SOME DISTRIBUTION PROBLEMS IN THE MULTIVARIATE COMPLEX GAUSSIAN CASE

by

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1. Introduction and Summary. Let X_1: p x n and X_2: p x n be real random variables having the joint density function

$$(1.1)\qquad (2\pi)^{-pn}|\Sigma_o|^{-\frac{1}{2}n} \exp\{-\tfrac{1}{2} \operatorname{tr} \Sigma_o^{-1}(X-\nu)(X-\nu)'\}, \quad -\infty \leq X \leq \infty$$

where

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \Sigma_o = \begin{pmatrix} \Sigma_1 & -\Sigma_2 \\ \Sigma_2 & \Sigma_1 \end{pmatrix}, \quad \nu = \begin{pmatrix} \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix},$$

Σ_1: p x p is a real symmetric positive definite (p.d.) matrix, Σ_2: p x p is a real skew-symmetric matrix, μ_j: p x q and M_j: q x n $(j = 1,2)$, are given matrices or their joint density does not contain Σ_1, Σ_2, μ_1, μ_2 as parameters. Then it has been shown by Goodman [10] that the distribution of the complex matrix $Z = X_1 + iX_2$, $(i = (-1)^{\frac{1}{2}})$, is complex Gaussian and its density function is given by

$$(1.2)\qquad N_c(\mu M, \Sigma) = \pi^{-pn}|\Sigma|^{-n} \exp\{-\operatorname{tr} \Sigma^{-1}(Z-\mu M)(Z-\overline{\mu M})'\}$$

where $\Sigma = \Sigma_1 + i\Sigma_2$ is Hermitian p.d., i.e. $\bar{\Sigma}' = \Sigma$, $\mu = \mu_1 + i\mu_2$ and $M = M_1 + iM_2$. Goodman [5], Wooding [17], James [6], Al-Ani [1], and Khatri [8], [9], [10], [11] have studied distributions derived from a sample of a complex p-variate normal distribution.

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Some concepts which are important and necessary notation are given below.

$$\tilde{\Gamma}_m(a) = \pi^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \Gamma(a-i+1)$$

$$[a]_\kappa = \prod_{i=1}^{m} (a-i+1)_{k_i} = \tilde{\Gamma}_m(a,\kappa)/\tilde{\Gamma}_m(a)$$

where $\kappa = (k_1, k_2, \ldots, k_p)$ is a partition of the integer k and

$$\tilde{\Gamma}_m(a,\kappa) = \pi^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \Gamma(a+k_i-i+1) \quad .$$

The hypergeometric functions are defined as

$${}_p\tilde{F}_q(a_1,\ldots,a_p;\, b_1,\ldots,b_q; \underset{\sim}{A}, \underset{\sim}{B}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\prod_{i=1}^{p} [a_i]_\kappa \tilde{C}_\kappa(\underset{\sim}{A})\, \tilde{C}_\kappa(\underset{\sim}{B})}{\prod_{i=1}^{q} [b_i]_\kappa\, \tilde{C}_\kappa(\underset{\sim}{I}_m) k!}$$

or when $\underset{\sim}{B} = \underset{\sim}{I}_m$ we denote it by

$${}_p\tilde{F}_q(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, \underset{\sim}{A})$$

and $\tilde{C}_\kappa(\underset{\sim}{A})$ is a zonal polynomial of a Hermitian matrix $\underset{\sim}{A}$ and is given as a symmetric function of the characteristic roots of $\underset{\sim}{A}$. (See Section 6)

The non-central distributions of the characteristic roots concerning the classical problems of the equality of two covariance matrices, MANOVA model, and canonical correlation coefficients have been found by James [6] and Khatri [8], [10]. Here for the three cases mentioned, we give the general moment and the density which is expressed in terms of Meijer's G-function [13], [14], for $W^{(p)} = \prod_{i=1}^{p} (1-w_i)$, where the w_i, $i = 1,2,\ldots,p$

are the characteristic roots in the above cases. The moments and densities are analogous to those given in the real case by Jouris [7]. Further the density functions of U and Pillai's V criteria in the complex central case are obtained for $p = 2$ and from the non-central complex multivariate F distribution various independence relationships are shown and independent beta variables are obtained. The last section is devoted to complex zonal polynomials. A method for computing them in terms of elementary symmetric function (esf's) is given and they are tabulated through degree 8 in Tables 1-4.

2. <u>Density Functions of $W^{(p)}$ in the Non-Central Case</u>. Case 1: Testing the Equality of Two Covariance Matrices.

Let $\underset{\sim}{X}: (p \times n_1) \sim N_c(\underset{\sim}{0}, \underset{\sim}{\Sigma}_1)$ and $\underset{\sim}{Y}: (p \times n_2) \sim N_c(\underset{\sim}{0}, \underset{\sim}{\Sigma}_2)$ be independent and $n_1 \geq p$. Then Khatri [10] has shown the density function of the characteristic roots, $0 < f_1 < \ldots < f_p < \infty$ of $(\underset{\sim}{X}\,\bar{\underset{\sim}{X}}')(\underset{\sim}{Y}\,\bar{\underset{\sim}{Y}}')^{-1}$ can be written as

$$(2.1) \qquad C(p)\,|\underset{\sim}{\Lambda}|^{-n_1}\,{}_1\tilde{F}_0(n\,;\underset{\sim}{I}_p-\underset{\sim}{\Lambda}^{-1},\underset{\sim}{F}(\underset{\sim}{I}_p+\underset{\sim}{F})^{-1})\,\frac{|\underset{\sim}{F}|^{n_1-p}}{|\underset{\sim}{I}_p+\underset{\sim}{F}|^{n}}\,\prod_{i>j}(f_i-f_j)^2$$

where

$$(2.2) \qquad C(p) = \frac{\pi^{p(p-1)}\tilde{\Gamma}_p(n)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}\,,\ n = n_1+n_2,\ \underset{\sim}{F} = \mathrm{diag}(f_1,\ldots,f_p)$$

and $\underset{\sim}{\Lambda}$ is a diagonal matrix whose diagonal elements are the characteristic roots of $(\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1})$. Transforming

$$(2.3) \qquad w_i = f_i/(1 + f_i)$$

we find the density of $0 < w_1 < \cdots < w_p$ is

$$(2.4) \qquad C(p)|\underset{\sim}{\Lambda}|^{-n_1}\,{}_1\tilde{F}_0(n;\ \underset{\sim}{I}_p-\underset{\sim}{\Lambda}^{-1},\ \underset{\sim}{W})|\underset{\sim}{W}|^{n_1-p}|\underset{\sim}{I}_p-\underset{\sim}{W}|^{n_2-p}\prod_{i>j}(w_i-w_j)^2$$

where

$$\underset{\sim}{W} = \mathrm{diag}\,(w_1,w_2,\ldots,w_p).$$

To find $E[W^{(p)}]^h$ where $W^{(p)} = \prod_{i=1}^{p}(1-w_i)$ we multiply (2.4) by $|\underset{\sim}{I}_p-\underset{\sim}{W}|^h$ and transform $\underset{\sim}{T} \to \underset{\sim}{U}\,\underset{\sim}{W}\,\bar{\underset{\sim}{U}}'$ where $\underset{\sim}{U}$ is unitary, i.e. $\underset{\sim}{U}\,\bar{\underset{\sim}{U}}' = \underset{\sim}{I}$, and $\underset{\sim}{T}$ is Hermitian p.d. . Using the Jacobian of transformation given by Khatri [8]

$$(2.5) \qquad J(\underset{\sim}{T};\underset{\sim}{U},\underset{\sim}{W}) = \prod_{i>j}(w_i-w_j)^2 h_2(\underset{\sim}{U})$$

and integrating out $\underset{\sim}{U}$ and $\underset{\sim}{W}$ using

$$(2.6) \qquad \int_{\underset{\sim}{U}\,\bar{\underset{\sim}{U}}'=\underset{\sim}{I}} h_2(\underset{\sim}{U}) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}$$

and

$$(2.7) \qquad \int_{\bar{\underset{\sim}{S}}=\underset{\sim}{S}>0}|\underset{\sim}{S}|^{q-p}|\underset{\sim}{I}_p-\underset{\sim}{S}|^{n+h-q-p}\tilde{C}_\kappa(\underset{\sim}{S})d\underset{\sim}{S} = \frac{\tilde{\Gamma}_p(q,\kappa)\tilde{\Gamma}_p(n+h-q)\tilde{C}_\kappa(\underset{\sim}{I}_p)}{\tilde{\Gamma}_p(n+h,\kappa)}$$

we get after simplifying

$$(2.8) \qquad E[W^{(p)}]^h = |\underset{\sim}{\Lambda}|^{-n_1}\ \frac{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(n_2+h)}{\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(n+h)}\ {}_2\tilde{F}_1(n,n_1;n+h;\underset{\sim}{I}_p-\underset{\sim}{\Lambda}^{-1}) .$$

Before finding the density of $W^{(p)}$, below are stated some needed results on Mellin's transforms [2], [3], [4], and Meijer's G-function [13], [14].

If s is any complex variate and $f(x)$ is a function of a real variable x, such that

$$F(s) = \int_0^\infty x^{s-1} f(x)\, dx \tag{2.9}$$

exists, then under certain regularity conditions

$$f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\, ds \quad . \tag{2.10}$$

$F(s)$ is called the Mellin transform of $f(x)$ and $f(x)$ is the inverse Mellin transform of $F(s)$. Meijer defined the G-function by

$$G_{p,q}^{m,n}\left(x \middle| {a_1,\ldots,a_p \atop b_1,\ldots,b_q}\right) = (2\pi i)^{-1} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} (1-a_j+s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+s) \prod_{j=n+1}^{p} \Gamma(a_j-s)}\, x^s ds \tag{2.11}$$

where C is a curve separating the singularities of $\prod_{j=1}^{m} \Gamma(b_j-s)$ from those of $\prod_{j=1}^{n} \Gamma(1-a_j+s)$, $q \geq 1$, $0 \leq n \leq p \leq q$, $0 \leq m \leq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$. Using (2.9) and (2.10) we see from (2.8) that the density of $f(W^{(p)})$ has the form

$$(2.12)\qquad f(W^{(p)}) = C_p \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa [n_1]_\kappa}{k!} \tilde{C}_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Lambda}^{-1}) \{W^{(p)}\}^{n_2-p}$$

$$\cdot\ (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \{W^{(p)}\}^{-r} \frac{\prod_{i=1}^{p} \Gamma(r+b_i)}{\prod_{i=1}^{p} \Gamma(r+a_i)} dr$$

where

$$(2.13)\qquad C_p = \frac{\tilde{\Gamma}_p(n)}{\tilde{\tilde{\Gamma}}_p(n_2)} |\underset{\sim}{\Lambda}|^{-n_1},\ b_i = i-1,\ a_i = n_1 + k_{p-i+1} + b_i \ .$$

Noting that the integral in (2.12) is in the form of Meijer's G-function we can write the density of $W^{(p)}$ as

$$(2.14)\qquad f(W^{(p)}) = C_p \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa [n_1]_\kappa}{k!} \tilde{C}_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Lambda}^{-1}) \{W^{(p)}\}^{n_2-p}$$

$$\cdot\ G^{p,0}_{p,p}\left(W^{(p)} \middle|\, {a_1, \ldots, a_p \atop b_1, \ldots, b_p}\right) \ .$$

Using the fact that

$$(2.15)\qquad G^{2,0}_{2,2}\left(x \middle|\, {a_1, a_2 \atop b_1, b_2}\right) = \frac{x^{b_1}(1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)}$$

$$\cdot\ {}_2F_1(a_2-b_2, a_1-b_2;\ a_1+a_2-b_1-b_2; 1-x) \quad 0 < x < 1$$

we find the density of $W^{(2)}$ to be

$$(2.16)\qquad f(W^{(2)}) = C_2 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa [n_1]_\kappa}{k!} \tilde{C}_\kappa(\underset{\sim}{I}_2 - \underset{\sim}{\Lambda}^{-1}) \{W^{(2)}\}^{n_2-2}$$

$$\cdot \frac{\{1-W^{(2)}\}^{2n_1+k-1}}{\Gamma(2n_1+k)} \; {}_2F_1(n_1+k_1, n_1+k_2-1; 2n_1+k; 1-W^{(2)})$$

where $\kappa = (k_1,k_2)$. Using the results of Consul [4] for $p = 3$ and Al-Ani [1] for $p = 4$ we could also write out the densities of $W^{(3)}$ and $W^{(4)}$.

Case 2: MANOVA Model. Suppose $\underset{\sim}{X}:(p \times m) \sim N_c(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ and $\underset{\sim}{Y}:(p \times n) \sim N_c(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ are independent with $m \geq p$. Then the joint density of the characteristic roots $0 < f_1 < \cdots < f_p$ of $(\underset{\sim}{X}\,\underset{\sim}{\bar{X}}')(\underset{\sim}{Y}\,\underset{\sim}{\bar{Y}}')^{-1}$ is given by Khatri [10] as

$$(2.15)\qquad C'(p) e^{-\mathrm{tr}\underset{\sim}{\Omega}} \; {}_1\tilde{F}_1(m+n; m; \underset{\sim}{\Omega}, (\underset{\sim}{I}_p + \underset{\sim}{F}^{-1})^{-1}) \frac{|\underset{\sim}{F}|^{m-p}}{|\underset{\sim}{I}_p + \underset{\sim}{F}|^{m+n}} \prod_{i>j} (f_i - f_j)^2$$

where

$$C'(p) = \frac{\tilde{\Gamma}_p(m+n)\pi^{p(p-1)}}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(p)} \quad , \quad \underset{\sim}{F} = \mathrm{diag}\ (f_1,\ldots,f_p)$$

and $\underset{\sim}{\Omega} = \mathrm{diag}\ (\omega_1,\ldots,\omega_p)$ where ω_i are the characteristic roots of $\underset{\sim}{\mu}\,\underset{\sim}{\bar{\mu}}'\,\underset{\sim}{\Sigma}^{-1}$. Now proceeding as in the previous case we obtain $E[W^{(p)}]^h$, $W^{(p)} = \prod_{i=1}^{p} (1-w_i)$ where

$$w_i = f_i/(1+f_i)$$

$$(2.16)\qquad E[W^{(p)}]^h = e^{-\mathrm{tr}\underset{\sim}{\Omega}} \frac{\tilde{\Gamma}_p(m+n)\tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(m+n+h)} \; {}_1\tilde{F}_1(m+n; m+n+h; \underset{\sim}{\Omega}) .$$

Using Mellin's transform and Meijer's G-function as in the previous case we get the density of $W^{(p)}$ as

$$(2.17) \qquad f(W^{(p)}) = e^{-\mathrm{tr}\underset{\sim}{\Omega}} \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa \tilde{C}_\kappa(\underset{\sim}{\Omega})}{k!} \{W^{(p)}\}^{n-p}$$

$$\cdot \; G^{p,0}_{p,p}\left(W^{(p)} \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}\right)$$

where

$$a_i = m+k_{p-i+1}+b_i, \quad b_i = i-1 \quad .$$

As in the covariance model case, we could also obtain the density explicitly for $p = 2,3,4$.

Case 3: Canonical Correlation. Let

$$(2.18) \qquad \begin{bmatrix} \underset{\sim}{X}: & p \times n \\ \underset{\sim}{Y}: & q \times n \end{bmatrix} \sim N_c \left[\underset{\sim}{0}, \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} \right]$$

$n \geq p+q$ and $q \geq p$. Then the joint density of the characteristic roots $0 < r_1^2 < \ldots < r_p^2$ $(\underset{\sim}{X}\,\bar{\underset{\sim}{Y}}')(\underset{\sim}{Y}\,\bar{\underset{\sim}{Y}}')^{-1}(\underset{\sim}{Y}\,\bar{\underset{\sim}{X}}')(\underset{\sim}{X}\,\bar{\underset{\sim}{X}}')^{-1}$ is given by Khatri [10] as

$$(2.19) \qquad C''(p)|\underset{\sim}{I}_p-\underset{\sim}{P}^2|^n \, {}_2\tilde{F}_1(n,n;q;\underset{\sim}{P}^2,\underset{\sim}{R}^2)|\underset{\sim}{R}^2|^{q-p}|\underset{\sim}{I}_p-\underset{\sim}{R}^2|^{n-q-p} \prod_{i>j} (r_i^2-r_j^2)^2$$

where

$$(2.20) \qquad C''(p) = \frac{\tilde{\Gamma}_p(n)\pi^{p(p-1)}}{\tilde{\Gamma}_p(n-q)\tilde{\Gamma}_p(q)\tilde{\Gamma}_p(p)} \, , \qquad \underset{\sim}{R}^2 = \text{diag}\,(r_1^2,\ldots,r_p^2)$$

and $\underset{\sim}{P}^2 = \text{diag}\,(\rho_1,\ldots,\rho_p)$ where ρ_i are the characteristic roots of $\underset{\sim}{\Sigma}_{11}^{-1}\,\underset{\sim}{\Sigma}_{12}\,\underset{\sim}{\Sigma}_{22}^{-1}\,\underset{\sim}{\Sigma}_{12}'$. Proceeding as in the previous cases we find $E[W^{(p)}]^h$, $W^{(p)} = \prod_{i=1}^{p}(1-r_i^2)$, by substituting in (2.8) as follows

$$(2.21) \qquad (n_1, n_2, \underset{\sim}{\Lambda}) \rightarrow (n, n-q, (\underset{\sim}{I}_p - \underset{\sim}{P}^2)^{-1}) \, .$$

Further the density of $W^{(p)}$ is obtained from (2.14) by making the above substitution and letting

$$a_i = q+k_{p-i+1}+b_i, \; b_i = i-1 \, .$$

As in the other cases the densities could be written out explicitly for $p = 2,3,4$.

3. <u>The Density Function of Pillai's V-Statistic in the Central Case For Two Roots</u>. If $\underset{\sim}{P}^2 = \underset{\sim}{0}$ in (2.19) we have the density function of the characteristic roots $r_1^2, r_2^2, \ldots, r_p^2$ in the central case. Letting $p = 2$ we have

$$(3.1) \qquad f_1(r_1^2, r_2^2) = C''(2)|\underset{\sim}{R}^2|^{q-2}|\underset{\sim}{I}_p - \underset{\sim}{R}^2|^{n-q-2}(r_1^2 - r_2^2)^2 \, .$$

Let $V = r_1^2+r_2^2$ and $G = r_1^2r_2^2$, $0 < V < 1$. To find the density function of V we make the above transformation and find

$$(3.2) \qquad f_2(V,G) = C''(2)G^{q-2}(1-V+G)^{n-q-2}(V^2-4G)^{\frac{1}{2}} .$$

Integrating G between the limits 0 to $V^2/4$, [16] and writing $(1-V+G)^{n-q-2}$ as a finite series we have

$$(3.3) \qquad f(V) = C''(2) \sum_{r=0}^{n-q-2} \binom{n-q-2}{r}(1-V)^{n-q-r-2} \cdot \left\{\int_0^{V^2/4} G^{q+r-2}(V^2-4G)^{\frac{1}{2}}dG\right\} .$$

Integrating the expression in the brackets by parts we find the density function of V to be

$$(3.4) \qquad f_3(V) = C''(2) \sum_{r=0}^{n-q-2} \binom{n-q-2}{2}(1-V)^{n-q-r-2} \cdot \frac{(q+r-2)!V^{2(q+r)-1}}{2^{q+r-1}3\cdot5\ldots(2(q+r)-1)} , \quad 0 < V < 1.$$

To obtain the density function of V in the range $1 \le V \le 2$ we change $r_i^2 \to 1-r_i^2$ in (3.1) and transform as before to get

$$(3.5) \qquad f_4(V,G) = C''(2)(1-V+G)^{q-2}G^{n-q-2}(V^2-4G)^{\frac{1}{2}} .$$

Writing $(1-V+G)^{q-2}$ as a series and integrating G between the limits 0 to $V^2/4$ we have

$$(3.6) \qquad f_5(V) = C''(2) \sum_{r=0}^{q-2} \binom{q-2}{r}(1-V)^{q-r-2}$$

$$\cdot \int_0^{V^2/4} G^{n+r-q-2}\,(V^2-4G)^{\frac{1}{2}}\,dG \quad .$$

Evaluating the integral by parts yields

$$(3.7) \qquad f_5(V) = C''(2) \sum_{r=0}^{q-2} \binom{q-2}{r}(1-V)^{q-r-2} \; \frac{(n+r-q-2)!V^{2(n+r-q)-1}}{2^{n+r-q-1}3\cdot5\ldots2(n+r-q)-1} \quad .$$

Transforming $V' = 2-V$, $1 \leq V \leq 2$ we find

$$(3.8) \qquad f_6(V) = C''(2) \sum_{r=0}^{q-2} \binom{q-2}{r}(V-1)^{q-r-2} \; \frac{(n+r-q-2)!(2-V)^{2(n+r-q)-1}}{2^{n+r-q-1}3\cdot5\ldots(2(n+r-q)-1)} \;,$$

$$1 \leq V \leq 2 \;.$$

By making the following changes in the parameters in (3.1)

$$(q, n-q, r_i^2) \rightarrow (m, n, w_i)$$

or

$$(q, n-q, r_i^2) \rightarrow (n_1, n_2, w_i)$$

we obtain the central density of the characteristic roots in the MANOVA or equality of two covariance matrices cases, respectively. Thus the results of this section and the next aren't restricted to the canonical correlation case, but extend to the two cases mentioned above as well.

4. <u>The Density Function of the U-Statistic in the Central Case for Two Roots.</u> To obtain the density function of U we make the transformation in (3.1)

$$r_i^2 = \lambda_i(1+\lambda_i)^{-1}$$

and find

$$g_1(\lambda_1,\lambda_2) = C''(2)|\underset{\sim}{Q}|^{q-2}|\underset{\sim}{I}_p+\underset{\sim}{Q}|^{-n}(\lambda_1-\lambda_2)^2 \tag{4.1}$$

where $\underset{\sim}{Q} = \text{diag}\,(\lambda_1,\lambda_2)$. Letting $U = \lambda_1+\lambda_2$ and $G = \lambda_1\lambda_2$ we see the joint density of U and G can be put in the form

$$g_2(U,G) = C''(2)\; G^{q-2}(1+\tfrac{U}{2})^{-2n}(U^2-4G)^{\frac{1}{2}} \cdot \left[1-\frac{U^2-4G}{4(1+\frac{U}{2})^2}\right]^{-n} . \tag{4.2}$$

Writing the part in brackets as a series and integrating G between the limits 0 to $U^2/4$ yields

$$(4.3) \qquad g_3(U) = C''(2)\left(1+\frac{U}{2}\right)^{-2n} \sum_{r=0}^{\infty} \frac{(-1)^r \binom{-n}{r}}{4^r\left(1+\frac{U}{2}\right)^{2r}}$$

$$\cdot \left\{ \int_0^{U^2/4} (U^2-4G)^{r+\frac{1}{2}} G^{q-2} \, dG \right\} .$$

Integrating the expression in the brackets by parts, we find the density of U for $p = 2$ is

$$(4.4) \qquad g_3(U) = C''(2) \sum_{r=0}^{\infty} (-1)^r \frac{\binom{-n}{r}(q-2)!\, U^{2r+2(q-3)+5}}{4^{r+q-1}\left(1+\frac{U}{2}\right)^{2r+2n}}$$

$$\cdot \frac{1}{(r+\frac{3}{2})(r+\frac{5}{2}) \cdots (r+\frac{2(q-3)+5}{2})} .$$

5. Complex Multivariate Beta Distribution and Independent Beta Variables. If $\underset{\sim}{X}:(p \times m)$ and $\underset{\sim}{Y}:(p \times n)$ are independent complex matrix variates $m \geq p$, whose columns are independent complex p-variate with covariance matrix $\underset{\sim}{\Sigma}$, and if $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$ and $E(\underset{\sim}{Y}) = \underset{\sim}{0}$, then the distribution of

$$(5.1) \qquad \underset{\sim}{F} = \bar{\underset{\sim}{X}}'(\underset{\sim}{Y}\,\bar{\underset{\sim}{Y}}')^{-1}\,\underset{\sim}{X}$$

depends on parameters

$$(5.2) \qquad \underset{\sim}{\Omega} = \bar{\underset{\sim}{\mu}}'\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{\mu}$$

and is [6]

$$(5.3) \qquad f(\underset{\sim}{F}) = k_1 e^{-\mathrm{tr}\underset{\sim}{\Omega}} \, {}_1\widetilde{F}_1(m+n;m;\underset{\sim}{\Omega}(\underset{\sim}{I}_p+\underset{\sim}{F}^{-1})^{-1})|\underset{\sim}{F}|^{m-p}|\underset{\sim}{I}_p+\underset{\sim}{F}|^{-(m+n)}(d\underset{\sim}{F})$$

where

$$(5.4) \qquad k_1 = \frac{\widetilde{\Gamma}_p(m+n)}{\widetilde{\Gamma}_p(m)\widetilde{\Gamma}_p(n)} \quad .$$

Since the density of $\underset{\sim}{F}$ for $p \geq m$ can be obtained from (5.3) by making the changes

$$(5.5) \qquad (p,m,n,) \rightarrow (m,p,m+n-p)$$

it suffices to work only with (5.3). Making the transformation

$$(5.6) \qquad \underset{\sim}{L} = (\underset{\sim}{I}_p+\underset{\sim}{F}^{-1})^{-1}$$

in (5.3) and noting $J(\underset{\sim}{L};\underset{\sim}{F}) = |\underset{\sim}{I}_p-\underset{\sim}{L}|^{-2p}$ [8] we have,

$$(5.7) \qquad f(\underset{\sim}{L}) = k_1 e^{-\lambda^2} \, {}_1\widetilde{F}_1(m+n;m;\lambda^2\underset{\sim}{L})|\underset{\sim}{L}|^{m-p}|\underset{\sim}{I}_p-\underset{\sim}{L}|^{n-p}(d\underset{\sim}{L}) \quad .$$

Proceeding in a manner similar to Khatri and Pillai [12] let

$$(5.8) \qquad \underset{\sim}{L} = \begin{array}{c} \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \underset{\sim}{\ell} & \underset{\sim}{L}_{11} \end{pmatrix} \\ \begin{matrix} 1 & \quad p-1 \end{matrix} \end{array} \begin{array}{c} 1 \\ p-1 \end{array}, \quad \underset{\sim}{L}_{22} = \underset{\sim}{L}_{11} - \underset{\sim}{\ell}\,\underset{\sim}{\ell}'/\ell_{11}$$

and note that $|\underset{\sim}{L}| = \ell_{11}|\underset{\sim}{L}_{22}|$ and

$$(5.9) \qquad |\underset{\sim}{I}_p - \underset{\sim}{L}| = (1-\ell_{11})|\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22} - \underset{\sim}{\ell}\,\underset{\sim}{\ell}'/[\ell_{11}(1-\ell_{11})]| \quad .$$

Now it can be shown that ℓ_{11} and $\{\underset{\sim}{L}_{22},\ \underset{\sim}{v} = \underset{\sim}{\ell}/[\ell_{11}(1-\ell_{11})^{\frac{1}{2}}]\}$ are independently distributed and their respective distributions are

$$(5.10) \qquad f_1(\ell_{11}) = [\beta(m,n)]^{-1} e^{-\lambda^2} {}_1\tilde{F}_1(m+n;m;\lambda^2\ell_{11})\ell_{11}^{m-1}(1-\ell_{11})^{n-1}$$

and

$$(5.11) \qquad f_2(\underset{\sim}{L}_{22},\underset{\sim}{v}) = k_2|\underset{\sim}{L}_{22}|^{m-p}|\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22} - \bar{\underset{\sim}{v}}'\underset{\sim}{v}|^{n-p} ,$$

where

$$(5.12) \qquad k_2 = k_1\ \beta(m,n) \quad .$$

For further independence, we can use the transformation

$$\underset{\sim}{u} = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-\frac{1}{2}}\ \underset{\sim}{v} \quad .$$

With Jacobian of transformation $|\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22}|^{-1}$ it can be shown that $\underset{\sim}{u}$ and $\underset{\sim}{L}_{22}$ are independently distributed and their respective distributions are

$$(5.13) \qquad f_3(\underset{\sim}{u}) = \pi^{-(p-1)}[\Gamma(n)/\Gamma(n-p+1)](1-\bar{\underset{\sim}{u}}'\underset{\sim}{u})^{n-p}$$

and

$$(5.14) \qquad f_4(\underset{\sim}{L}_{22}) = k_3|\underset{\sim}{L}_{22}|^{m-(p-1)-1}|\underset{\sim}{I}_{p-1}-\underset{\sim}{L}_{22}|^{n+1-(p-1)-1}$$

where

$$k_3 = \pi^{(p-1)}[\Gamma(n-p+1)/\Gamma(n)]k_2 \quad .$$

Notice that $\underset{\sim}{L}_{22}$: $(p-1) \times (p-1)$ is the central complex multivariate beta distribution with m and $n+1$ degrees of freedom. Making the transformation

$$(5.15) \qquad x_i = u_i/(1-\bar{u}_1u_1 - \dots - \bar{u}_{p-1}u_{p-1}), \; i = 1,2,\dots p-1, u_0 = 0$$

in (5.13) with Jacobian of transformation $\prod_{i=1}^{p-1} (1-\bar{x}_ix_i)^{p-i-1}$, we obtain the density of $\underset{\sim}{X} = (x_1,x_2,\dots,x_{p-1})'$ as

$$(5.16) \qquad f(\underset{\sim}{X}) = \pi^{-(p-1)} \prod_{i=1}^{p-1} \frac{\Gamma(n-i+1)}{\Gamma(n-i)} (1-\bar{x}_ix_i)^{n-i-1} \quad .$$

After making the transformation of $x_j = a_j + ib_j$ to polar coordinates (r_j, θ_j), we find with $\underset{\sim}{r} = (r_1,\ldots,r_{p-1})'$

$$(5.17) \qquad f(\underset{\sim}{r}) = \prod_{i=1}^{p-1} \frac{\Gamma(n-i+1)}{\Gamma(n-i)} (1-r_i^2)\ 2r_i\, dr_i \quad .$$

Finally the transformation $w_i = r_i^2$ yields independent real beta variates and their respective densities are given by

$$(5.18) \qquad f_i(w_i) = [B(1,n-i)]^{-1} (1-w_i)^{n-i-1} \quad .$$

6. <u>Complex Zonal Polynomials</u>. The zonal polynomials of a Hermitian matrix $\underset{\sim}{A}$ [6], are given by

$$(6.1) \qquad \tilde{C}_\kappa(\underset{\sim}{A}) = \chi_{[\kappa]}(1)\ \chi_{\{\kappa\}}(\underset{\sim}{A}) \quad ,$$

where $\kappa = (k_1,k_2,\ldots,k_m)$ is a partition of the integer k and $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group and is given by

$$(6.2) \qquad \chi_{[\kappa]}(1) = k! \prod_{i<j}^{m} (k_i-k_j-i+j) / \prod_{i=1}^{m} (k_i+m-i)! \quad ,$$

$\chi_{\{\kappa\}}(\underset{\sim}{A})$ is the character of the representation $\{\kappa\}$ of the linear group and is given as a symmetric function of the characteristic roots $e_1,e_2,\ldots,e_m$ of $\underset{\sim}{A}$ by

$$\chi_{\{\kappa\}}(\underset{\sim}{A}) = |(e_i^{k_j+m-j})|/|(e_i^{m-j})| \tag{6.3}$$

where the determinants are Vandermonde type. Further the following equality is satisfied

$$\sum_{\kappa} \tilde{C}_{\kappa}(\underset{\sim}{A}) = (a_1)^k \tag{6.4}$$

where a_i is the ith esf of the e_i's. Using the following lemma obtained by Pillai [15] we can get the zonal polynomials as a linear combination of the esf's. Tables 1-4 give $\chi_{\{\kappa\}}(\underset{\sim}{A})$ and $\chi_{[k]}(1)$ through degree 8.

Lemma: Let $D(g_s, g_{s-1}, \ldots, g_1)$, $(g_j \geq 0,\ j = 1,2,\ldots,s)$, denote the determinant

$$D(g_s, g_{s-1}, \ldots, g_1) = \begin{vmatrix} e_s^{g_s} & e_s^{g_{s-1}} & \cdots & e_s^{g_1} \\ \vdots & & & \\ e_1^{g_s} & e_1^{g_{s-1}} & & e_1^{g_1} \end{vmatrix} . \tag{6.5}$$

If $a_r (r \leq s)$ denotes the rth esf in s e's, then

$$\text{i)}\quad a_r D(g_s, g_{s-1}, \ldots, g_1) = \Sigma' \, D(g_s', g_{s-1}', \ldots, g_1') \tag{6.6}$$

where $g_j' = g_j + \delta$, $j = 1,2,\ldots,s$, $\delta = 0,1$ and Σ' denotes the sum over the $\binom{s}{r}$ combinations of s g's taken r at a time for which r indices $g_j' = g_j + 1$ such that $\delta = 1$ while for other indices $g_j' = g_j$ such that $\delta = 0$.

ii) $(a_r)^k (a_h)^\ell D(g_s, g_{s-1}, \ldots, g_1)$, $k, \ell \geq 0$, can be expressed as a sum of $\binom{s}{r}^k \binom{s}{h}^\ell$ determinants obtained by performing on $D(g_s, g_{s-1}, \ldots, g_1)$ in any order (i) k times and (i) ℓ times with $r=h$. However, if at least two of the indices in any determinant are equal, the corresponding term in the summation vanishes.

An example will suffice to show how $\chi_{\{\kappa\}}(\underset{\sim}{A})$ for any degree can be obtained from those of lower degree. Here we obtain $\chi_{\{\kappa\}}(\underset{\sim}{A})$ for $k = 3$. Let

$$(6.7) \qquad D = |(e_i^{m-j})|$$

and

$$(6.8) \qquad D(k_1+m-1, k_2+m-2, \ldots, 1) = |(e_i^{k_j+m-j})| \quad .$$

When $k = 2$ we have

$$(6.9) \qquad (a_1^2 - a_2)D = D(m+1, m-2, m-3, \ldots, 1) \quad \text{for} \quad \kappa = (2)$$

and

$$(6.10) \qquad a_2 D = D(m, m-1, m-3, \ldots, 1) \quad \text{for} \quad \kappa = (1^2) \quad .$$

Multiplying (6.9) and (6.10) by a_1, using Pillai's lemma, gives

$$(6.11) \qquad (a_1^3 - a_1 a_2)D = D(m+2, m-2, \ldots, 1) + D(m+1, m-1, m-3, \ldots, 1)$$

and

$$(6.12) \qquad a_1a_2D = D(m+1,m-1,m-3,\ldots,1) + D(m,m-1,m-2,m-4,\ldots,1).$$

But since

$$(6.13) \qquad a_3D = D(m,m-1,m-2,m-4,\ldots,1)$$

we have substituting in (6.12)

$$(6.14) \qquad (a_1a_2 - a_3)D = D(m+1,m-1,m-3,\ldots,1) \quad .$$

When $\kappa = (1^3)$ and $\kappa = (21)$ in (6.8), we obtain (6.13) and (6.14) respectively. Thus

$$\chi_{\{1^3\}}(\underset{\sim}{A}) = a_3 \quad \text{and} \quad \chi_{\{21\}}(\underset{\sim}{A}) = a_1a_2-a_3 \quad .$$

Substituting (6.14) in (6.11) we find

$$(6.15) \qquad (a_1^3-2a_1a_2+a_3)D = D(m+2,m-2,\ldots,1)$$

and thus

$$\chi_{\{3\}}(\underset{\sim}{A}) = a_1^3-2a_1a_2+a_3 \quad .$$

Table 1. Complex Zonal Polynomials of 1st - 5th Degree

	In terms of elementary symmetric functions of the latent roots of $\underset{\sim}{A}$	$\chi_{[\kappa]}(1)$
	1st Degree	
$\chi_{\{1\}}$	a_1	1
	2nd Degree	
$\chi_{\{2\}}$	$a_1^2 - a_2$	1
$\chi_{\{1^2\}}$	a_2	1
	3rd Degree	
$\chi_{\{3\}}$	$a_1^3 - 2a_1a_2 + a_3$	1
$\chi_{\{21\}}$	$a_1a_2 - a_3$	2
$\chi_{\{1^3\}}$	a_3	1

4th Degree

	a_1^4	$a_1^2a_2$	a_2^2	a_1a_3	a_4	$\chi_{[\kappa]}(1)$
$\chi_{\{4\}}$	1	-3	1	2	-1	1
$\chi_{\{31\}}$		1	-1	-1	1	3
$\chi_{\{2^2\}}$			1	-1	0	2
$\chi_{\{21^2\}}$				1	-1	3
$\chi_{\{1^4\}}$					1	1

5th Degree

	a_1^5	$a_1^3a_2$	$a_1a_2^2$	$a_1^2a_3$	a_2a_3	a_1a_4	a_5	$\chi_{[\kappa]}(1)$
$\chi_{\{5\}}$	1	-4	3	3	-2	-2	1	1
$\chi_{\{41\}}$		1	-2	-1	2	1	-1	4
$\chi_{\{32\}}$			1	-1	-1	1	0	5
$\chi_{\{31^2\}}$				1	-1	-1	1	6
$\chi_{\{2^21\}}$					1	-1	0	5
$\chi_{\{21^3\}}$						1	-1	4
$\chi_{\{1^5\}}$							1	1

Table 2. Complex Zonal Polynomials for 6th Degree

	a_1^6	$a_1^4a_2$	$a_1^2a_2^2$	$a_1^3a_3$	a_2^3	$a_1a_2a_3$	$a_1^2a_4$	a_3^2	a_2a_4	a_1a_5	a_6	$\chi_{[\kappa]}(1)$
$\chi_{\{6\}}$	1	-5	6	4	-1	-6	-3	1	2	2	-1	1
$\chi_{\{51\}}$		1	-3	-1	1	4	1	-1	-2	-1	1	5
$\chi_{\{42\}}$			1	-1	-1	0	1	0	1	-1	0	9
$\chi_{\{41^2\}}$				1	0	-2	-1	1	1	1	-1	10
$\chi_{\{3^2\}}$					1	-2	1	1	-1	0	0	5
$\chi_{\{321\}}$						1	-1	-1	0	1	0	16
$\chi_{\{31^3\}}$							1	0	-1	-1	1	10
$\chi_{\{2^3\}}$								1	-1	0	0	5
$\chi_{\{2^21^2\}}$									1	-1	0	9
$\chi_{\{21^4\}}$										1	-1	5
$\chi_{\{1^6\}}$											1	1

Table 3. Complex Zonal Polynomials for 7th Degree

	a_1^7	$a_1^5a_2$	$a_1^3a_2^2$	$a_1^4a_3$	$a_1a_2^3$	$a_1^2a_2a_3$	$a_1^3a_4$	$a_2^2a_3$	$a_1a_3^2$	$a_1a_2a_4$	$a_1^2a_5$	a_3a_4	a_2a_5	a_1a_6	a_7	$\chi_{[\kappa]}(1)$
$\chi_{\{7\}}$	1	-6	10	5	-4	-12	-4	3	3	6	3	-2	-2	-2	1	1
$\chi_{\{61\}}$		1	-4	-1	3	6	1	-3	-2	-4	-1	2	2	1	-1	6
$\chi_{\{52\}}$			1	-1	-2	1	1	2	-1	0	-1	0	-1	1	0	14
$\chi_{\{51^2\}}$				1	0	-3	-1	1	2	2	1	-2	-1	-1	1	15
$\chi_{\{43\}}$					1	-2	1	-1	2	0	-1	-1	1	0	0	14
$\chi_{\{421\}}$						1	-1	-1	-1	1	1	1	0	-1	0	35
$\chi_{\{41^3\}}$							1	0	0	-2	-1	1	1	1	-1	20
$\chi_{\{3^21\}}$								1	-1	-1	1	1	-1	0	0	21
$\chi_{\{32^2\}}$									1	-1	0	-1	1	0	0	21
$\chi_{\{321^2\}}$										1	-1	-1	0	1	0	35
$\chi_{\{31^4\}}$											1	0	-1	-1	1	15
$\chi_{\{2^31\}}$												1	-1	0	0	14
$\chi_{\{2^21^3\}}$													1	-1	0	14
$\chi_{\{21^5\}}$														1	-1	6
$\chi_{\{1^7\}}$															1	1

Table 4. Complex Zonal Polynomials for 8th Degree

	a_1^8	$a_1^6a_2$	$a_1^4a_2^2$	$a_1^5a_3$	$a_1^2a_2^3$	$a_1^3a_2a_3$	$a_1^4a_4$	a_2^4	$a_1a_2^2a_3$	$a_1^2a_3^2$	$a_1^2a_2a_4$	$a_1^3a_5$
$X_{\{8\}}$	1	-7	15	6	-10	-20	-5	1	12	6	12	4
$X_{\{71\}}$		1	-5	-1	6	8	1	-1	-9	-3	-6	-1
$X_{\{62\}}$			1	-1	-3	2	1	1	3	-2	-1	-1
$X_{\{61^2\}}$				1	0	-4	-1	0	3	3	3	1
$X_{\{53\}}$					1	-2	1	-1	1	2	-1	-1
$X_{\{521\}}$						1	-1	0	-2	-1	2	1
$X_{\{51^3\}}$							1	0	0	0	-3	-1
$X_{\{4^2\}}$								1	-3	1	2	-1
$X_{\{431\}}$									1	-1	-1	1
$X_{\{42^2\}}$										1	-1	0
$X_{\{421^2\}}$											1	-1
$X_{\{41^4\}}$												1
$X_{\{3^22\}}$												
$X_{\{3^21^2\}}$												
$X_{\{32^21\}}$												
$X_{\{321^3\}}$												
$X_{\{31^5\}}$												
$X_{\{2^4\}}$												
$X_{\{2^31^2\}}$												
$X_{\{2^21^4\}}$												
$X_{\{21^6\}}$												
$X_{\{1^8\}}$												

Table 4. (Cont'd.)

	$a_2a_3^2$	$a_2^2a_4$	$a_1a_3a_4$	$a_1a_2a_5$	$a_1^2a_6$	a_4^2	a_3a_5	a_2a_6	a_1a_7	a_8	$\chi_{[\kappa]}(1)$
$\chi_{\{8\}}$	-3	-3	-6	-6	-3	1	2	2	2	-1	1
$\chi_{\{71\}}$	3	3	4	4	1	-1	-2	-2	-1	1	7
$\chi_{\{62\}}$	-1	-2	2	0	1	0	0	1	-1	0	20
$\chi_{\{61^2\}}$	-2	-1	-4	-2	-1	1	2	1	1	-1	21
$\chi_{\{53\}}$	-1	2	-2	0	1	0	1	-1	0	0	28
$\chi_{\{521\}}$	2	0	0	-1	-1	0	-1	0	1	0	64
$\chi_{\{51^3\}}$	0	1	2	2	1	-1	-1	-1	-1	1	35
$\chi_{\{4^2\}}$	2	-2	-2	2	0	1	-1	0	0	0	14
$\chi_{\{431\}}$	-1	0	3	-1	-1	-1	0	1	0	0	70
$\chi_{\{42^2\}}$	-1	1	-1	1	0	0	1	-1	0	0	56
$\chi_{\{421^2\}}$	0	-1	-1	1	1	1	0	0	-1	0	90
$\chi_{\{41^4\}}$	0	0	0	-2	-1	0	1	1	1	-1	35
$\chi_{\{3^22\}}$	1	-1	-1	1	0	1	-1	0	0	0	42
$\chi_{\{3^21^2\}}$		1	-1	-1	1	0	1	-1	0	0	56
$\chi_{\{32^21\}}$			1	-1	0	-1	0	1	0	0	70
$\chi_{\{321^3\}}$				1	-1	0	-1	0	1	0	64
$\chi_{\{31^5\}}$					1	0	0	-1	-1	1	21
$\chi_{\{2^4\}}$						1	-1	0	0	0	14
$\chi_{\{2^31^2\}}$							1	-1	0	0	28
$\chi_{\{2^21^4\}}$								1	-1	0	20
$\chi_{\{21^6\}}$									1	-1	7
$\chi_{\{1^8\}}$										1	1

REFERENCES

[1] Al-Ani, S. (1968). Some distribution problems concerning characteristic roots and vectors in multivariate analysis. Mimeograph Series No. 162, Department of Statistics, Purdue University.

[2] Consul, P.C. (1966). On Some Inverse Mellin Integral Transforms. Academie Royale Des Science de Belgique. 52, 547-561.

[3] Consul, P.C. (1967). On the Exact Distributions of Likelihood Ratio Criteria for Testing Independence of Sets of Variates Under the Null Hypothesis. Ann. Math. Stat. 38, 1160-1169.

[4] Consul, P.C. (1967). On the Exact Distribution of the W Criterion for Testing the Sphericity in a p-variate Normal Distribution. Ann. Math. Stat. 38, 1170-1174.

[5] Goodman, N.R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution. (An introduction) Ann. Math. Stat. 34, 152-176.

[6] James, A.T. (1964). Distribution of matrix and latent roots derived from normal samples. Ann. Math. Stat. 35, 475-501.

[7] Jouris, G.M. (1968). On the non-central distributions of Wilks'-Λ for tests of three hypotheses. Mimeograph Series No. 164, Department of Statistics, Purdue University.

[8] Khatri, C.G. (1965). Classical statistical analysis based on a certain multivariate complex Gaussian distribution. Ann. Math. Stat. 36, 98-114.

[9] Khatri, C.G. (1965). A test for reality of a covariance matrix in certain complex Gaussian distributions. Ann. Math. Stat. 36, 115-119.

[10] Khatri, C.G. (1968). Noncentral distributions of the ith largest characteristic roots of three matrices concerning complex multivariate normal populations. Mimeograph Series No. 149, Department of Statistics, Purdue University.

[11] Khatri, C.G. (1968). On the moments of two matrices in three situations for complex multivariate normal populations. Mimeograph Series No. 161, Department of Statistics, Purdue University.

[12] Khatri, C.G. and Pillai, K.C.S. (1965). Some results on the non-central multivariate beta distribution and moments of traces of two matrices. Ann. Math. Stat. 36, 1511-1520.

[13] Meijer, C.S. (1946). Nederl. Akad. Wetensch. Proc., 49.

[14] Meijer, C.S. (1946). On the G-function I. Indagationes Mathematicae, 8, 124-134.

[15] Pillai, K.C.S. (1964). On the moments of elementary symmetric functions of the roots of two matrices. Ann. Math. Stat. 35, 1704-1712.

[16] Pillai, K.C.S. and Jayachandran, K. (1967). Power comparisons of tests of two multivariate hypotheses based on four criteria. Biometrika 54, 195-210.

[17] Wooding, R.A. (1956). The multivariate distribution of complex normal variables. Biometrika 43, 212-215.