}^{I}(K^{-1}_{i\beta_1} + K^{-1}_{i\beta_2}))^2)\sigma^4_e$$

$$+ 16I^{-1}\,\sigma^2_B\sigma^2_{AB} + 8I^{-3}\sum_{i=1}^{I}(K^{-1}_{i\beta_1} + K^{-1}_{i\beta_2})\,\sigma^2_{AB}\,\sigma^2_e$$

$$+ 8I^{-2}\sum_{i=1}^{I}(K^{-1}_{i\beta_1}+K^{-1}_{i\beta_2})\,\sigma^2_B\,\sigma^2_e + 6I^{-3}\,\sigma^2_\tau + 6I^{-3}\,(I-1)\,\sigma^2_\nu$$

$$+ 12I^{-3}\,(I-1)\,\sigma^2_\omega + 24I^{-1}\,\sigma^2_\chi + 12I^{-1}\,\sigma^2_\xi] ,$$

$$\zeta_{I;2}^{(1,1)} = (1/2)\,[(\gamma_B+4)\,\sigma_B^4 + I^{-2}\,(I^{-1}\,\gamma_{AB}+4)\,\sigma_{AB}^4$$

$$+ (I^{-3}\,C\,\gamma_e + 2(I^2\,(J-1))^{-1}\,((J-2)\,D + J\,E^2))\,\sigma_e^4 + 8I^{-1}\,\sigma_B^2\,\sigma_{AB}^2$$

$$+ 8I^{-2}\,E\,\sigma_{AB}^2\,\sigma_e^2 + 8I^{-1}\,E\,\sigma_B^2\,\sigma_e^2 + 3I^{-3}\,\sigma_\tau^2 + 3I^{-3}\,(I-1)\,\sigma_\nu^2$$

$$+ 6I^{-3}\,(I-1)\,\sigma_\omega^2 + 12I^{-1}\,\sigma_\chi^2 + 6I^{-1}\,\sigma_\xi^2\,].$$

Next, note that var $(U^{(2)})$ is as given in (3.6), except that

$$\zeta_{c_1;c_2}^{(2,2)} = \left[\binom{I}{c_1}\binom{I}{c_2}\right]^{-1} \sum \zeta_{c_1(\alpha_1,\ldots,\alpha_{c_1})\,;\,c_2(\beta_1,\ldots,\beta_{c_2})}^{(2,2)} \tag{4.9}$$

where $\sum$ is over all combinations $(\alpha_1,\ldots,\alpha_{c_1})$ of c_1 integers chosen from $(1,\ldots,I)$ and $(\beta_1,\ldots,\beta_{c_2})$ of c_2 integers chosen from $(1,\ldots,J)$, and

$$\zeta_{c_1}^{(2,2)}\,(\alpha_1,\ldots,\alpha_{c_1})\,;\,c_2(\beta_1,\ldots,\beta_{c_2})$$

$$= \text{var}\,(f^{*(2)}_{c_1(\alpha_1,\ldots,\alpha_{c_1})\,;\,c_2(\beta_1,\ldots,\beta_{c_2})}\,(u_{\alpha_1},\ldots,u_{\alpha_{c_1}}; v_{\beta_1},\ldots,v_{\beta_2}))\,,$$

and

$$f^{*(2)}_{c_1(\alpha_1,\ldots,\alpha_{c_1});c_2(\beta_1,\ldots,\beta_{c_2})}\,(u_{\alpha_1},\ldots,u_{\alpha_{c_1}}\,;\,v_{\beta_1},\ldots,v_{\beta_{c_2}})$$

$$= E\left\{ f^{*(2)}\,(u_{\alpha_1},u_{\alpha_2}\,;\,v_{\beta_1},v_{\beta_2})\;\middle|\;u_{\alpha_1},\ldots,u_{\alpha_{c_1}}\,;\,v_{\beta_1},\ldots,v_{\beta_{c_2}}\right\}.$$

One then obtains,

$$
\begin{aligned}
\zeta^{(2,2)}_{1;0} &= (1/4)\,\sigma^2_\tau\,, \\
\zeta^{(2,2)}_{0;1} &= (1/4)\,\sigma^2_\nu\,, \\
\zeta^{(2,2)}_{1\,(\alpha_1);1\,(\beta_1)} &= (1/16)\Big[(\gamma_{AB}+2)\,\sigma^4_{AB} + (K^{-3}_{\alpha_1\beta_1}\,\gamma_e + 2K^{-2}_{\alpha_1\beta_1})\,\sigma^4_e + 4K^{-1}_{\alpha_1\beta_1}\,\sigma^2_{AB}\,\sigma^2_e \\
&\qquad + 3\sigma^2_\tau + 3\sigma^2_\nu\Big]\,, \\
\zeta^{(2,2)}_{1;1} &= (1/16)\Big[(\gamma_{AB}+2)\sigma^4_{AB} + (C\,\gamma_e + 2F)\sigma^4_e + 4E\,\sigma^2_{AB}\,\sigma^2_e + 3\sigma^2_\tau + 3\sigma^2_\nu\Big]\,, \\
\zeta^{(2,2)}_{2;0} &= (1/2)\,(\sigma^2_\tau + 2\sigma^2_\omega)\,, \\
\zeta^{(2,2)}_{0;2} &= (1/2)\,(\sigma^2_\nu + 2\sigma^2_\omega)\,, \qquad\qquad (4.10) \\
\zeta^{(2,2)}_{2(\alpha_1\alpha_2);1(\beta_1)} &= (1/16)\Big[2\,(\gamma_{AB}+4)\sigma^4_{AB} + \Big((K^{-3}_{\alpha_1\beta_1}+K^{-3}_{\alpha_2\beta_1})\,\gamma_e + 2\,(K^{-1}_{\alpha_1\beta_1}+K^{-1}_{\alpha_2\beta_1})^2\Big)\sigma^4_e \\
&\qquad + 8\,(K^{-1}_{\alpha_1\beta_1}+K^{-1}_{\alpha_2\beta_1})\,\sigma^2_{AB}\,\sigma^2_e + 3\,\sigma^2_\tau + 3\,\sigma^2_\nu + 6\,\sigma^2_\omega\Big]\,, \\
\zeta^{(2,2)}_{2;1} &= (1/8)\Big[(\gamma_{AB}+4)\sigma^4_{AB} + (C\,\gamma_e + 2(I-1)^{-1}\,((I-2)F + ID))\sigma^4_e + 8\,E\,\sigma^2_{AB}\,\sigma^2_e \\
&\qquad + 3\,\sigma^2_\tau + 3\,\sigma^2_\nu + 6\,\sigma^2_\omega\Big]\,, \\
\zeta^{(2,2)}_{1;2} &= (1/8)\Big[(\gamma_{AB}+4)\sigma^4_{AB} + (C\,\gamma_e + 2(J-1)^{-1}\,((J-2)F + JD)\,\sigma^4_e + 8\,E\,\sigma^2_{AB}\,\sigma^2_e \\
&\qquad + 3\,\sigma^2_\tau + 3\,\sigma^2_\nu + 6\,\sigma^2_\omega\Big]\,,
\end{aligned}
$$

$$\zeta^{(2,2)}_{2(\alpha_1,\alpha_2)\,;\,2(\beta_1,\beta_2)} = (1/16)\Big[\, 4(\gamma_{AB}+8)\sigma^4_{AB} + \Big((K^{-3}_{\alpha_1\beta_1} + K^{-3}_{\alpha_1\beta_2} + K^{-3}_{\alpha_2\beta_1} + K^{-3}_{\alpha_2\beta_2})\,\gamma_e$$

$$+ 2\,(K^{-1}_{\alpha_1\beta_1} + K^{-1}_{\alpha_1\beta_2} + K^{-1}_{\alpha_2\beta_1} + K^{-1}_{\alpha_2\beta_2})^2 \Big)\sigma^4_e$$

$$+ 4(K^{-1}_{\alpha_1\beta_1} + K^{-1}_{\alpha_1\beta_2} + K^{-1}_{\alpha_2\beta_1} + K^{-1}_{\alpha_2\beta_2})\sigma^2_{AB}\,\sigma^2_e$$

$$+ 12\sigma^2_\tau + 12\sigma^2_\nu + 24\sigma^2_\omega \,\Big]\,,$$

$$\zeta^{(2,2)}_{2;2} = (1/4)\Big[\, (\gamma_{AB}+8)\sigma^4_{AB} + \Big\{ C\,\gamma_e + 2((I-1)(J-1))^{-1}((I-2)(J-2)\,F$$

$$+ J(I-2)\,D' + I(J-2)D + IJE^2) \Big\}\,\sigma^4_e$$

$$+ 16\,E\,\sigma^2_{AB}\,\sigma^2_e + 3\sigma^2_\tau + 3\sigma^2_\nu + 6\sigma^2_\omega \,\Big]\,.$$

Finally, note that cov $(U^{(1)}, U^{(2)})$ is as in (3.8), except that

$$\zeta^{(1,2)}_{2;c} = \Big[\binom{I}{2}\binom{J}{c} \Big]^{-1} \sum \zeta^{(1,2)}_{2(\alpha_1,\alpha_2)\,;\,c(\beta_1,\ldots,\beta_c)} \tag{4.11}$$

where $\sum$ is over all combinations (α_1,α_2) of 2 integers chosen from $(1,\ldots,I)$ and $(\beta_1,\ldots,\beta_c)$ of c integers chosen from $(1,\ldots,J)$, and

$$\zeta^{(1,2)}_{2(\alpha_1,\alpha_2)\,;\,c(\beta_1,\ldots,\beta_c)} = \mathrm{cov}\Big(f^{*(1)}_{I\,;\,c(\beta_1,\ldots,\beta_c)}(u_1,\ldots,u_I\,;\,v_{\beta_1},\ldots,v_{\beta_c})\,,$$

$$f^{*(2)}_{2(\alpha_1,\alpha_2)\,;\,c(\beta_1,\ldots,\beta_c)}(u_{\alpha_1},u_{\alpha_2}\,;\,v_{\beta_1},\ldots,v_{\beta_c}) \Big)\,.$$

One then obtains,

$$\zeta^{(1,2)}_{2;0} = I^{-2}(\sigma_\tau^2 - 2\sigma_\omega^2) ,$$

$$\zeta^{(1,2)}_{2(\alpha_1,\alpha_2);1(\beta_1)} = (1/8)\, I^{-2}\Big[2\gamma_{AB}\sigma_{AB}^4 + \{(K^{-3}_{\alpha_1\beta_1} + K^{-3}_{\alpha_2\beta_1})\gamma_e + 2(K^{-1}_{\alpha_1\beta_1} - K^{-1}_{\alpha_2\beta_1})^2\}\sigma_e^4$$

$$+ 6\sigma_\tau^2 + 2(I-3)\sigma_\nu^2 - 12\sigma_\omega^2 + 2I^2\sigma_\xi^2\Big] ,$$

$$\zeta^{(1,2)}_{2;1} = (1/4)\, I^{-2}\Big[\gamma_{AB}\sigma_{AB}^4 + \{C\,\gamma_e + 2I(I-1)^{-1}(F-D)\}\,\sigma_e^4 + 3\sigma_\tau^2$$

$$+ (I-3)\sigma_\nu^2 - 6\sigma_\omega^2 + I^2\sigma_\xi^2\Big] , \qquad (4.12)$$

$$\zeta^{(1,2)}_{2(\alpha_1,\alpha_2);2(\beta_1,\beta_2)} = (1/8)\, I^{-2}\Big[4\gamma_{AB}\sigma_{AB}^4 + \{(K^{-3}_{\alpha_1\beta_1} + K^{-3}_{\alpha_1\beta_2} + K^{-3}_{\alpha_2\beta_1} + K^{-3}_{\alpha_2\beta_2})\gamma_e$$

$$+ 2(K^{-1}_{\alpha_1\beta_1} - K^{-1}_{\alpha_2\beta_1})^2$$

$$+ 4(K^{-1}_{\alpha_1\beta_1}K^{-1}_{\alpha_1\beta_2} - K^{-1}_{\alpha_1\beta_1}K^{-1}_{\alpha_2\beta_2} - K^{-1}_{\alpha_1\beta_2}K^{-1}_{\alpha_2\beta_1} + K^{-1}_{\alpha_2\beta_1}K^{-1}_{\alpha_2\beta_2})$$

$$+ 2(K^{-1}_{\alpha_1\beta_2} - K^{-1}_{\alpha_2\beta_2})^2\}\,\sigma_e^4$$

$$+ 12\sigma_\tau^2 + 4(I-3)\sigma_\nu^2 + 8(I-3)\sigma_\omega^2 + 8I^2\sigma_\chi^2 + 4I^2\sigma_\xi^2\Big] ,$$

$$\zeta^{(1,2)}_{2;2} = (1/2)\, I^{-2}\Big[\gamma_{AB}\sigma_{AB}^4 + \{C\,\gamma_e + 2I(I-1)(J-1))^{-1}((J-2)(F-D) + J(D'-E^2))\}\,\sigma_e^4$$

$$+ 3\sigma_\tau^2 + (I-3)\sigma_\nu^2 + 2(I-3)\sigma_\omega^2 + 2I^2\sigma_\chi^2 + I^2\sigma_\xi^2\Big] .$$

Combining (4.8), (4.10), and (4.12) with the expression for $\tilde{\sigma}^2_B$ in (4.5) one obtains,

$$
\begin{aligned}
\operatorname{var}(\tilde{\sigma}^2_B) = {} & (J^{-1}\gamma_B + 2(J-1)^{-1})\sigma^4_B + 2(I(I-1)(J-1)^{-1}\sigma^4_{AB} \\
& + 2(I^3(J-1)^2(I-1)^2)^{-1}\left[(I^3(J-2)+(I-1)(I-2))D+I^3E^2-I^2(J-2)F-(2I^2-3I+2)D'\right]\sigma^4_e \\
& + 4(I(J-1))^{-1}\sigma^2_B\sigma^2_{AB} + 4(E\,I(I-1)(J-1))^{-1}\sigma^2_{AB}\sigma^2_e \qquad (4.13) \\
& + 4(EI(J-1))^{-1}\sigma^2_B\sigma^2_e + 2(I(I-1)J)^{-1}\sigma^2_\nu + 2(I(I-1))^{-1}(J(J-1)^{-1} - 2J^{-1})\sigma^2_\omega \\
& + 4(IJ)^{-1}((J-1) + (J-1)^{-1})\sigma^2_\chi + 4(IJ)^{-1}\sigma^2_\xi \,.
\end{aligned}
$$

This expression agrees with (3.10) when $K_{ij} \equiv K$, and the only difference is found in coefficients of terms involving the error variance.

5. Concluding remarks.

The present author has examined variance expressions for variance components estimates using Hoeffding's U-statistic approach. It is felt that the results themselves are not of interest, but rather the method in which they were obtained. might be. That is, one might really be interested in some "optimum" selection of weights in either (2.3), (4.4) or some even more complicated design. Tukey [1956b] has done a good job of answering this question for (2.3). However, in a two-way classification, the results for arbitrary weighting schemes would be very difficult to interpret in any general sense. Even the results of Harville [1969], and the above results in section 4 are difficult to interpret.

But it is for these reasons that the U-statistic approach is felt to offer some promise. Computer calculation of terms like (4.7), (4.9), or (4.11) would be relatively simple in any specific case, and would be much more meaningful to a practitioner than an expression like (2.19), (4.12) or something undoubtedly more complicated. It is also by means of such an approach that one may possibly be able to eliminate the requirement that $K_{ij} \geq 1$ for each pair (i,j) as mentioned in section 4.

REFERENCES

[1] Arvesen, J. N. (1969). Jackknifing U-statistics. Ann. Math. Statist. 40, 2076-2100.

[2] Cornfield, J. and Tukey, J. W. (1956). Average values of mean squares in factorials. Ann. Math. Statist. 27, 907-949.

[3] Harville, D. (1969). Variances of variance-component estimators for the unbalanced 2-way cross classification with application to balanced incomplete block designs. Ann. Math. Statist. 40, 408-416.

[4] Henderson, C. R. (1953). Estimation of variance and covariance components. Biometrics 9, 226-252.

[5] Hoeffding, W. (1948). A class of statistics with asymptotically normal distribution. Ann. Math. Statist. 19, 293-325.

[6] Hooke, R. (1956a). Symmetric functions of a two-way array. Ann. Math. Statist. 27, 55-79.

[7] Hooke, R. (1956b). Some applications of bipolykays to the estimation of variance components and their moments. Ann. Math. Statist. 27, 80-98.

[8] Scheffé, H. (1959). Analysis of Variance. Wiley, New York.

[9] Searle, S. R. (1968). Another look at Henderson's methods of estimating variance components. Biometrics 24, 749-778.

[10] Tukey, J. W. (1956a). Keeping moment-like sampling computations simple. Ann. Math. Statist. 27, 37-54.

[11] Tukey, J. W. (1956b). Variances of variance components: I. Balanced designs. Ann. Math. Statist. 27, 722-736.

[12] Tukey, J. W. (1957). Variances of variance components: II. The unbalanced single classification. Ann. Math. Statist. 28, 43-56.