Inequalities and Asymptotic Bounds for Ramsey Numbers *

by

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1. We consider Ramsey's Theorem, [3], as it pertains to partitions of pairs of elements of a finite set into disjoint classes denoted by $A_1, A_2, \ldots, A_c$.

Definition 1: A partition of the pairs of elements of a finite set S will be called an $(n_1, n_2, \ldots, n_c)$ partition if n_i is larger than the largest number of elements of S all of whose pairs are in the class A_i for $i = 1, 2, \ldots, c$.

Definition 2: $R(n_1, n_2, \ldots, n_c)$ is the greatest integer such that an $(n_1, n_2, \ldots, n_c)$ partion exists on a set of $R(n_1, n_2, \ldots, n_c)$ elements. (According to Ramsey's Theorem such a finite integer exists.)

Definition 3: If a set $H \subseteq S$ is specified then for each $e \in S$ the j support of e in H is the set of elements of H whose pairs with e are in the class A_j .

Definition 4: An i element of j support with respect to H is an element whose j support in H is an i subset of H.

2. We consider now an $(n_1, n_2, \ldots, n_c)$ partition of the pairs of elements of a set S. Without loss of generality we consider a subset $H \subseteq S$ all of whose pairs of elements are in the class A_j and for which $|H| = n_j - 1$.

With respect to this set H the elements of S-H are partitioned into classes according to their j support in H. We will let p_i denote the number of i elements of j support with respect to H.

Thus we have that

$$|S| = \sum_{i=0}^{n_j-1} p_i \quad . \tag{1}$$

Our purpose here will be to give bounds for certain linear combinations of the p_i .

Proposition 1: Let p_i, $i = 0, 1, \ldots$ be the number of i points of j support with respect to H.

Then

$$\sum_{\nu=n_j-i+1}^{n_j-1} p_\nu \binom{\nu}{k-n_j+1+\nu} \leq R(n_1, \ldots, n_{j-1}, k+1, n_{j+1}, \ldots, n_c) \binom{n_j-1}{k} \tag{2}$$

for all $k \geq n_j-i-1$.

Proof: Let S admit an $(n_1, n_2, \ldots, n_c)$ partition P and let $H \subseteq S$, $|H| = n_j-1$, such that all pairs of elements of H are in the class A_j .

For each $K \subseteq H$, $|K| = k$, we consider all elements whose j support in H is an n_j-i set or larger and whose support contains H-K, call this set S(K). The partition P when restricted to the elements of S-H forms a $(n_1, n_2, \ldots, n_{j-1}, k+1, n_{j+1}, \ldots, n_c)$ partition of the pairs of elements of S(K) otherwise when P is restricted to the elements of H-K those elements, together with S(K), would contain a set of n_j or more elements all of whose pairs are in the restriction of the class A_j to the subset $(H-K) \cup S(K)$.

Thus if one considers all k-subsets of H and observes that each ν element will be counted $\binom{\nu}{k-n_j+1+\nu}$ times and that on each k-subset there can be at most $R(n_1,\ldots,n_{j-1}, k+1, n_{j+1},\ldots, n_c)$ elements, the bound (2) results.

Proposition 2: With the numbers p_ν as in proposition 1, the following inequality is satisfied

$$\sum_{\nu=n_j-1-k}^{n_j-1} (\nu-(n_j-1-k)) \binom{\nu}{n_j-1-k} p_\nu \leq k\binom{n_j-1}{k} R(n_1,\ldots,n_{j-1},k, n_{j+1},\ldots,n_c) \quad (3)$$

for all k .

Proof: Proceeding as in proposition 1 we now will count the number of pairs of elements in class A_j for which one element of the pair is an element of H.

Specifically, for any subset $K \subseteq H$, $|K| = k$ there are at most $R(n_1,\ldots, n_{j-1},k, n_{j+1},\ldots, n_c)$ elements of S-H whose j support contains H-K and whose pairs with a fixed element of K are in the class A_j. Each ν element of S-H of j support in H will have pairs in the class A_j with $\nu-(n_j-1-k)$ elements of H.

Combining the two facts above, we see that each $K \subseteq H$, $|K| = k$ will allow at most $k\,R(n_1,\ldots, n_{j-1},k, n_{j+1},\ldots, n_c)$ such pairs.

The inequality (3) then results from the consideration of all k-subsets of H to obtain the term on the right hand side and from adding up the number of such pairs for each ν element of j support in H to obtain the left hand side. The proposition is thus proved.

In the special case of a partition into two classes we can obtain another inequality analogous to (3) but for which the right hand side differs.

Proposition 3: For a partition of a set S into two classes to obtain an (n_1, n_2) partition we have

$$\sum_{\nu=n_2-k}^{n_2-1} (n_2-1-\nu) \binom{\nu}{\nu-(n_2-1-k)} p_\nu \leq k\, R(n_1-1, k+1) \binom{n_2-1}{k} . \qquad (4)$$

Proof: To see the truth of (4) we again consider an (n_2-1)-subset $H \subseteq S$ for which all pairs of elements of H are in A_2. Then for each ν element of S-H with 2 support in H for which that support contains a set H-K for some k-subset $K \subseteq H$ there are at most $n_2-1-\nu$ elements of K whose pairs with that ν element are in class A_1. But for each element of K there can be at most $R(n_1-1, k+1)$ such elements of S-H whose support contains H-K. Thus if we now impose this restriction for all k subsets of H the inequality (4) is obtained as stated.

3. Applications of the inequalities to $R(n_1, N)$, $n_1 \leq N$.

In our application we assume without loss of generality the existence of a set $S(n_1)$, for each n_1, which admits an $(n_1+1, N+1)$ partition of the pairs of its elements and which contains an N set $H(n_1)$ all of whose pairs are in the second class. We will denote the number of i elements of 1 support in $H(n_1)$ by p_i (please note that this number in section 2 was denoted by p_{N-i}). p_i is simply the number of elements of S-H which have i pairs with elements of H in the first partition class.

With this change in our notation the inequality (4) is

$$\sum_{V=1}^{K} V \binom{N-V}{K-V} \; p_V \leq K \, R(n_1, K+1) \binom{N}{K} \tag{5}$$

and for K=N becomes simply

$$\sum_{V=1}^{N} V p_V \leq N \, R(n_1, N+1) \; . \tag{6}$$

The pair of inequalities (5) and (6) in the variables $p_0, p_1, \ldots, p_N$ will next be used to find $\max \sum_{V=0}^{N} p_V$ subject to the constraints imposed by those inequalities and $p_V \geq 0$ for $V = 0, 1, 2, \ldots, N$.

We will now formulate the dual to this linear programming problem. The dual problem is to find the minimum of

$$N \, R(n_1, N+1) \, X_1 + K \binom{N}{K} R(n_1, K+1) \, X_2$$

subject to the conditions that the variables X_1, X_2 satisfy the inequalities

$$i \, X_1 + i \binom{N-i}{K-i} X_2 \geq 1 \quad \text{for} \quad i = 1, \ldots, N \; . \tag{7}$$

If one considers the geometry of the first problem it is natural to conjecture that the $\max \sum_{V=0}^{N} p_V$ for the variables $p_0, \ldots, p_N$ constrained to the given convex region will occur at the extreme point in the coordinate plane of $p_\nu, p_{\nu+1}$ where ν is that value of V for which inequality

(5) places a greater restriction on p_V than (6) and inequality (6) places a greater restriction on p_{V+1} than (5). ν is easily found by minimizing p_V subject to (5) over all values of K and taking the largest value of V for which that minimum is achieved by a value of $K < N$.

To carry out that procedure one must use a value of $R(n_1, K+1)$ which is known to be an upper bound. For that purpose we appeal to [1] and observe that $\binom{K+n_1-1}{n_1-1}$ will be an upper bound for all n_1, K. With this bound we find that p_V is minimized by (5) when

$$K = \text{integer} \left(\frac{n_1(V-1)}{n_1-V}\right) \tag{8}$$

and this will be less than N if

$$V < \frac{n_1(N+1)}{N+n_1} \tag{9}$$

Thus we define ν to be the largest integer satisfying (9). We now apply this preceding information to show that

Theorem 1.

$$\text{Max } \Sigma\, p_i \leq \frac{K\, R(n_1,\ K+1)\binom{N}{K}(N-\nu)}{\nu(\nu+1)(N-K)\binom{N-\nu}{K-\nu}} + \frac{[(\nu+1)(N-K)-(N-\nu)]}{\nu(\nu+1)(N-K)}\, N\, R(n_1,\ N+1) \tag{10}$$

where $p_1, p_2, \cdots, p_N$ satisfy the inequalities (5) and (6).

Proof: We will establish (10) by observing that with $p_i=0$, $i \neq \nu, \nu+1$ and solving (5) and (6) as equations in p_ν and $p_{\nu+1}$ yields the value given on the r. h. s. of (10). That establishes the max Σp_i to be at least as large as the asserted value.

Next for the dual problem if we let

$$X_1 = \frac{(\nu+1)(N-K)-(N-\nu)}{\nu(\nu+1)(N-K)} \quad \text{and} \quad X_2 = \frac{N-\nu}{\nu(\nu+1)(N-K)\binom{N-\nu}{K-\nu}}$$

we find that

$$[X_1 \; N \; R(n_1, N+1) + X_2 \; K\binom{N}{K} \; R(n_1, K+1)]$$

is the same value as that given in (10) and hence the minimum for the dual problem is at most this value provided X_1, X_2 satisfy conditions (7). Thus if we demonstrate that we have a feasible solution to the dual problem we can appeal to the duality theorem for linear programming which would then assert that (10) gives the max Σp_i .

To verify that X_1, X_2 is a feasible solution we must show that (7) holds. Let N, ν, K be fixed and denote the left hand side of (7) by a_i . Then the first differences

$$a_i - a_{i+1} = -1 \left(\frac{(\nu+1)(N-K)-(N-\nu)}{\nu(\nu+1)(N-K)}\right) + \binom{N-i}{K-i}\left(i - \frac{(i+1)(K-i)}{N-i}\right)\frac{(N-\nu)}{\binom{N-\nu}{K-\nu}\nu(\nu+1)(N-K)}$$

and the second differences

$$(a_i - a_{i+1})-(a_{i+1} - a_{i+2})=(N-K)(i(N-i)-2(i+1)) > 0$$

hence the first differences are decreasing in i, i.e., $a_i - a_{i+1} > a_{i+1} - a_{i+2}$ but it is easily verified that $a_\nu - a_{\nu+1} = 0$ and that $a_\nu = a_{\nu+1} = 1$. Therefore, the terms $a_i \geq 1$ for all i. This concludes the proof of the theorem.

4. Upper bounds for $R(n_1+1, N+1)$ extrapolated from $R(3, K)$.

We will consider (10) and the bound for $R(n_1, N+1)$ given by $f(n_1, N) \binom{N+n_1-1}{n_1-1}$ where $f(3, N) = A \text{ loglog } N/\log N$, see [2, p. 154]. Thus, we must consider the ratio

$$\frac{\binom{K+n_1-1}{n_1-1}\binom{N}{K}}{\binom{N-\nu}{K-\nu}\binom{N+n_1}{n_1}} \tag{11}$$

in estimating the bound for $R(n_1+1, N+1)$ given by the bound for $R(n_1, N+1)$ and $(R(n_1, K+1)$. Expanding the binomial coefficients in (11) it follows immediately that a useful upper bound for this ratio is

$$\frac{n_1}{N+n_1}\left[1 + \frac{C_1 \ \nu(N-K)n_1}{K\ N}\right] \tag{12}$$

for which if $K = N/2$ and $n_1 \leq \sqrt{N}/(\log N)^{\frac{1}{2}}$ we can write (12) as

$$\frac{n_1}{N+n_1}\left[1 + \frac{C_1 \nu n_1}{N}\right] \tag{13}$$

where C_1 is an absolute constant which is independent of N and n_1.

For values of $n_1 \geq \sqrt{N}/(\log N)^{\frac{1}{2}}$ we let $N-K = N \log\log N/(\nu+1)$ and in this case the ratio (11) can be written as

$$\frac{n_1}{K+n_1} \prod_{j=1}^{N-K} \left(1 + \frac{(\nu-n_1)(K+j) + \nu(n_1-2)}{(K+n_1+j)(K-\nu+j)}\right)$$

and the product is bounded by $\left(1 + \frac{C_3}{N}\right)$ (14)

with C_3 being a constant independent of n_1 and N.

Lemma 1.

Let $R(n_1, L+1) \leq f(n_1, L)\binom{L+n_1-1}{n_1-1}$

where

$$f(n_1, N/2)/f(n_1, N) \leq 1 + C_4/\log N\ , \quad n_1 \leq \sqrt{N}/(\log N)^{\frac{1}{2}}\ ,$$

then

$$R(n_1+1, N+1) \leq f(n_1, N) \binom{N+n_1}{n_1}\left(1 + \frac{C_5}{\nu \log N}\right) \quad .$$

Proof: Using $K = N/2$ and the bound obtained in (13) whenever $n_1 \leq \sqrt{N}/(\log N)^{\frac{1}{2}}$, Theorem 1 asserts

$$R(n_1+1,\ N+1) \le \frac{f(n_1,\ N)\binom{N+n_1}{n_1} Nn_1}{(N+n_1)(\nu+1)(\nu N)} \left\{(1 + \frac{C_4}{\log N})(1 + \frac{C_1\nu n_1}{N})(N-\nu)+(\nu+1)N-2(N-\nu)\right\}$$

$$\le f(n_1,\ N)\binom{N+n_1}{n_1} \left\{1 + \frac{1}{N} + \frac{\frac{C_4}{\log N}(1 + \frac{C_1\nu n_1}{N}) + \frac{C_1}{\log N}(1 + \frac{C_4}{\log N})}{\nu}\right\}$$

$$\le f(n_1,\ N)\binom{N+n_1}{n_1}\{1 + \frac{C_5}{\nu \log N}\}$$

where C_5 is independent of n_1 and N.

Lemma 2.

Let $R(n_1,\ L+1) \le f(n_1,\ L)\binom{L+n_1-1}{n_1-1}$ where

$$f(n_1,\ K)/f(n_1,\ N) \le 1 + \frac{C_6 \log\log N}{\nu \log N}$$

for $K = N(\frac{\nu+1-\log\log N}{\nu+1})$ and ν as defined in section 3, then

$$R(n_1+1,\ N+1) \le f(n_1,\ N)\binom{N+n_1}{n_1}(1 + \frac{C_8}{\nu \log N}) .$$

Proof: From Theorem 1 and the bound (14) with $n_1 \geq \sqrt{N}/(\log N)^{\frac{1}{2}}$ and $N-K = N \log\log N/(\nu+1)$ we have

$$R(n_1+1,\ N+1) \leq f(n_1,\ N)\binom{N+n_1}{n_1}\left\{\frac{\frac{Kn_1(N-\nu)}{K+n_1}\left(1+\frac{C_3}{N}\right)\left(1+\frac{C_6\log\log N}{\nu\log N}\right)}{\nu\ N\ \log\log N}\right.$$

$$\left. + \frac{Nn_1}{N+n_1}\ \frac{[N\log\log N - (N-\nu)]}{\nu\ N\ \log\log N}\right\} .$$

Replacing the factor $\frac{Nn_1}{(\nu+1)(N+n_1)}$ by 1 and collecting terms this expression simplifies to

$$R(n_1+1,\ N+1) \leq f(n_1,\ N)\binom{N+n_1}{n_1}\left\{1 + \frac{1}{N} + \frac{(\nu+1-\log\log N)(N-\nu)C_7\log\log N}{\nu^2\ N\ \log N\ \log\log N}\right\}$$

$$\leq f(n_1,\ N)\binom{N+n_1}{n_1}\left(1 + \frac{C_8}{\nu\log N}\right) .$$

We now use lemmas 1 and 2 to prove

Theorem 2.

$$R(n_1+1,\ N+1) \leq C^* \frac{\log\log N}{\log N}\ \binom{N+n_1}{n_1}$$

where C^* is independent of N and $n_1 \leq N$.

Proof: From lemmas 1 and 2 we see that the bound

$$R(3, N+1) \leq \frac{A \log\log N}{\log N} \binom{N+2}{2}$$

obtained in [2, p. 154] can be extrapolated using Theorem 1 to give us

$$R(n_1+1, N+1) \leq f(n_1, N)\binom{N+n_1}{n_1}\left(1 + \frac{C'}{\nu \log N}\right)$$

for some C' independent of N and n_1 provided we can show that $f(n_1, K)/f(n_1, N)$ satisfies the hypothesis of the lemmas for each n_1.

If we take

$$f(3, N) = \frac{A \log\log N}{\log N}$$

then

$$f(n_1, N) = \frac{A \log\log N}{\log N} \prod_{j=4}^{n_1}\left(1 + \frac{C'}{j \log N}\right)$$

and the ratio

$$f(n_1, N/2)/f(n_1, N) = \left(1 + \frac{\log 2}{\log(N/2)}\right) \prod_{j=4}^{n_1} \left(1 + \frac{\log 2}{\log(N/2)}\right)\left(1 - \frac{j \log 2}{j \log N + C'}\right)$$

$$\leq \left(1 + \frac{\log 2}{\log(N/2)}\right) \prod_{j=4}^{n_1} \left(1 + \frac{C' \log 2}{j \log N \log(N/2)}\right)$$

and with $n_1 < \sqrt{N}$ this product is bounded by

$$\left(1 + \frac{1}{\log(N/2)}\right) \frac{C' \log 2}{2} + 1$$

hence if $\frac{c^* \log 2}{2} + 1 < c^*$ this bound can be iterated and we will perpetuate the bound $1 + \frac{c^*}{\log N}$ for the ratio. Note that c^* is independnet of n_1 and N for $n_1 \le \sqrt{N}$.

This gives us

$$R(n_1+1,\ N+1) \le \frac{A \log\log N}{\log N} \binom{N+n_1}{n_1} \prod_{j=1}^{n_1} \left(1 + \frac{c^*}{j \log N}\right)$$

for all $n_1 \le \sqrt{N}/(\log N)^{\frac{1}{2}}$.

Similarly, from Lemma 2, the function $f(n_1+1, L)$ satisfies the same bound on the ratio and for $K = N\left(\frac{\nu+1-\log\log N}{\nu+1}\right)$ this ratio is decreasing with increasing values of n_1 so there is a constant c^* independent of n_1 and N so that

$$R(n_1+1,\ N+1) \le \frac{A \log\log N}{\log N} \binom{N+n_1}{n_1} \prod_{j=1}^{n_1} \left(1 + \frac{c^*}{j \log N}\right)$$

for all $n_1 \le N$.

Since $\prod_{j=1}^{n_1} \left(1 + \frac{c^*}{j \log N}\right)$ is uniformly bounded in N for all $n_1 \le N$ the conclusion of the theorem follows with

$$C^* = A \sup_N \prod_{j=1}^{N} \left(1 + \frac{c^*}{j \log N}\right) .$$

Conjecture: In [2, section 3] a rather sharp bound was obtained for the number of pairs that an element must share in class 1 with elements of a maximal set all of whose pairs are in class 2 but only in the case $n_1 = 3$. Comparison with the results of section 3 of this paper indicates that the order of magnitude of $R(n_1, N)$ for $n_1 = 4, 5, \ldots, N$ is much smaller than is indicated by any work to date. One should be able to extend the sharper bounds of [2] for larger values of n_1 to obtain asymptotic bounds of smaller order of magnitude than we have obtained here however I have not been able to do so.

References

[1] P. Erdos and G. Szekeres, On a combinatorial problem in geometry, Composition Math. 2 (1935), 463-470.

[2] Jack E. Graver and James Yackel, Some Graph Theoretic Results Associated with Ramsey's Theorem, Journal of Combinatorial Theory 4(1968), 125-175.

[3] F. P. Ramsey, On a problem of formal logic, Proc. London Math. Soc. (2) 30(1930), 264-286.

[4] C. Frasnay, Sur les fonctions d'entiers se rapportant au théorème de Ramsey, Comptes Rendus, 256(1963), 2507-2510.