On the distribution of the sphericity test criterion in classical and complex normal populations having unknown covariance matrices

by

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1. Introduction and Summary. Let $\underset{\sim}{x}$: pxl be distributed N $(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ where $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ are both unknown. Let $\underset{\sim}{S}$ be the sum of product matrix of a sample of size N. To test the hypothesis of sphericity, namely, $H_o : \underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{I}_p$, where $\sigma^2 > o$ is unknown, against $H_1 : \underset{\sim}{\Sigma} \neq \sigma^2 \underset{\sim}{I}_p$, Mauchly [10] obtained the likelihood ratio test criterion for H_o in the form $W = |\underset{\sim}{S}| / [(\text{tr}\, \underset{\sim}{S}) / p]^p$. Thus the criterion W is a power of the ratio of the geometric mean and the arithmetic mean of the roots $\theta_1, \theta_2, \ldots, \theta_p$ of $|\underset{\sim}{S} - \theta \underset{\sim}{I}| = \underset{\sim}{0}$ (see Anderson [1]). For p = 2, Mauchly [10] showed that the density of W is

$$(1.1) \qquad f(w) = \tfrac{1}{2} (n-1) w^{\frac{1}{2}(n-3)}, \qquad o \leq w \leq 1 ,$$

where n = N-1. The exact distribution in the null case was obtained by Consul [3], [4], in the form

$$(1.2) \quad f(w) = k(p,n) w^{\frac{1}{2}(n-p-1)} G^{p,o}_{p,p} \left(w \,\middle|\, \begin{matrix} \frac{1}{2}(p-1) + (p-1)/p, \ldots, \frac{1}{2}(p-1) \\ o, \frac{1}{2}, 1, \ldots, \frac{1}{2}(p-1) \end{matrix} \right) ,$$

where

$$k(p,n) = (2\pi)^{\frac{1}{2}(p-1)} \; p^{\frac{1}{2} - \frac{1}{2}pn} \; \Gamma(\tfrac{1}{2}pn) \Big/ \prod_{j=1}^{p} \Gamma[\tfrac{1}{2}(n-j-1)] ,$$

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and $G^{m,n}_{p,q}\left(x \middle| \begin{matrix} a_1,\ldots, a_p \\ b_1,\ldots, b_q \end{matrix}\right)$ is the G-function defined in the next section.

In this paper we have obtained the general moments of W both in real and complex cases for arbitrary covariance matrices and also the corresponding distributions of W in terms of G-function.

2. Some definitions and results. In this section we give a few definitions and some lemmas which are needed in the sequel. First we define Meijer's G-function by [11]

$$(2.1)\quad G^{m,n}_{p,q}\left(x \middle| \begin{matrix} a_1,\ldots, a_p \\ b_1,\ldots, b_q \end{matrix}\right) = (2\pi i)^{-1} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)} x^s \, ds ,$$

where an empty product is interpreted as unity and C is a curve separating the singularities of $\prod_{j=1}^{m} \Gamma(b_j-s)$ from those of $\prod_{j=1}^{n} \Gamma(1-a_j+s)$, $q \geq 1$,

$o \leq n \leq p \leq q$, $o \leq m \leq q$; $x \neq o$ and $|x| < 1$ if $q = p$; $x \neq o$ if $q > p$.

The G-function of (2.1) can be expressed as a finite number of generalised hypergeometric functions (see Pillai, Al-Ani and Jouris [12] and Luke [9]) and in particular we have

$$(2.3)\quad G^{2,o}_{2,2}\left(x \middle| \begin{matrix} a_1,a_2 \\ b_1,b_2 \end{matrix}\right) =$$

$$\frac{x^{b_1} (1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)} \cdot {}_2F_1 (a_2-b_2, a_1-b_2, a_1+a_2-b_1-b_2, 1-x) \qquad o < x < 1$$

where ${}_2F_1$ here is the Gauss hypergeometric function.

Now we state the Gauss and Legendre's multiplication formula for gamma functions as

$$(2.4) \qquad \prod_{r=1}^{n} \Gamma[\, z + (r-1)/n \,] = (2\pi)^{\frac{1}{2}(n-1)} \, n^{\frac{1}{2}-nz} \, \Gamma(nz) \; .$$

Further, the hypergeometric function of matrix variates is defined by

$${}_pF_q \,(a_1, \ldots, a_p;\, b_1, \ldots, b_q;\, \underset{\sim}{S}, \underset{\sim}{T} = \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa \, C_\kappa\,(\underset{\sim}{S})\, C_\kappa\,(\underset{\sim}{T})}{(b_1)_\kappa \cdots (b_q)_\kappa \, C_\kappa\,(\underset{\sim}{I}_m)\; k\,!}$$

where the zonal polynomials, $C_\kappa\,(\cdot)$, and $(\cdot)_\kappa$ are defined in [6] .

Lemma 2.1. Let $\underset{\sim}{Z}$: mxm be a complex symmetric matrix whose real part is p.d. and let $\underset{\sim}{T}$: mxm be an arbitrary complex symmetric matrix. Then

$$(2.5) \quad \int_{\underset{\sim}{S} > o} \exp\,(-\mathrm{tr}\; \underset{\sim}{Z}\, \underset{\sim}{S})\; |\underset{\sim}{S}|^{t-\frac{1}{2}(m+1)} C_\kappa(\underset{\sim}{T}\, \underset{\sim}{S})\; d\, \underset{\sim}{S} = \Gamma_m\,(t,\kappa)\; |\underset{\sim}{Z}|^{-t}\; C_\kappa\,(\underset{\sim}{T}\, \underset{\sim}{Z}^{-1})$$

where $\Gamma_m(t,\kappa)$ is defined in (15) of Constantine [2] and $R(t) > \frac{1}{2}\,(m-1)$. (See Constantine [2]).

Lemma 2.2. Let $\underset{\sim}{T}$ be as in lemma 2.1. Then

$$(2.6) \quad \int_{\underset{\sim}{S} > o} \exp\,(-\,\tfrac{1}{2}\; \mathrm{tr}\; \underset{\sim}{S})\; |\underset{\sim}{S}|^{t\,-\,\frac{1}{2}\,(m+1)} (\mathrm{tr}\; \underset{\sim}{S})^q\; C_\kappa\,(\underset{\sim}{T}\, \underset{\sim}{S})\; d\, \underset{\sim}{S}$$

$$= \Gamma_m\,(t,\; \kappa)\; 2^{tm+k+q}\; \Gamma(mt+k+q)\; C_\kappa\,(\underset{\sim}{T}) \;\; / \; \Gamma(mt + k).$$

Proof. We shall consider the cases when (1) $q \geq o$ and (2) $q < o$.

(1) $q \geq o$. From (2.5), for $u > o$ we have

$$(2.7) \quad \int_{S > 0} \exp\left(-\tfrac{1}{2} u \operatorname{tr} S\right) |S|^{t-\frac{1}{2}(m+1)} C_\kappa(T S)\, dS = 2^{tm+k}\, u^{-tm-k}\, \Gamma_m(t,\kappa)\, C_\kappa(T) .$$

To prove this case we differentiate (2.7) q times w.r.t. u under the integral sign and let u = 1 to obtain

$$(2.8) \quad \int_{S > 0} \exp\left(-\tfrac{1}{2} \operatorname{tr} S\right) |S|^{t-\frac{1}{2}(m+1)} (\operatorname{tr} S)^q\, C_\kappa(T S)\, dS = 2^{tm+k+q}\, \Gamma_m(t,\kappa)\, \Gamma(tm+k+q)\, C_\kappa(T) \,/\, \Gamma(mt+k).$$

which is also (19) of Khatri [7].

(2) $q < o$. To prove this case, we integrate (2.7) successively r times w.r.t. u, change the order of integration and let u = 1, yielding

$$(2.9) \quad \int_{S > 0} \exp\left(-\tfrac{1}{2} \operatorname{tr} S\right) |S|^{t-\frac{1}{2}(m+1)} (\operatorname{tr} S)^{-r}\, C_\kappa(T S)\, dS = 2^{tm+k-r}\, \Gamma(tm+k-r)\, \Gamma_m(t,\kappa)\, C_\kappa(T) \,/\, \Gamma(tm+k).$$

Since $\Gamma(tm+k-r) \,/\, \Gamma(tm+k) = \prod_{j=1}^{n} (tm+k-j)^{-1}$, (2.9) holds if $tm+k-r > o$. This proves the lemma.

Lemma 2.3. Let Z : mxm be a complex symmetric matrix whose real part is p.d. and let T : mxm be an arbitrary complex symmetric matrix and S : pxp be a hermitian matrix. Then

$$(2.10) \quad \int_{\bar{S} = S > 0} \exp(-\operatorname{tr} Z S)\, |S|^{a-p}\, \tilde{C}_\kappa(T S)\, dS = \tilde{\Gamma}_p(a,\kappa)\, |Z|^{-a}\, \tilde{C}_\kappa(T Z^{-1})$$

where $\tilde{\Gamma}_p(a,\kappa)$ is defined in [6].

Lemma 2.4. Let $\underset{\sim}{T}$ and $\underset{\sim}{S}$ be as in lemma 2.3. Then

$$(2.11) \qquad \int_{\underset{\sim}{S}'=\underset{\sim}{S}>0} \exp(-\operatorname{tr}\underset{\sim}{S}) \, |\underset{\sim}{S}|^{a-p} (\operatorname{tr}\underset{\sim}{S})^{j} \, \tilde{C}_{\kappa}(\underset{\sim}{T}\underset{\sim}{S}) \, d\underset{\sim}{S}$$

$$= \tilde{\Gamma}_p(a,\kappa)\, \Gamma(ap+k+j)\, \tilde{C}_{\kappa}(\underset{\sim}{T}) \,/\, \Gamma(ap+k) .$$

Proof. The proof is exactly similar to the proof of lemma 2.2 and hence omitted.

Lemma 2.5. If s is any complex variate and f(x) is a function of a real variable x, such that

$$F(s) = \int_0^{\infty} x^{s-1} \, f(x) \, dx$$

exists, then under certain regularity conditions

$$f(x) = (2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} x^{-s} \, F(s) \, ds.$$

F(s) is called the Mellin transform of f(x), and f(x) is the inverse Mellin transform of F(s). (See Titchmarsh [13])

3. Distribution of W in the real case. Let $\underset{\sim}{S}$: pxp be distributed as Wishart (n, p, $\underset{\sim}{\Sigma}$). Then the distribution of the latent roots $g_1, g_2, \ldots, g_p$ of $\underset{\sim}{S}$ has been shown by James [6] to depend only on the latent roots of $\underset{\sim}{\Sigma}$ and is given by

$$(3.1) \quad k(p, n, \underset{\sim}{\Sigma}) \, |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \, {}_0F_0(-\tfrac{1}{2}\underset{\sim}{\Sigma}^{-1}, \underset{\sim}{G}) \, |\underset{\sim}{G}|^{\frac{1}{2}(n-p-1)} \prod_{i<j} (g_i - g_j) \prod_{i=1}^{p} dg_i ,$$

where

$$k(p, n, \underset{\sim}{\Sigma}) = |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \pi^{\frac{1}{2}p^2} / \{2^{\frac{1}{2}pn} \Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}p)\} ,$$

$$\underset{\sim}{G} = \text{diag}(g_1, g_2, \ldots, g_p) , \quad \infty > g_1 \geq g_2 \geq \cdots \geq g_p > o .$$

The distribution (3.1) is not convenient for further development and the convergence of the series is slow. But the convergence may be improved by writing (3.1) in the form suggested by Pillai (See Pillai, Al-Ani and Jouris [12]),

$$(3.2) \quad k(p, n, \underset{\sim}{\Sigma}) \, |\underset{\sim}{G}|^{\frac{1}{2}(n-p-1)} \exp(-\tfrac{1}{2} \text{tr } \underset{\sim}{G}) \prod_{i<j} (g_i - g_j) \; {}_oF_o(\underset{\sim}{M}, \underset{\sim}{G})$$

where

$$\underset{\sim}{M} = \tfrac{1}{2} (\underset{\sim}{I} - \underset{\sim}{\Sigma}^{-1}) .$$

<u>Theorem 3.1</u>. Let $\underset{\sim}{G}$ be distributed as in (3.2) and let $W = |\underset{\sim}{G}|/\{(\text{tr } \underset{\sim}{G})/p\}^p$ be the sphericity criterion. Then the h-th moment of W is given by

$$(3.3) \quad E(W^h) = \frac{p^{ph} |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}}{\Gamma_p(\frac{1}{2}n)} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{M})}{k!} 2^k \frac{\Gamma_p(\frac{1}{2}n + h, \kappa)\, \Gamma(\frac{1}{2}pn + k)}{\Gamma(\frac{1}{2}pn + ph + k)} .$$

<u>Proof</u>. To find $E(W^h)$ we multiply (3.2) by $|\underset{\sim}{G}| / [(\text{tr } \underset{\sim}{G}) / p]^p$, transform $\underset{\sim}{G} \to \underset{\sim}{H}\,\underset{\sim}{V}\,\underset{\sim}{H}'$ where $\underset{\sim}{H}$ is an orthogonal and $\underset{\sim}{V}$ a symmetric matrix, integrate out $\underset{\sim}{H}$ and $\underset{\sim}{V}$ using (44) and (22) of Constantine [2]. We get

$$(3.4) \quad E(W^h) = p^{ph} k(p, n, \underset{\sim}{\Sigma}) \Gamma_p(\tfrac{1}{2}p) \pi^{-\frac{1}{2}p^2}$$

$$\cdot \sum_{k=o}^{\infty} \sum_{\kappa} [C_\kappa(\underset{\sim}{M}) / C_\kappa(\underset{\sim}{I}_p)\, k!]$$

$$\int_{\underset{\sim}{V} > o} \exp(-\tfrac{1}{2} \text{tr } \underset{\sim}{V}) |\underset{\sim}{V}|^{(\frac{1}{2}n+h) - \frac{1}{2}(p+1)} (\text{tr } \underset{\sim}{V})^{-ph} C_\kappa(\underset{\sim}{V}) d\underset{\sim}{V}.$$

Applying lemma (2.2) to the integral on the R.H.S. of (3.4) we get (3.3).

Theorem 3.2. For any finite p, the p.d.f. of W is

$$(3.5)\quad f(w) = C(p, n, \underset{\sim}{\Sigma}) \sum_{k=o}^{\infty} \sum_{\kappa} \frac{2^k C_\kappa(\underset{\sim}{M})}{k!} p^{\frac{1}{2} - \frac{1}{2} pn-k} \Gamma(\tfrac{1}{2} pn + k)$$

$$\cdot\, w^{\frac{1}{2}(n-p-1)} G^{p,o}_{p,p}\left(w \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}\right),$$

where

$$C(p, n, \underset{\sim}{\Sigma}) = \pi^{\frac{1}{4}p(p-1)} |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} (2\pi)^{\frac{1}{2}(p-1)} / \Gamma_p(\tfrac{1}{2}n),$$

$$a_j = (k + j-1) / p + \tfrac{1}{2}(p-1)\ ;\ b_j = k_j + \tfrac{1}{2}(p-j).$$

For p = 2, (3.5) reduces to

$$(3.6)\quad f(w) = \frac{|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}}{2\,\Gamma(n-1)} w^{\frac{1}{2}(n-3)} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{\Gamma(n + k)}{k!} C_\kappa(\underset{\sim}{M}) w^{k_1 + \frac{1}{2}}$$

$$\cdot\, {}_2F_1(a_2-b_2, a_1-b_2; a_1+a_2-b_1-b_2, 1-w).$$

Proof. Applying (2.4) on $\Gamma[p(\frac{1}{2} n + h + k/p)]$ we have from (3.3)

$$E(W^h) = C(p, n, \underset{\sim}{\Sigma}) \sum_{k=o}^{\infty} \sum_{\kappa} [\{2^k C_\kappa(\underset{\sim}{M}) p^{\frac{1}{2} - \frac{1}{2} pn-k} \Gamma(\tfrac{1}{2} pn + k)\}/k!]$$

$$\prod_{j=1}^{p} [\, \Gamma\{\tfrac{1}{2} n+h+k_j-\tfrac{1}{2}(j-1)\} / \Gamma\{\tfrac{1}{2} n + ((k+j-1)/p) + h\}\,].$$

Using Lemma 2.5, the density of W has the form

$$(3.7)\quad f(w) = C\,(p, n, \underset{\sim}{\Sigma}) \sum_{k=o}^{\infty} \sum_{\kappa} \frac{2^k C_\kappa(\underset{\sim}{M})}{k!} p^{\frac{1}{2}-\frac{1}{2}pn-k} \Gamma(\tfrac{1}{2}pn + k)\, w^{\frac{1}{2}(n-p-1)}$$

$$\cdot\,(2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} w^{-r} \frac{\prod_{i=1}^{p} \Gamma(r + b_i)}{\prod_{i=1}^{p} \Gamma(r + a_i)}\, dr\,,$$

where

$$r = \tfrac{1}{2}n + h - \tfrac{1}{2}(p-1),\; b_j = k_j + \tfrac{1}{2}(p-j),\; a_j = (k+j-1)/p + \tfrac{1}{2}(p-1)\,.$$

Noting that the integral in (3.7) is in the form of Meijer's G-function, we can write the density of W as in (3.5).

(3.6) can be obtained easily from (3.5) by putting $p = 2$ in (3.5) and using (2.3).

<u>Remark</u>. Putting $\underset{\sim}{\Sigma} = \underset{\sim}{I}$ in (3.5) and (3.6), we can easily deduce the result of Consul in (1.2) [3], [4]. amd Mauchly in (1.1), [10].

4. <u>Distribution of W in the complex case</u>. Let $\underset{\sim}{\tilde{S}}$: pxp be distributed as a Complex Wishart $(n, p, \underset{\sim}{\tilde{\Sigma}})$ (see Goodman [4]). Then the distribution of the latent roots $g_1, g_2, \ldots, g_p$ of $\underset{\sim}{\tilde{S}}$ is (James [5])

$$(4.1)\quad k(p, n, \underset{\sim}{\tilde{\Sigma}})\; {}_0\tilde{F}_0(-\underset{\sim}{\tilde{\Sigma}}^{-1}, \underset{\sim}{G})\; |G|^{n-p} \prod_{i<j} (g_i - g_j)^2 \prod_{i=1}^{p} dg_i$$

where

$$k(p, n, \underset{\sim}{\tilde{\Sigma}}) = \frac{|\underset{\sim}{\tilde{\Sigma}}|^{-n}\, \pi^{p(p-1)}}{\tilde{\Gamma}_p(n)\, \tilde{\Gamma}_p(p)}\;;\quad \tilde{\Gamma}_p(n) \text{ and}$$

${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; \underset{\sim}{S}, \underset{\sim}{T})$ are defined in (83) and (88) of James [6].

As in the real case, the convergence of (4.1) may be improved by writing it in the form

$$(4.2)\quad k\,(p,\, n,\, \tilde{\Sigma})\ {}_0\tilde{F}_0\,(\tilde{M}_1\, G)\ \exp\,(-\mathrm{tr}\ G)\ |G|^{n-p} \prod_{i<j} (g_i - g_j)^2 \prod_{i=1}^{p} dg_i \ ,$$

where $\tilde{M}_1 = I_p - \tilde{\Sigma}^{-1}$.

<u>Theorem 5.1</u>. Let G be distributed as in (4.2) and let $W = |G| / [\ (\mathrm{tr}\ G) / p\ ]^p$. Then h-th moment of W is

$$(4.3)\quad \frac{p^{ph}}{\tilde{\Gamma}_p(n)} |\tilde{\Sigma}|^{-n} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa\,(\tilde{M}_1)}{k\,!}\ \Gamma\,(np+k)\ \tilde{\Gamma}_p\,(n+h,\ \kappa)\ /\ \Gamma\,(np+k+ph).$$

<u>Proof</u>: Multiplying (4.2) by $|G|\ /\ [\ (\mathrm{tr}\ G)\ /\ p)\ ]^p$, using the transformation $G \to U\, V\, \bar{U}'$ where U is unitary and V is hermitian p.d. we have on integrating out U and using the results (see Khatri [8])that the Jacobian of transformation is

$$J\ (G,\ U\, V) = \prod_{i<j} (g_i - g_j)^2\quad h_2\ (U)$$

and that
$$\int_{U\bar{U}' = I} h_2\ (U) = \frac{\pi^{p\,(p-1)}}{\tilde{\Gamma}_p(p)}\ ,\ \text{we have}$$

$$E\ (W^h) = \frac{p^{ph}\ |\tilde{\Sigma}|^{-n}}{\tilde{\Gamma}_p(n)} \sum_{k=0} \sum_{\kappa} \frac{\tilde{C}_\kappa\ (\tilde{M}_1)}{\tilde{C}_\kappa\ (I_p)\ k!} \int_{V>0} \exp\ (-\mathrm{tr}\ V) |V|^{n+h-p}\ (\mathrm{tr}\ V)^{-ph}\ \tilde{C}_\kappa(V) dV\ .$$

Using lemma (2.4) to the integral on the right, we get (4.3).

Theorem 5.2. The density of W is

$$f(w) = \frac{\pi^{\frac{1}{2}p(p-1)} |\tilde{\Sigma}|^{-n} (2\pi)^{\frac{1}{2}(p-1)}}{\tilde{\Gamma}_p(n)} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{\tilde{C}_\kappa(\tilde{M}_1)}{k!} \Gamma(pn+k)$$

$$\cdot p^{\frac{1}{2}-pn-k} w^{n-p} G^{p,o}_{p,p} \left(w \,\middle|\, \begin{matrix} a_1 a_2, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}\right),$$

where $a_j = (k/p) + (j-1) / p + (p-1)$, and $b_j = k_j - j + p$.

Proof. The proof is exactly similar to that of theorem 3.2 and hence is omitted.

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