An Exact Comparison of the Waitingtimes under Three Priority Rules

by

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1. Introduction

This paper is a sequel to "A Priority Rule Based on the Ranking of the Service Times for the M|G|1 Queue" which appeared in Operations Research, Vol. 17, 1969, 466-477. We use the notation and concepts defined there without repeating their formal definitions.

Informally summarized, we are considering the classical M|G|1 queue with Poisson input of rate λ and service time distribution $H(\cdot)$ and we distinguish between successive "generations" of customers as in Kendall [1] and Neuts [3]. The customers present at $t = 0$ form the first generation; the new arrivals during the total time required to process them form the second generation, the third generation consists of those arriving during the service of those in the second generation and so on. This continues until the initial busy period comes to an end and starts over (regeneratively) with the arrival of the first customer in the next busy period.

The priority rules discussed in [2] consist of serving within each generation the customers in the order of shortest (SPT) or longest(LPT) service times first.

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In [2] a number of comparisons between these two rules and the first come, first served (FCFS) discipline in regards to expected waitingtimes in the equilibrium state were carried out.

Many questions involving more than the expected values can be asked. In order to answer them an exact comparison of the waitingtimes as random variables needs to be made. For example, a referee of [2], asked for the (limiting) probability that a customer "does better" under one priority rule than under each of the other two. We attempt to answer such questions in the present paper.

We denote by $\eta(t)$ the virtual waitingtime at time t, i.e. the waitingtime of a (virtual) customer joining the queue at time t. By $\bar{\eta}(t,x)$ we mean the virtual waitingtime under the LPT priority rule of a customer, arriving at t whose service time is x. Likewise $\underline{\eta}(t,x)$ is the waitingtime of this customer under the SPT rule. Clearly under the FCFS rule the waitingtime of a customer does not depend on the amount of service he requests, but it does under the other two disciplines.

In this paper we discuss the joint distribution of the random variables $\eta(t)$, $\bar{\eta}(t,x)$, $\underline{\eta}(t,x)$ and their limiting joint distribution as $t \to \infty$.

We can "visualize" the definition of these three random variables on a common probability space as follows. Imagine that a customer joining the queue at time t consists of three identical parts 1, 2, 3 each requiring a processing time $x \geq 0$. Part 1 waits in front of a server operating under the FCFS rule, part 2 in front of a server operating under the LPT rule and finally part 3 waits in front of a unit governed by the SPT rule. Then $\eta(t)$, $\bar{\eta}(t,x)$ and $\underline{\eta}(t,x)$ are the waitingtimes of parts 1, 2 and 3 respectively.

2. An Auxiliary Calculation

Consider the time points t and $t + t'$, $t > 0$, $t' > 0$. The probability that during the interval $(0, t)$, j_1 customers arrive whose service times are less than

x, j_2 whose service times are greater than x and that during $(t, t + t')$, j_3 and j_4 arrive with service times respectively less and greater than x is given by:

$$(1) \qquad e^{-\lambda t - \lambda t'} \frac{[\lambda t\, H(x)]^{j_1}}{j_1!} \frac{[\lambda t'\, H(x)]^{j_3}}{j_3!} \cdot \frac{\{\lambda t\,[1 - H(x)]\}^{j_2}}{j_2!} \cdot \frac{\{\lambda t'\,[1 - H(x)]\}^{j_4}}{j_4!}$$

We assume that x is a point of continuity of $H(\cdot)$ so that the probability that one or more customers have service times exactly equal to x is zero.

The distributions $\tilde{H}(\cdot)$ and $\tilde{\tilde{H}}(\cdot)$ are defined by:

$$(2) \qquad \tilde{H}(y) = \frac{H(y)}{H(x)}, \qquad 0 \leq y \leq x,$$

$$= 1, \qquad y \geq x,$$

$$\tilde{\tilde{H}}(y) = 0, \qquad y \leq x,$$

$$= \frac{H(y) - H(x)}{1 - H(x)}, \qquad y \geq x,$$

$\tilde{H}(\cdot)$ and $\tilde{\tilde{H}}(\cdot)$ are clearly the conditional service time distributions given the information that the required processing time of a customer is respectively less or greater than x.

Next, let U_1' and U_2' be the <u>total</u> service time of all customers in $(0,t)$ with service time respectively less and greater than x. Similarly U_3' and U_4' are the corresponding quantities for the customers arriving in $(t, t + t')$.

The following auxiliary probability mass function is of importance in the sequel. We define $W(t, t'; x_1, x_2, x_3, x_4)$ as the probability that for given $t > 0$ and $t' > 0$, the random variables U_1', U_2', U_3', and U_4' satisfy:

$$(3) \qquad U_1' \leq x_1,\ U_2' \leq x_2,\ U_3' \leq x_3,\ U_4' \leq x_4,$$

It follows readily, using (1), that:

(4) $W(t, t'; x_1, x_2, x_3, x_4) =$

$$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} e^{-\lambda t-\lambda t'} \frac{[\lambda t\ H(x)]^{j_1}}{j_1!} \frac{[\lambda t'\ H(x)]^{j_3}}{j_3!}$$

$$\cdot \frac{\{\lambda t\ [1 - H(x)]\}^{j_2}}{j_2!} \cdot \frac{\{\lambda t'\ [1 - H(x)]\}^{j_4}}{j_4!}\ \tilde{H}^{(j_1)}(x_1)\ \tilde{H}^{(j_3)}(x_3)$$

$$\cdot\ \tilde{\tilde{H}}^{(j_2)}(x_2)\ \tilde{\tilde{H}}^{(j_4)}(x_4)\ ,$$

where $\tilde{H}^{(j_1)}(x_1)$ is the j_1-th convolution power of $\tilde{H}(\cdot)$ evaluated at x_1. Similar interpretations are given to the other factors.

This expression does not simplify directly. As many formulae in applied probability it involves series in the convolution powers of distribution functions. Upon taking Laplace-Stieltjes transforms:

(5) $W^*(t, t'; s_1, s_2, s_3, s_4) =$

$$\int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-s_1x_1 - s_2x_2 - s_3x_3 - s_4x_4} d_{x_1,\dots,x_4} W(t,t'; x_1,x_2,x_3,x_4)$$

we obtain a more familiar series.

(6) $W^*(t,t'; s_1,s_2,s_3,s_4) =$

$$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} e^{-\lambda t-\lambda t'} \frac{[\lambda t\ H(x)]^{j_1}}{j_1!} \frac{[\lambda t'\ H(x)]^{j_3}}{j_3!}\ .$$

$$\cdot \frac{\{ \lambda t [1 - H(x)] \}^{j_2}}{j_2!} \frac{\{ \lambda t' [1 - H(x)] \}^{j_4}}{j_4!} \cdot \tilde{h}^{j_1}(s_1) \tilde{h}^{j_3}(s_3) \tilde{\tilde{h}}^{j_2}(s_2)$$

$$\cdot \tilde{\tilde{h}}^{j_4}(s_4) ,$$

where $\tilde{h}(s)$ and $\tilde{\tilde{h}}(s)$ are the L. S. transforms of $\tilde{H}(\cdot)$ and $\tilde{\tilde{H}}(\cdot)$.

Summing we obtain:

(7) $W^*(t,t'; s_1, s_2, s_3, s_4) =$

$$\exp \{ - \lambda t - \lambda t' + \lambda t H(x) \tilde{h}(s_1) + \lambda t [1 - H(x)] \tilde{\tilde{h}}(s_2)$$

$$+ \lambda t' H(x) \tilde{h}(s_3) + \lambda t' [1 - H(x)] \tilde{\tilde{h}}(s_4) \} ,$$

We now return to the M|G|1 queue, which we consider at time t. We define the following five random variables. U_o is the length of time beyond t until the generation of customers in service at time t completes its service. U_1 and U_2 are respectively the total service times of the customers with processing times less than and greater than x, who joined the queue between the beginning of the service of the current generation but before t. U_3 and U_4 are respectively the total service times of the customers with processing times less than and greater than x, who join the queue during the time interval $(t, t + U_o)$.

If at time t the server is idle all five variables are zero.

We express the joint distribution of the waitingtimes $\bar{\eta}(t,x)$, $\eta(t)$ and $\underline{\eta}(t,x)$ in terms of the joint distribution of the random variables U_j, $j = 0, \ldots, 4$.

3. The Joint Distribution of the U_j, $j = 0,\ldots, 4$.

$R_i^o(t, x_o, x_1, x_2, x_3, x_4)$ is the probability that in $(0,t)$ the queue has never become empty and the variables U_j, $j = 0,\ldots, 4$ associated with the timepoint t satisfy $U_j \leq x_j$, $j = 0,\ldots, 4$, given that at $t = 0$ $i \geq 1$ customers were in the queue, one beginning service at that time.

This probability is given by:

(8) $R_i^o(t, x_o, x_1, x_2, x_3, x_4) =$

$$\sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \int_0^{x_o} \int_0^{t} d\ {}_oQ_{i\nu}^{(n)}(\tau)\, d H^{(\nu)}(t + t' - \tau)$$

$$\cdot\ W(t-\tau, t'; x_1, x_2, x_3, x_4),$$

by the following argument. At some time τ prior to t, the generation in service at t enters service. There are some number $\nu \geq 1$ customers in it, so that the duration of the total service time distribution of these ν customers is the ν-fold convolution $H^{(\nu)}(\cdot)$ of $H(\cdot)$. If $U_o \leq x_o$ must hold, the total service time of these ν customers cannot exceed $t + x_o$. The other requirements $U_1 \leq x_1$, $U_2 \leq x_2$, $U_3 \leq x_3$, $U_4 \leq x_4$ account for the factor $W(t-\tau, t'; x_1, x_2, x_3, x_4)$. The probabilities ${}_oQ_{i\nu}^{(n)}(\cdot)$ were defined in [2].

Taking Laplace-Stieltjes transforms in (8), i.e. evaluating:

(9) $\tilde{R}_i^o(\xi, s_o, s_1, s_2, s_3, s_4) =$

$$\int_0^{\infty} e^{-\xi t}\, dt \int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-s_o x_o - s_1 x_1 - s_2 x_2 - s_3 x_3 - s_4 x_4}$$

$$d_{x_o, x_1, x_2, x_3, x_4}\, R_i^o(t, x_o, x_1, x_2, x_3, x_4),$$

and recalling (7) we obtain:

(10) $\tilde{R}_i^o (\xi, s_o, s_1, s_2, s_3, s_4) =$

$$\sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \int_o^{\infty} e^{-\xi t} dt \int_o^{\infty} \int_o^{\infty} \int_o^{\infty} \int_o^{\infty} \int_o^{\infty} e^{-s_o x_o - s_1 x_1 - s_2 x_2 - s_3 x_3 - s_4 x_4}$$

$$\cdot \int_o^t d_o Q_{i\nu}^{(n)} (\tau) \cdot d H^{(\nu)} (t + x_o - \tau) \cdot W (t-\tau, x_o; x_1, x_2, x_3, x_4)$$

$$= \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} {}_o q_{i\nu}^{(n)} (\xi) \cdot \int_o^{\infty} \int_o^{\infty} \exp \Big\{ - \xi t_1 - \lambda t_1 - \lambda x_o - s_o x_o$$

$$+ \lambda t_1 H(x) \tilde{h} (s_1) + \lambda t_1 [1 - H (x)] \tilde{\tilde{h}} (s_2) + \lambda x_o H (x) \tilde{h} (s_3)$$

$$+ \lambda x_o [1 - H (x)] \tilde{\tilde{h}} (s_4) \Big\} d H^{(\nu)} (t_1 + x_o) dt_1$$

$$= \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} {}_o q_{i\nu}^{(n)} (\xi) . \Big\{ h^{\nu} [\lambda + s_o - \lambda H(x) \tilde{h} (s_3) - \lambda [1 - H(x)] \tilde{\tilde{h}} (s_4)]$$

$$- h^{\nu} [\xi + \lambda - \lambda H (x) \tilde{h} (s_1) - \lambda [1 - H (x)] \tilde{\tilde{h}} (s_2)] \Big\} . \Big\{ \xi - s_o$$

$$- \lambda H (x) [\tilde{h} (s_1) - \tilde{h} (s_3)] - \lambda [1 - H (x)] [\tilde{\tilde{h}} (s_2) - \tilde{\tilde{h}} (s_4)] \Big\}^{-1}$$

$$= \Big\{ \xi - s_o - \lambda H (x) [\tilde{h} (s_1) - \tilde{h} (s_3)] - \lambda [1 - H (x)] [\tilde{\tilde{h}} (s_2) - \tilde{\tilde{h}} (s_4)] \Big\}^{-1} .$$

$$\cdot \sum_{n=0}^{\infty} \Big\{ {}_o q_i^{(n)} \Big\{ \xi, h [\lambda + s_o - \lambda H(x) \tilde{h} (s_3) - \lambda [1 - H (x)] \tilde{\tilde{h}} (s_4)] \Big\}$$

$$- {}_o q_i^{(n)} \Big\{ \xi, h [\xi + \lambda - \lambda H (x) \tilde{h} (s_1) - \lambda [1-H (x)] \tilde{\tilde{h}} (s_2)] \Big\} \Big\}$$

in terms of the functions ${}_o q_i^{(n)} (\xi, Z)$ defined in [3] .

Next, let $R_i\ (t, x_o, x_1, x_2, x_3, x_4)$ be the probability that at time t, the random variables U_j associated with t satisfy $U_j \leq x_j$, j = 0, 1, 2, 3, 4 given that at time t = 0 there were i customers in the queue.

The standard regeneration argument relates the function R_i $(t, x_o, x_1, x_2, x_3, x_4)$ to the functions $R_i^o\ (t, x_o, x_1, x_2, x_3, x_4)$ as follows:

(11) $R_i\ (t, x_o, x_1, x_2, x_3, x_4) =$

$$R_i^o\ (t, x_o, x_1, x_2, x_3, x_4) + P\left\{ \xi\ (t) = 0 \mid \xi\ (0) = i\right\} U\ (x_o, x_1, x_2, x_3, x_4)$$

$$+ \int_o^t R_1^o\ (t-u, x_o, x_1, x_2, x_3, x_4)\ d\ M_1\ (u)\ ,$$

where $P\left\{ \xi\ (t) = 0 \mid \xi\ (0) = i \right\}$ is the (conditional) probability that the queuelength $\xi\ (t) = 0$, i.e. that the server is idle at time t. $M_1\ (\cdot)$ is the renewal function of the (general) renewal process of beginnings of busy periods. $U\ (x_o, x_1, x_2, x_3, x_4)$ is the distribution degenerate at zero in five variables.

We recall that:

$$(12)\quad m_1\ (\xi) = \int_o^\infty e^{-\xi t}\ d\ M_1\ (t) = \lambda\ \gamma^i\ (\xi)\ .\ [\xi + \lambda - \lambda\gamma\ (\xi)\ ]^{-1}\ ,$$

where $\gamma\ (\xi)$ is the L. S. transform of the distribution of the busy period of the M|G|1 queue.

Also:

$$(13)\quad \int_o^\infty e^{-\xi t}\ P\ \left\{ \xi\ (t) = 0 \mid \xi\ (0) = i \right\} dt = \frac{1}{\lambda}\ m_1\ (\xi)\ ,$$

Upon taking transforms in (11) we obtain:

(14) $\tilde{R}_i\ \ (\xi, s_o, s_1, s_2, s_3, s_4) =$

$$\int_o^\infty e^{-\xi t}\ E_t \left\{ e^{-s_oU_o\ -\ s_1U_1\ -\ s_2U_2\ -\ s_3U_3\ -\ s_4U_4} \right\}\ dt$$

$$= \tilde{R}_i^{o} (\xi, s_o, s_1, s_2, s_3, s_4)$$

$$+ \gamma^{i} (\xi) [\xi + \lambda - \lambda \gamma (\xi)]^{-1} \left\{ 1 + \lambda \tilde{R}_1^{o} (\xi, s_o, s_1, s_2, s_3, s_4) \right\},$$

$\tilde{R}_i^{o} (\xi, s_o, s_1, s_2, s_3, s_4)$, $i \geq 1$, is given in (10).

When $1 - \lambda \alpha > 0$; α being the mean of $H (\cdot)$, the existence of a joint limiting distribution for U_j, $j = 0, 1, 2, 3, 4$ is guaranteed by the main limit theorem for regenerative processes and the existence of a stationary version of the imbedded Markov renewal process of the M'G|1 queue [3]. When $1 - \lambda \alpha \leq 0$, the main limit theorem for regenerative processes guarantees that $R_i (t, x_o, x_1, x_2, x_3, x_4)$ tends to zero for all i, $x_j \geq 0$, $j = 0, \ldots, 4$. Since the limiting distribution exists, when $1 - \lambda \alpha > 0$, its transform is given by:

$$(15) \quad \tilde{R} (s_o, s_1, s_2, s_3, s_4) = \lim_{\xi \to 0+} \xi \tilde{R}_i (\xi, s_o, s_1, s_2, s_3, s_4)$$

$$= (1 - \lambda \alpha) \left\{ 1 + \lambda \tilde{R}_1^{o} (0+, s_o, s_1, s_2, s_3, s_4) \right\}.$$

4. The Joint Distribution of $\overline{\eta} (t,x)$, $\eta (t)$ and $\underline{\eta} (t,x)$

The random variables $\overline{\eta} (t,x)$, $\eta (t)$ and $\underline{\eta} (t,x)$ are for each $t > o$ related to the random variables U_j, $j = 0, 1, 2, 3, 4$ associated with the timeinstant t by:

$$(16) \quad \begin{aligned} \overline{\eta} (t,x) &= U_o + U_2 + U_4, \\ \eta (t) &= U_o + U_1 + U_2, \\ \underline{\eta} (t,x) &= U_o + U_1 + U_3 \end{aligned}$$

That this is indeed so, we argue for $\overline{\eta} (t,x)$. The other cases are similar. Consider a virtual customer with service time x arriving at time t. He has to wait until all customers of the present generation, if any, have been served. This is a length of time U_o. Next, in the next generation, all customers with service time greater than x are served ahead of him. Regardless of the actual order of service the total amount of processing time required by all customers with service time exceeding x is $U_2 + U_4$. U_2 is the processing time of those who preceded him and U_4 that of those who succeeded him in the arrival sequence. We have:

(17) $$\bar{\eta}(t,x)\,\zeta_1 + \eta(t)\,\zeta_2 + \underline{\eta}(t,x)\,\zeta_3 = (\zeta_1+\zeta_2+\zeta_3)\,U_o + (\zeta_2+\zeta_3)\,U_1 + (\zeta_1+\zeta_2)\,U_2 + \zeta_3 U_3 + \zeta_1 U_4\,,$$

which implies that:

(18) $$\tilde{S}_i(\xi,\zeta_1,\zeta_2,\zeta_3) = \int_o^\infty e^{-\xi t}\, E\left[e^{-\bar{\eta}(t,x)\zeta_1 - \eta(t)\zeta_2 - \underline{\eta}(t,x)\zeta_3}\right] dt = \tilde{R}_i(\xi,\zeta_1+\zeta_2+\zeta_3,\zeta_2+\zeta_3,\zeta_1+\zeta_2,\zeta_3,\zeta_1)\,,$$

where $\tilde{R}_i(.,.,.,.,.,.)$ is given by (14).

Formula (18) shows how the joint distribution of $\bar{\eta}(t,x)$, $\eta(t)$ and $\underline{\eta}(t,x)$ is related to the basic parameters of the M|G|1 queue. We discuss the limiting joint distribution of these three variables as $t \to \infty$, in some detail.

5. The Limiting Joint Distribution

The limiting joint distribution of the three virtual waitingtimes exists if and only if $1 - \lambda\alpha > 0$. Its Laplace-Stieltjes transform is given by:

(19) $$\tilde{S}(\zeta_1,\zeta_2,\zeta_3) = (1-\lambda\alpha)\left\{1 + \lambda\,\tilde{R}_1^{\infty}(0,\zeta_1+\zeta_2+\zeta_3,\zeta_2+\zeta_3,\zeta_1+\zeta_2,\zeta_3,\zeta_1)\right\}$$

$$= (1 - \lambda\alpha) \left\{ 1 - \frac{\lambda}{\zeta_1+\zeta_2+\zeta_3 + \lambda\, H(x) [\, \tilde{h}(\zeta_2+\zeta_3) - \tilde{h}(\zeta_3)\,] + \lambda [\, 1 - H(x)\,]\, [\, \tilde{\tilde{h}}(\zeta_1+\zeta_2) - \tilde{\tilde{h}}(\zeta_1)\,]} \right.$$

$$\cdot \sum_{n=0}^{\infty} \left\{ {}_0q_1^{(n)} \left\{ 0, h\, [\lambda + \zeta_1 + \zeta_2 + \zeta_3 - \lambda H(x)\, \tilde{h}(\zeta_3) - \lambda[1-H(x)]\, \tilde{\tilde{h}}(\zeta_1)] \right\} \right.$$

$$\left. \left. - {}_0q_1^{(n)} \left\{ 0, h\, [\, \lambda - \lambda H(x)\, \tilde{h}(\zeta_2+\zeta_3) - \lambda\, [1 - H(x)]\, \tilde{\tilde{h}}(\zeta_1+\zeta_2)\,] \right\} \right\} \right\},$$

6. Moments of the Limiting Distribution of the Basic Variables U_j, $j=0,1,2,3,4$.

Let us denote:

$$\underset{\sim}{s} = (s_0, s_1, s_2, s_3, s_4)$$

$$\underset{\sim}{0} = (0, 0, 0, 0, 0)$$

$$\theta \equiv \theta(x, \underset{\sim}{s})$$

(20)
$$= s_0 + \lambda H(x)\, [\, \tilde{h}(s_1) - \tilde{h}(s_3)\,] + \lambda\, [\, 1-H(x)\,]\, [\, \tilde{\tilde{h}}(s_2) - \tilde{\tilde{h}}(s_4)\,]$$

$$Y \equiv Y(x, \underset{\sim}{s})$$

(21)
$$= h\, [\, \lambda + s_0 - \lambda H(x)\, \tilde{h}(s_3) - \lambda(1-H(x))\, \tilde{\tilde{h}}(s_4)\,]$$

$$\tilde{Y} \equiv \tilde{Y}(x, \underset{\sim}{s})$$

(22)
$$= h\, [\, \lambda - \lambda H(x)\, \tilde{h}(s_1) - \lambda\, (1-H(x))\, \tilde{\tilde{h}}(s_2)\,]$$

$$\psi_n \equiv \psi_n(x, \underset{\sim}{s})$$

(23)
$$= h_n(0, Y) - h_n(0, \tilde{Y}),\quad n \geq 0,$$

where the functions $h_n(\cdot,\cdot)$, $n \geq 0$ are defined in [2].

From (15) the L.S. transform of the limiting joint distribution of U_j, j = 0, 1, 2, 3, 4 is given by:

$$(24)\quad \tilde{R}(\underset{\sim}{s}) = (1-\lambda\alpha)\left\{1 - \frac{\lambda}{\theta} \sum_{n=0}^{\infty} \left[{}_{o}q_1^{(n)}(0,Y) - {}_{o}q_1^{(n)}(0,\tilde{Y}) \right] \right\},$$

The theorem in Appendix I of [2] implies:

$$(25)\quad {}_{o}q_1^{(n)}(0,Y) - {}_{o}q_1^{(n)}(0,\tilde{Y}) = h_n(0,Y) - h_n(0,\tilde{Y}) = \psi_n(x,\underset{\sim}{s})$$

Substitution of (25) in (24) yields:

$$(26)\quad \tilde{R}(\underset{\sim}{s}) = (1-\lambda\alpha)\left\{1 - \frac{\lambda}{\theta} \sum_{n=0}^{\infty} \psi_n \right\},$$

$E_\infty U$ denotes the expected value of the limiting distribution of U and further denote:

$$\alpha = \int_0^\infty u \, dH(u) \qquad \alpha_x = \int_0^x u \, dH(u)$$

$$\beta = \int_0^\infty u^2 \, dH(u) \qquad \beta_x = \int_0^x u^2 \, dH(u)$$

$$\gamma = \int_0^\infty u^3 \, dH(u) \qquad \gamma_x = \int_0^x u^3 \, dH(u)$$

$$\alpha_x^* = \alpha - \alpha_x \qquad \beta_x^* = \beta - \beta_x \qquad \gamma_x^* = \gamma - \gamma_x$$

Then from (26) we have:

$$E_\infty U_j = - \left.\frac{\partial \tilde{R}}{\partial s_j}\right]_{\underset{\sim}{s} = \underset{\sim}{0}}$$

(27) $$= \lambda\ (1-\lambda\alpha) \sum_{n=0}^{\infty} \left[\frac{\theta \frac{\partial\psi_n}{\partial s_j} - \frac{\partial\theta}{\partial s_j} \psi_n}{\theta^2} \right]_{\underset{\sim}{s} = 0}$$

where term by term differentiation is justified as in [2]. Applying l' Hopital's rule twice on the right hand side of (27) we get:

(28) $$E_\infty U_j = \frac{\lambda\ (1-\lambda\alpha)}{2 \left(\frac{\partial\theta}{\partial s_j}\right)^2} \sum_{n=0}^{\infty} \left[\frac{\partial\theta}{\partial s_j} \frac{\partial^2\psi_n}{\partial s_j^2} - \frac{\partial^2\theta}{\partial s_j^2} \frac{\partial\psi_n}{\partial s_j} \right] \Bigg|_{\underset{\sim}{s} = \underset{\sim}{0}},$$

$$j = 0, 1, 2, 3, 4.$$

Similarly:

$$E_\infty U_j^2 = \frac{\partial^2 \tilde{R}}{\partial s_j^2} \Big]_{\underset{\sim}{s} = \underset{\sim}{0}}$$

(29) $$= \frac{-\lambda\ (1-\lambda\alpha)}{6 \left(\frac{\partial\theta}{\partial s_j}\right)^3} \sum_{n=0}^{\infty} \left[2 \left(\frac{\partial\theta}{\partial s_j}\right)^2 \frac{\partial^3\psi_n}{\partial s_j^3} - 3 \frac{\partial\theta}{\partial s_j} \frac{\partial^2\theta}{\partial s_j^2} \frac{\partial^2\psi_n}{\partial s_j^2} \right.$$

$$\left. + \left\{ 3 \left(\frac{\partial^2\theta}{\partial s_j^2}\right)^2 - 2 \frac{\partial\theta}{\partial s_j} \frac{\partial^3\theta}{\partial s_j^3} \right\} \frac{\partial\psi_n}{\partial s_j} \right] \Bigg|_{\underset{\sim}{s} = \underset{\sim}{0}},$$

$$j = 0, 1, 2, 3, 4$$

and

$$E_\infty (U_i U_j) = \frac{\partial^2 \tilde{R}}{\partial s_i \partial s_j} \Big]_{\underset{\sim}{s} = \underset{\sim}{0}}$$

(30) $$= \frac{-\lambda\ (1-\lambda\alpha)}{6 \left(\frac{\partial\theta}{\partial s_i}\right)^3} \sum_{n=0}^{\infty} \left[- \left\{ 3 \left(\frac{\partial^2\theta}{\partial s_i^2}\right)^2 + 4 \frac{\partial\theta}{\partial s_i} \frac{\partial^3\theta}{\partial s_i^3} \right\} \frac{\partial\psi_n}{\partial s_j} \right.$$

$$\cdot + 5 \frac{\partial^3\theta}{\partial s_i^3} \frac{\partial\theta}{\partial s_j} \frac{\partial\psi_n}{\partial s_i} - 3 \frac{\partial\theta}{\partial s_i} \cdot \frac{\partial^2\theta}{\partial s_i^2} \cdot \frac{\partial^2\psi_n}{\partial s_i \partial s_j}$$

$$+ 3 \frac{\partial\theta}{\partial s_j} \cdot \frac{\partial^2\theta}{\partial s_i^2} \cdot \frac{\partial^2\psi_n}{\partial s_i^2} + 3 \left(\frac{\partial\theta}{\partial s_i}\right)^2 \frac{\partial^3\psi_n}{\partial s_i^2 \partial s_j}$$

$$- \frac{\partial\theta}{\partial s_i} \frac{\partial\theta}{\partial s_j} \frac{\partial^3\psi_n}{\partial s_i^3} \Big] \Bigg|_{\underset{\sim}{s} = \underset{\sim}{0}} , \quad i \neq j = 0, 1, 2, 3, 4$$

To compute (28), (29) and (30) we need the following:

$$\frac{\partial\theta}{\partial s_o} = 1 \quad , \quad \frac{\partial^2\theta}{\partial s_o^2} = 0 \quad , \quad \frac{\partial^3\theta}{\partial s_o^3} = 0$$

$$\frac{\partial\theta}{\partial s_1} \Big]_{s_1=0} = - \lambda\alpha_x \quad , \quad \frac{\partial^2\theta}{\partial s_1^2} \Big]_{s_1=0} = \lambda\beta_x$$

$$\frac{\partial^3\theta}{\partial s_1^3} \Big]_{s_1=0} = - \lambda\gamma_x \quad , \quad \frac{\partial\theta}{\partial s_2} \Big]_{s_2=0} = - \lambda\alpha_x^*$$

$$\frac{\partial^2\theta}{\partial s_2^2} \Big]_{S_2=0} = \lambda\beta_x^* \quad , \quad \frac{\partial^3\theta}{\partial s_2^3} \Big]_{s_2=0} = - \lambda\gamma_x^*$$

$$\frac{\partial^r\theta}{\partial s_3^r} \Big]_{s_3=0} = - \frac{\partial^r\theta}{\partial s_1^r} \Big]_{s_1=0} \quad , \quad \frac{\partial^r\theta}{\partial s_4^r} \Big]_{s_4=0} = - \frac{\partial^r\theta}{\partial s_2^r} \Big]_{s_2=0} \quad ,$$

$$r = 1, 2, 3 .$$

(31) $$\frac{\partial\psi_n}{\partial s_j}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}} = h_n'(0,1)\left[\frac{\partial Y}{\partial s_j} - \frac{\partial\tilde{Y}}{\partial s_j}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}, \quad j = 0, 1, 2, 3, 4 .$$

$$\frac{\partial\psi_n}{\partial s_o}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\alpha\, h_n'(0,1)$$

$$\frac{\partial\psi_n}{\partial s_1}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}} = \lambda\alpha\alpha_x\, h_n'(0,1)$$

$$\frac{\partial\psi_n}{\partial s_2}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}} = \lambda\alpha\alpha_x^*\, h_n'(0,1)$$

$$\frac{\partial\psi_n}{\partial s_3}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\frac{\partial\psi_n}{\partial s_1}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}}, \quad \frac{\partial\psi_n}{\partial s_4}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\frac{\partial\psi_n}{\partial s_2}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

(32) $$\frac{\partial^2\psi_n}{\partial s_j^2}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}} = h_n'(0,1)\left[\frac{\partial^2 Y}{\partial s_j^2} - \frac{\partial^2\tilde{Y}}{\partial s_j^2}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

$$+ h_n''(0,1)\left[\left(\frac{\partial Y}{\partial s_j}\right)^2 - \left(\frac{\partial\tilde{Y}}{\partial s_j}\right)^2\right]_{\underset{\sim}{s}=\underset{\sim}{0}},$$

$$j = 0, 1, 2, 3, 4$$

$$\frac{\partial^2\psi_n}{\partial s_o^2}\Big]_{\underset{\sim}{s}=\underset{\sim}{0}} = \beta\, h_n'(0,1) + \alpha^2 h_n''(0,1)$$

$$\left.\frac{\partial^2 \psi_n}{\partial s_1^{\,2}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\,[\lambda^2 \alpha_x^2 \beta + \lambda\alpha\beta_x]\, h_n'\,(0,1) - \lambda^2 \alpha^2 \alpha_x^2\, h_n''\,(0,1)$$

$$\left.\frac{\partial^2 \psi_n}{\partial s_2^{\,2}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\,[\,\lambda^2 \alpha_x^{*2}\,\beta + \lambda\alpha\beta_x^*\,]\, h_n'\,(0,1) - \lambda^2 \alpha^2 \alpha_x^{*2}\, h_n''(0,1)$$

$$\left.\frac{\partial^2 \psi_n}{\partial s_3^{\,2}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\left.\frac{\partial^2 \psi_n}{\partial s_1^{\,2}}\right|_{\underset{\sim}{s}=\underset{\sim}{0}}, \qquad \left.\frac{\partial^2 \psi_n}{\partial s_4^{\,2}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\left.\frac{\partial^2 \psi_n}{\partial s_2^{\,2}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

$$\left.\frac{\partial^2 \psi_n}{\partial s_i\,\partial s_j}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = h_n'\,(0,1)\left[\frac{\partial^2 Y}{\partial s_i\,\partial s_j} - \frac{\partial^2 \tilde{Y}}{\partial s_i\,\partial s_j}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

$$+\, h_n''\,(0,1)\left[\frac{\partial Y}{\partial s_i}\,\frac{\partial Y}{\partial s_j} - \frac{\partial \tilde{Y}}{\partial s_i}\,\frac{\partial \tilde{Y}}{\partial s_j}\right]_{\underset{\sim}{s}=\underset{\sim}{0}},$$

$$i \neq j = 0,\ 1,\ 2,\ 3,\ 4$$

$$\left.\frac{\partial^2 \psi_n}{\partial s_1\,\partial s_2}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\,\lambda^2 \beta\ \ \alpha_x \alpha_x^*\, h_n'\,(0,1) - \lambda^2 \alpha^2 \alpha_x \alpha_x^*\ \ h_n''\,(0,1)$$

$$\left.\frac{\partial^2 \psi_n}{\partial s_1\,\partial s_3}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0, \qquad \left.\frac{\partial^2 \psi_n}{\partial s_1\,\partial s_4}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0$$

$$\left.\frac{\partial^2 \psi_n}{\partial s_2\,\partial s_3}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0, \qquad \left.\frac{\partial^2 \psi_n}{\partial s_2\,\partial s_4}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0\,.$$

$$\left.\frac{\partial^2 \psi_n}{\partial s_3\,\partial s_4}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\left.\frac{\partial^2 \psi_n}{\partial s_1\,\partial s_2}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

(33) $$\left.\frac{\partial^3\psi_n}{\partial s_i^{\,2}\,\partial s_j}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = h_n'(0,1)\left[\frac{\partial^3 Y}{\partial s_i^{\,2}\,\partial s_j} - \frac{\partial^3\tilde{Y}}{\partial s_i^{\,2}\,\partial s_j}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

$$+ h_n''(0,1)\left[2\frac{\partial^2 Y}{\partial s_i\,\partial s_j}\frac{\partial Y}{\partial s_i} - 2\frac{\partial^2\tilde{Y}}{\partial s_i\partial s_j}\frac{\partial\tilde{Y}}{\partial s_i} + \frac{\partial^2 Y}{\partial s_i^{\,2}}\frac{\partial Y}{\partial s_j} - \frac{\partial^2\tilde{Y}}{\partial s_i^{\,2}}\frac{\partial\tilde{Y}}{\partial s_j}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

$$+ h_n'''(0,1)\left[\left(\frac{\partial Y}{\partial s_i}\right)^2\frac{\partial Y}{\partial s_j} - \left(\frac{\partial\tilde{Y}}{\partial s_i}\right)^2\frac{\partial\tilde{Y}}{\partial s_j}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

$i \neq j = 0, 1, 2, 3, 4.$

$$\left.\frac{\partial^3\psi_n}{\partial s_o^{\,2}\,\partial s_1}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0 \quad , \quad \left.\frac{\partial^3\psi_n}{\partial s_o^{\,2}\,\partial s_2}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0$$

$$\left.\frac{\partial^3\psi_n}{\partial s_o^{\,2}\,\partial s_3}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\lambda\alpha_x\,\gamma\,h_n'(0,1) - 3\,\lambda\alpha\alpha_x\,\beta\,h_n''(0,1) - \lambda\alpha^3\alpha_x\,h_n'''(0,1)$$

$$\left.\frac{\partial^3\psi_n}{\partial s_o^{\,2}\,\partial s_4}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\lambda\alpha_x^*\gamma\,h_n'(0,1) - 3\,\lambda\alpha\alpha_x^*\,\beta\,h_n''(0,1) - \lambda\alpha^3\alpha_x^*\,h_n'''(0,1)$$

$$\left.\frac{\partial^3\psi_n}{\partial s_1^{\,2}\,\partial s_2}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = \lambda^2\alpha_x^*\,(\lambda\gamma\alpha_x^2 + \beta\beta_x)\,h_n'(0,1) + \lambda^2\alpha\alpha_x^*\,(3\lambda\alpha_x^2\beta + \alpha\beta_x)\,h_n''(0,1)$$

$$+ \lambda^3\alpha^3\alpha_x^*\alpha_x^2\,h_n'''(0,1)$$

$$\left.\frac{\partial^3\psi_n}{\partial s_1^{\,2}\,\partial s_3}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0 \quad , \quad \left.\frac{\partial^3\psi_n}{\partial s_1^{\,2}\,\partial s_4}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0$$

$$\left.\frac{\partial^3\psi_n}{\partial s_2^{\,2}\,\partial s_3}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0 \quad , \quad \left.\frac{\partial^3\psi_n}{\partial s_2^{\,2}\,\partial s_4}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = 0$$

$$\left.\frac{\partial^3\psi_n}{\partial s_3^{\,2}\,\partial s_4}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = - \left.\frac{\partial^3\psi_n}{\partial s_1^{\,2}\,\partial s_2}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

(34) $$\left.\frac{\partial^3\psi_n}{\partial s_j^{\,3}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = h_n'(0,1)\left[\frac{\partial^3 Y}{\partial s_j^{\,3}} - \frac{\partial^3\tilde{Y}}{\partial s_j^{\,3}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

$$+ 3\,h_n''(0,1)\left[\frac{\partial Y}{\partial s_j}\,\frac{\partial^2 Y}{\partial s_j^{\,2}} - \frac{\partial\tilde{Y}}{\partial s_j}\,\frac{\partial^2\tilde{Y}}{\partial s_j^{\,2}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}}$$

$$+ h_n'''(0,1)\left[\left(\frac{\partial Y}{\partial s_j}\right)^3 - \left(\frac{\partial\tilde{Y}}{\partial s_j}\right)^3\right]_{\underset{\sim}{s}=\underset{\sim}{0}} ,$$

$j = 0,\ 1,\ 2,\ 3,\ 4.$

$$\left.\frac{\partial^3\psi_n}{\partial s_o^{\,3}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = -\gamma\, h_n'(0,1) - 3\,\alpha\,\beta\, h_n''(0,1) - \alpha^3\, h_n'''(0,1)$$

$$\left.\frac{\partial^3\psi_n}{\partial s_1^{\,3}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = h_n'(0,1)\left[\lambda^3\alpha_x^{\,3}\gamma + 3\,\lambda^2\alpha_x\beta\beta_x + \lambda\,\alpha\,\gamma_x\right]$$

$$+ 3\,h_n''(0,1)\,\lambda\alpha\alpha_x\left[\lambda^2\alpha_x^{\,2}\beta + \lambda\,\alpha\,\beta_x\right]$$

$$+ h_n'''(0,1)\,(\lambda\,\alpha\,\alpha_x)^3$$

$$\left.\frac{\partial^3\psi_n}{\partial s_2^{\,3}}\right]_{\underset{\sim}{s}=\underset{\sim}{0}} = h_n'(0,1)\left[\lambda^3\,\alpha_x^{*3}\,\gamma + 3\lambda^2\alpha_x^{*}\beta\beta_x^{*} + \lambda\,\alpha\,\gamma_x^{*}\right]$$

$$+ 3\, h_n''(0,1)\, \lambda\alpha\alpha_x^* \left[\lambda^2 \alpha_x^{*2} \beta + \lambda\, \alpha\, \beta_x^*\right]$$

$$+ h_n'''(0,1)\, (\lambda\, \alpha\, \alpha_x^*)^3$$

$$\left.\frac{\partial^3 \psi_n}{\partial s_3^{\,3}}\right]_{\underset{\sim}{s} = \underset{\sim}{0}} = - \left.\frac{\partial^3 \psi_n}{\partial s_1^{\,3}}\right]_{\underset{\sim}{s} = \underset{\sim}{0}} , \quad \left.\frac{\partial^3 \psi_n}{\partial s_4^{\,3}}\right]_{\underset{\sim}{s} = \underset{\sim}{0}} = - \left.\frac{\partial^3 \psi_n}{\partial s_2^{\,3}}\right]_{\underset{\sim}{s} = \underset{\sim}{0}}$$

And from reference [2] we have:

$$\sum_{n=0}^{\infty} h_n'(0,1) = \frac{1}{1 - \lambda\, \alpha}$$

$$\sum_{n=0}^{\infty} h_n''(0,1) = \beta\, \lambda^2 \Big| \ (1-\lambda\alpha) \quad (1-\lambda^2\alpha^2)$$

$$\sum_{n=0}^{\infty} h_n'''(0,1) = \frac{1}{1-\lambda\alpha} \left[\frac{\gamma\lambda^3}{1-\lambda^3\, \alpha^3} + \frac{3\, \lambda^5\, \alpha\, \beta^2}{(1-\lambda^2\, \alpha^2)\, (1-\lambda^3\, \alpha^3)}\right]$$

Substituting the above calculations in (28), (29) and (30) and simplifying, we get

$$E_\infty U_o = + \frac{\lambda\, (1-\lambda\alpha)}{2} \sum_{n=0}^{\infty} \left[\beta\, h_n'(0,1) + \alpha^2\, h_n''\, (0,1)\right]$$

$$= \frac{\lambda\, \beta}{2\, (1-\lambda^2\, \alpha^2)}$$

$$E_\infty U_1 = \frac{\lambda^2\, (1-\lambda\alpha)\, \alpha_x}{2} \sum_{n=0}^{\infty} \left[\beta\, h_n'(0,1) + \alpha^2\, h_n''\, (0,1)\right]$$

$$= \frac{\lambda^2\, \alpha_x\, \beta}{2\, (1-\lambda^2\, \alpha^2)}$$

$$E_\infty U_2 = \frac{\lambda^2\ (1-\lambda\alpha)\ \alpha_x^*}{2} \sum_{n=0}^{\infty} \left[\beta\ h_n^{'}(0,1) + \alpha^2\ h_n^{''}\ (0,1)\right]$$

$$= \frac{\lambda^2\ \alpha_x^*\ \beta}{2\ (1-\lambda^2\ \alpha^2)}$$

By Symmetry it follows that,

$$E_\infty U_3 = \frac{\lambda^2\ \alpha_x\ \beta}{2\ (1-\lambda^2 \alpha^2)} \quad ,$$

and

$$E_\infty U_4 = \frac{\lambda^2\ \alpha_x^*\ \beta}{2\ (1-\lambda^2\ \alpha^2)} \quad ,$$

Denote:

$$\underset{\sim}{U} = \begin{pmatrix} U_o \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}$$

Then:

(35) $$E_\infty \underset{\sim}{U} = \begin{pmatrix} a \\ \lambda\alpha_x a \\ \lambda\alpha_x^* a \\ \lambda\alpha_x a \\ \lambda\alpha_x^* a \end{pmatrix}$$

where $a = \dfrac{\lambda\ \beta}{2\ (1-\lambda^2\ \alpha^2)}$

$$E_\infty U_o^{\ 2} = \frac{\lambda\ (1-\lambda\ \alpha)}{3} \sum_{n=0}^{\infty} [\ \gamma\ h_n^{'}(0,1) + 3\ \alpha\ \beta\ h_n^{''}(0,1) + \alpha^3\ h_n^{'''}\ (0,1)\ ]$$

$$= \frac{\lambda}{3\ (1-\lambda^3\alpha^3)} \left[\ \gamma + \frac{3\ \lambda^2\ \alpha\ \beta^2}{1 - \lambda^2\alpha^2}\ \right]$$

$$E_\infty U_1^2 = \frac{\lambda\,(1-\lambda\alpha)}{6} \sum_{n=0}^{\infty} [\ (2\,\lambda^2\,\alpha_x^2\,\gamma + 3\,\lambda\,\beta\,\beta_x)\,h_n'\,(0,1)$$

$$+ 3\lambda\alpha\,(2\lambda\alpha_x^2\beta + \alpha\beta_x)\,h_n''(0,1) + 2\lambda^2\alpha^3\alpha_x^2\,h_n'''(0,1)\ ]$$

$$= \frac{\lambda^3\,\alpha_x^2}{3\,(1-\lambda^3\alpha^3)}\ (\gamma + \frac{3\lambda^2\,\alpha\,\beta^2}{1-\lambda^2\alpha^2}) + \frac{\lambda^2\,\beta\,\beta_x}{2\,(1-\lambda^2\alpha^2)}$$

$$E_\infty U_2^2 = \frac{\lambda(1-\lambda\alpha)}{6} \sum_{n=0}^{\infty} [\ (2\,\lambda^2\,\alpha_x^{*2}\,\gamma + 3\lambda\beta\beta_x^*)\,h_n'(0,1)$$

$$+ 3\lambda\alpha\,(2\lambda\alpha_x^{*2}\,\beta + \alpha\,\beta_x^*)\,h_n''(0,1) + 2\,\lambda^2\,\alpha^3\,\alpha_x^{*2}\,h_n'''(0,1)\ ]$$

$$= \frac{\lambda^3\alpha_x^{*2}}{3\,(1-\lambda^3\alpha^3)}\ (\gamma + \frac{3\lambda^2\ \alpha\,\beta^2}{1-\lambda^2\,\alpha^2}) + \frac{\lambda^2\,\beta\,\beta_x}{2\,(1-\lambda^2\,\alpha^2)}$$

$$E_\infty U_3^2 = E_\infty U_1^2\ , \qquad E_\infty U_4^2 = E_\infty U_2^2$$

$$E_\infty(U_o U_1) = + \frac{\lambda^2(1-\lambda\alpha)\alpha_x}{6} \sum_{n=0}^{\infty} [\gamma\,h_n'(0,1) + 3\alpha\beta\,h_n''(0,1) + \alpha^3\,h_n'''(0,1)\ ]$$

$$= \frac{\lambda^2\,\alpha_x}{6(1-\lambda^3\,\alpha^3)}\ [\ \gamma + \frac{3\,\lambda^2\,\alpha\,\beta^2}{1-\lambda^2\,\alpha^2}\ ]$$

$$E_\infty(U_o U_2) = \frac{\lambda^2(1-\lambda\,\alpha)\,\alpha_x^*}{6} \sum_{n=0}^{\infty} [\gamma\,h_n'(0,1) + 3\alpha\beta\,h_n''(0,1) + \alpha^3\,h_n'''(0,1)\ ]$$

$$= \frac{\lambda^2\,\alpha_x^*}{6\,(1-\lambda^3\alpha^3)}\ [\ \gamma + \frac{3\,\lambda^2\,\alpha\,\beta^2}{1-\lambda^2\,\alpha^2}\ ]$$

$$E_\infty(U_oU_3) = \frac{\lambda^2(1-\lambda\alpha)\,\alpha_x}{3} \sum_{n=0}^{\infty} [\gamma\, h_n^{'}(0,1) + 3\alpha\beta\, h_n^{''}(0,1) + \alpha^3\, h_n^{'''}(0,1)\,]$$

$$= \frac{\lambda^2\,\alpha_x}{3(1-\lambda^3\,\alpha^3)}\quad [\,\gamma + \frac{3\,\lambda^2\,\alpha\,\beta^2}{1-\lambda^2\,\alpha^2}\,]$$

$$E_\infty(U_oU_4) = \frac{\lambda^2\,(1-\lambda\,\alpha)\,\alpha_x^*}{3} \sum_{n=0}^{\infty} [\gamma\, h_n^{'}(0,1) + 3\alpha\beta\, h_n^{''}(0,1) + \alpha^3\, h_n^{'''}(0,1)\,]$$

$$= \frac{\lambda^2\,\alpha_x^*}{3(1-\lambda^3\,\alpha^3)}\quad [\,\gamma + \frac{3\,\lambda^2\,\alpha\,\beta^2}{1-\lambda^2\,\alpha^2}\,]$$

$$E_\infty(U_1U_2) = \frac{\lambda^3\,(1-\lambda\,\alpha)\,\alpha_x\,\alpha_x^*}{3} \sum_{n=0}^{\infty} [\gamma\, h_n^{'}(0,1) + 3\alpha\beta\, h_n^{''}(0,1) + \alpha^3\, h_n^{'''}(0,1)\,]$$

$$= \frac{\lambda^3\,\alpha_x\,\alpha_x^*}{3\,(1-\lambda^3\,\alpha^3)}\quad [\,\gamma + \frac{3\,\lambda^2\,\alpha\,\beta^2}{1-\lambda^2\,\alpha^2}\,]$$

$$E_\infty(U_1U_3) = \frac{\lambda^3\,(1-\lambda\,\alpha)\,\alpha_x^2}{6} \sum_{n=0}^{\infty} [\,\gamma\, h_n^{'}(0,1) + 3\,\alpha\,\beta\, h_n^{''}(0,1)\,]$$

$$= \frac{\lambda^3\,\alpha_x^2}{6\,(1-\lambda^3\,\alpha^3)}\quad [\,\gamma + \frac{3\,\lambda^2\,\alpha\,\beta^2}{1-\lambda^2\,\alpha^2}\,]$$

$$E_\infty(U_1U_4) = \frac{\lambda^3\,(1-\lambda\,\alpha)\,\alpha_x\,\alpha_x^*}{6} \sum_{n=0}^{\infty} [\,\gamma\, h_n^{'}(0,1) + 3\,\alpha\,\beta\, h_n^{''}(0,1)$$

$$+ \alpha^3\, h_n^{'''}\,(0,1)\,]$$

$$= \frac{\lambda^3\,\alpha_x\,\alpha_x^*}{6\,(1-\lambda^3\,\alpha^3)}\quad [\,\gamma + \frac{3\,\lambda^2\,\alpha\,\beta^2}{1-\lambda^2\,\alpha^2}\,]$$

$$E_\infty(U_2U_3) = E_\infty(U_1U_4)$$

$$E_\infty(U_2U_4) = \frac{\lambda^3\ (1-\lambda\ \alpha)\ \alpha_x^{*2}}{6} \sum_{n=0}^{\infty} \left[\gamma\ h_n'(0,1) + 3\alpha\beta\ h_n''(0,1) + \alpha^3\ h_n'''(0,1)\right]$$

$$= \frac{\lambda^3\ \alpha_x^{*2}}{6\ (1-\lambda^3\ \alpha^3)} \left[\gamma + \frac{3\ \lambda^2\ \alpha\ \beta^2}{1 - \lambda^2\ \alpha^2}\right]$$

$$E_\infty(U_3U_4) = E_\infty(U_1U_2)$$

$$E_\infty(\underset{\sim}{U}\ \underset{\sim}{U}') = \begin{pmatrix} b & & & & \\ \frac{\lambda}{2}\ \alpha_x\ b & \lambda^2\ \alpha_x^2\ b + \lambda\ \beta_x\ a & & & \\ \frac{\lambda}{2}\ \alpha_x^*\ b & \lambda^2\ \alpha_x\ \alpha_x^*\ b & \lambda^2\ \alpha_x^{*2}\ b + \lambda\ \beta_x^*\ a & & \\ \lambda\ \alpha_x\ b & \frac{\lambda^2}{2}\ \alpha_x^2\ b & \frac{\lambda^2}{2}\ \alpha_x\alpha_x^* b & \lambda^2\alpha_x^2 b + \lambda\beta_x a & \\ \lambda\ \alpha_x^*\ b & \frac{\lambda^2}{2}\ \alpha_x\ \alpha_x^*\ b & \frac{\lambda^2}{2}\ \alpha_x^{*2}\ b & \lambda^2\alpha_x\alpha_x^* b & \lambda^2\alpha_x^{*2}\ b + \lambda\beta_x^* a \end{pmatrix} \tag{36}$$

where $a = E_\infty U_o$, $\quad b = E_\infty U_o^2$.

That is, a is the steady state expected residual life length of the generation serving at time t, and $b - a^2$, its variance.

<u>Remark</u>:

For x = 0+ the second and the fourth rows as well as the second and the fourth columns of the matrix $E_\infty\ (\underset{\sim}{U}\ \underset{\sim}{U}')$ tend to zero. While for sufficiently large x, the third and the fifth rows and columns tend to zero.

If we define:

$$r_{ij} = \frac{E_x(E\,U_i\,U_j) - (E_x\,E\,U_i)(E_x\,E\,U_j)}{\sqrt{E_x\,EU_i^2 - (E_x\,EU_i)^2}\,\sqrt{E_x\,EU_j^2 - (E_x\,EU_j)^2}} \quad ,$$

$$i, j = 0, 1, 2, 3, 4$$

then in the case of an M|M|1 queue as the traffic intensity tends to one it can be seen that r_{23} and r_{14} are negative, and $r_{04} > r_{03} > r_{02} >_{01}$

7. Moments of the Limiting Distribution of the Virtual Waiting Times $\underline{\eta}(t,x)$, $\eta(t)$ and $\overline{\eta}(t,x)$

Let us denote:

$$\Lambda = \begin{pmatrix} \underline{\eta}\,(t,x) \\ \eta\,(t) \\ \overline{\eta}\,(t,x) \end{pmatrix} , \qquad A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Then from (16) we get

$$\Lambda = A\,\underset{\sim}{U}$$

(37) $E_\infty\,\Lambda = A\,E_\infty\,\underset{\sim}{U}$

(38) $E_\infty\,(\Lambda\,\Lambda') = A\,E_\infty\,(\underset{\sim}{U}\,\underset{\sim}{U}')\,A'$

Direct computation of $E_\infty\,\Lambda$ was given in [2]. Further a direct computation of $E_\infty\,\Lambda\,\Lambda'$ is possible from (19). However, here we compute (37) and (38) in terms of the steady state moments of the basic variables U's. Substituting (35) in (37) and (36) in (38) and simplifying we get:

(39) $E_\infty\,\Lambda = \begin{pmatrix} (1 + 2\,\lambda\,\alpha_x)\,a \\ (1 + \lambda\,\alpha)\,a \\ (1 + 2\,\lambda\,\alpha_x^*)\,a \end{pmatrix}$

$$E_\infty(\Lambda\Lambda') = \begin{pmatrix} \begin{array}{l} b(1+3\lambda\alpha_x+3\lambda^2\alpha_x^2) \\ \quad + 2\lambda\beta_x\ a \end{array} & & \\ \begin{array}{l} b(1+\frac{\lambda}{2}\alpha+\frac{3}{2}\lambda\alpha_x \\ \quad +\frac{3}{2}\lambda^2\alpha\alpha_x)+\lambda\beta_x a \end{array} & \begin{array}{l} b(1+\lambda\alpha+\lambda^2\alpha^2) \\ \quad + \lambda\ \beta\ a \end{array} & \\ b(1+\frac{3}{2}\lambda\alpha+3\lambda^2\alpha_x\alpha_x^*) & \begin{array}{l} b(1+\frac{\lambda}{2}\alpha+\frac{3}{2}\lambda\alpha_x^* \\ \quad +\frac{3}{2}\lambda^2\alpha\alpha_x^*)+\lambda\beta_x^*\ a \end{array} & \begin{array}{l} b(1+3\lambda\alpha_x^*+3\lambda^2\alpha_x^{*2}) \\ \quad + 2\lambda\ \beta_x^*\ a \end{array} \end{pmatrix} \tag{40}$$

We define:

(41) $$\rho_{12} = [E_x\,(E\,\underline{\eta}\,(\infty,X)\ \eta(\infty)) - (E_x\,E\,\underline{\eta}\,(\infty,X))\ (E\,\eta\,(\infty))\ ]$$

$$[E_x\,(E\,\underline{\eta}^2\,(\infty,X)) - (E_x\,E\,\underline{\eta}\,(\infty,X))^2]^{-\frac{1}{2}}$$

$$[E\,\eta^2\,(\infty) - (E\,\eta\,(\infty))^2]^{-\frac{1}{2}}$$

(42) $$\rho_{13} = [E_x\,(E\,\underline{\eta}\,(\infty,X)\ \overline{\eta}\,(\infty,X)) - (E_x\,E\,\underline{\eta}\,(\infty,X))\ (E_x\,E\,\overline{\eta}\,(\infty,X))]$$

$$[E_x\,(E\,\underline{\eta}^2\,(\infty,X)) - (E_x\,E\,\underline{\eta}\,(\infty,X))^2]^{-\frac{1}{2}}$$

$$[E_x\,(E\,\overline{\eta}^2\,(\infty,X)) - (E_x\,E\,\overline{\eta}\,(\infty,X))^2]^{-\frac{1}{2}}$$

(43) $$\rho_{23} = [E_x\,(E\,\eta\,(\infty)\ \overline{\eta}\,(\infty,X)) - (E\,\eta\,(\infty))\ (E_x\,E\,\overline{\eta}\,(\infty,X))]$$

$$[E\,\eta^2\,(\infty) - (E\,\eta\,(\infty))^2]^{-\frac{1}{2}}$$

$$[E_x\,(E\,\overline{\eta}^2\,(\infty,X)) - (E_x\,E\,\overline{\eta}\,(\infty,X))^2]^{-\frac{1}{2}}$$

In the case of an $M|M|1$ queue when the traffic intensity tends to one it can be shown that:

(44) $\rho_{12} = 0.79$, $\rho_{23} = 0.90$ and $\rho_{13} \approx \rho_{12}\, \rho_{23}$

8. The Probability $P\{\underline{\eta}(t,x) < \eta(t)\}$

From (16) we have:

(45) $P\{\underline{\eta}(t,x) < \eta(t)\} = P\{U_3 < U_2\}$

$$= \int_0^\infty \int_0^{x_2} d_{x_2 x_3}\ \Lambda_i(t, x_2, x_3)$$

where $\Lambda_i(t, x_2, x_3)$ is the joint distribution of U_2 and U_3 given t.
As in § 2, if given t and t', the joint distribution of U_2 and U_3 is given by:

(46) $$\phi(t, t'; x_2, x_3) = \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} e^{-\lambda t(1-H(x))-\lambda t' H(x)} \frac{\{\lambda t[1-H(x)]\}^{j_2}}{j_2!}$$

$$\frac{[\lambda t' H(x)]^{j_3}}{j_3!} \quad \tilde{\tilde{H}}^{(j_2)}(x_2) \quad \tilde{H}^{(j_1)}(x_3)$$

Let $\overset{\circ}{\Lambda}_i(t, x_2, x_3)$ be the probability that in $(0,t)$ the queue has never become empty and that the variables U_2 and U_3 associated with the time point t satisfy $U_2 \leq x_2$, $U_3 \leq x_3$, given that at $t = 0$ there were $i \geq 1$ customers in the queue, one of who was beginning his service at that time. Then:

(47) $$\overset{\circ}{\Lambda}_i(t, x_2, x_3) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \int_{0\,(t')}^{\infty} \int_{0\,(\tau)}^{t} d_0 Q_{i\nu}^{(n)}(\tau) \quad dH^{(\nu)}(t+t'-\tau)$$

$$\cdot\ \phi\ (t-\tau,\ t',\ x_2,\ x_3)$$

$$= \sum_{\nu=1}^{\infty} \int_0^t d\,{}_oR_{i\nu}^{(\tau)}\ F_\nu\ (t-\tau,\ x_2,\ x_3)\ ,$$

where ${}_oR_{i\nu}(\cdot) = \sum_{n=0}^{\infty} {}_oQ_{i\nu}^{(n)}(\cdot)$ and:

(48) $$F_\nu(t-\tau,\ x_2,\ x_3) = \int_{\substack{o\\(t')}}^{\infty} dH^{(\nu)}(t+t'-\tau)\ \phi\ (t-\tau,\ t',\ x_2,\ x_3)$$

By the renewal argument as in (11):

(49) $$\Lambda_i\ (t,\ x_2,\ x_3) = \Lambda_i^\circ\ (t,\ x_2,\ x_3) + \int_o^t \Lambda_1^\circ\ (t-u,\ x_2,\ x_3)\ d\ M_1\ (u)$$

$$+ P\ \{\ \xi(t) = 0\ \big|\ \xi(0) = i\ \}\ U\ (x_2,\ x_3)$$

where:

(50) $$U(x_2,\ x_3) = 1 \text{ if } x_2 \geq 0 \text{ and } x_3 \geq 0,$$

$$= 0 \text{ otherwise.}$$

Substitution of (47) in (49) yields:

(51) $$\Lambda_i(t,\ x_2, x_3) = \sum_{\nu=1}^{\infty} \int_o^t d\,{}_oR_{i\nu}(\tau)\ F_\nu(t-\tau,\ x_2,\ x_3)$$

$$+ \int_o^t \Lambda_1^\circ\ (t-u,\ x_2,\ x_3)\ d\ M_1(u)$$

$$+ P\ \{\ \xi(t) = 0\ \big|\ \xi(0) = i\ \}\ U\ (x_2,\ x_3)$$

If $\Lambda\ (x_2, x_3)$ is the limiting value of $\Lambda_i\ (t, x_2, x_3)$ as $t \to \infty$, then similar to the renewal argument in appendix II of [2] we have:

$$\Lambda(x_2,x_3) = (1-\lambda\alpha)\ \{\ U(x_2,x_3) + \lambda \sum_{j=1}^{\infty}\ {}_oR_{ij}(+\infty) \int_o^{\infty} F_j\ (\tau,\ x_2,\ x_3)\ d\tau\ \}$$

if $1 - \lambda\,\alpha > 0$,

$= 0$ otherwise.

Finally from (45) if follows that:

$$\lim_{t\to\infty} P\ \{\ \underline{\eta}(t,x) < \eta(t)\ \} = (1-\lambda\alpha)\ \{\ 1 + \lambda \sum_{j=1}^{\infty}\ {}_oR_{1j}(+\infty) \int_{o\,(x_2)}^{\infty} \int_{o\,(x_3)}^{x_2} \int_o^{\infty} d_{x_2 x_3} F_j\ (\tau,\ x_2,\ x_3)\ d\tau\ \}$$

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13. ABSTRACT

The waitingtimes of customers under three priority rules are compared. The first is the order of arrival rule. The second is the shortest processing time (within a generation) first and the third is longest processing time (within a generation) first.

In particular the covariance matrix of the trivariate distribution of the equilibrium waitingtimes is obtained explicitly.

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