An Approximation to the Distribution of the Largest Root of a Matrix and Percentage Points*

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K.C.S. Pillai and G.M. Jouris

Department of Statistics
Division of Mathematical Sciences

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Purdue University

1. Introduction

Let X_1: pxn and X_2: pxn be real random variables having the joint density function

$$(2\pi)^{-pn} |\Sigma_o|^{-\frac{1}{2}n} \exp \{-\tfrac{1}{2} \operatorname{tr} \Sigma_o^{-1}(X - \nu)(X - \nu)'\} \qquad -\infty \leq X \leq \infty$$

where

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \Sigma_o = \begin{pmatrix} \Sigma_1 & -\Sigma_2 \\ \Sigma_2 & \Sigma_1 \end{pmatrix}, \quad \nu = \begin{pmatrix} \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix},$$

Σ_1: pxp is a real symmetric positive definite (p.d.) matrix, Σ_2: pxp is a real skew symmetric matrix, μ_j: pxq and M_j: qxn (j=1,2) are given matrices or their joint density does not contain $\Sigma_1, \Sigma_2, \mu_1, \mu_2$ as parameters. Then it has been shown by Goodman [1] that the distribution of the complex matrix $Z = X_1 + iX_2$ $(i=(-1)^{\frac{1}{2}})$, is complex Gaussian and its density function is given by

$$N_c(\mu M, \Sigma) = \pi^{-pn} |\Sigma|^{-n} \exp\{-\operatorname{tr} \Sigma^{-1}(Z-\mu M)(Z-\mu M)'\}.$$

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where $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}_1 + i\underset{\sim}{\Sigma}_2$ is Hermitian p.d., i.e. $\bar{\underset{\sim}{\Sigma}}' = \underset{\sim}{\Sigma}$, $\underset{\sim}{\mu} = \underset{\sim}{\mu}_1 + i\underset{\sim}{\mu}_2$ and $\underset{\sim}{M} = \underset{\sim}{M}_1 + i\underset{\sim}{M}_2$.

Khatri [4],[5] has suggested the largest root for testing the hypotheses

$$H_o: \underset{\sim}{\mu} = 0 \quad \text{vs} \quad H_1: \underset{\sim}{\mu} \neq 0$$

and

$$H_o: \underset{\sim}{\Sigma}_2 = 0 \quad \text{vs} \quad H_2: \underset{\sim}{\Sigma}_2 \neq 0$$

and has shown the joint distribution of the latent roots $w_1, w_2, \ldots, w_q$ under the above null hypotheses can be written as given below [3]. From the form of the density (1.1) it is easy to see that the largest root serves as a criterion for testing the equality of covariance matrices from two complex normal populations, the independence of two sets of normal variates and other hypotheses similar to those usually discussed in the real case, in view of the fact that the densities under these null hypotheses reduce to (1.1) with different definitions of m and n [2].

$$(1.1) \qquad C_1\left\{\prod_{j=1}^{q} w_j^m(1-w_j)^n\right\}\left\{\prod_{i>j}(w_i - w_j)^2\right\} \; dw_1\ldots dw_q,$$

where

$$(1.2) \qquad C_1 = \prod_{j=1}^{q} \Gamma(m+n+q+j)/\{\Gamma(n+j)\Gamma(m+j)\Gamma(j)\}$$

and

$$0 \leq w_1 \leq w_2 \leq \cdots \leq w_q \leq 1.$$

Further, Khatri [3] has derived the distribution of w_q (or w_1) in a determinant form as follows

$$(1.3) \qquad \Pr\{w_q \leq x;m,n\} = C_1 \begin{vmatrix} \beta_o & \beta_1 & \cdots & \beta_{q-1} \\ \beta_1 & \beta_2 & \cdots & \beta_q \\ \cdot & \cdot & \cdots & \cdot \\ \beta_{q-1} & \beta_q & & \beta_{2q-2} \end{vmatrix} = C_1 |(\beta_{i+j-2})|,$$

where C_1 is defined in (1.2),

$$(1.4) \qquad \beta_{i+j-2} = \int_o^x w^{m+i+j-2}(1-w)^n \, dw$$

for $i,j=1,2,\ldots,q$, and (β_{i+j-2}) is a $q \times q$ matrix. $\Pr\{w_1 \leq x;m,n\}$ can be obtained from (1.3) using

$$(1.5) \qquad \Pr\{w_1 \leq x;m,n\} = 1-\Pr\{w_q \leq 1-x;n,m\}.$$

In this paper an approximation to the cdf of w_q at the upper end is obtained and upper 1% and 5% points are given for $q=2,3,4,5,6$ in Tables 2-11. A general form of the approximation is given, with the results for $q=2$ and 3 written out explicitly.

2. Case For Two Roots

It is easily seen from (1.3) that letting $q=2$ and expanding the determinant

$$(2.1)\quad \Pr\{w_2 \le x;m,n\} = C_1\left\{\int_0^x w^m(1-w)^n\,dw \int_0^x w^{m+2}(1-w)^n\,dw - \left(\int_0^x w^{m+1}(1-w)^n dw\right)^2\right\}.$$

Integrating by parts for integral values of m and n, we can write

$$(2.2)\qquad \int_0^x w^{m+\ell}(1-w)^n\,dw = \beta(m+\ell+1,n+1) - \sum_{j=0}^{m+\ell} \frac{(m+\ell)!}{j!}\,\frac{x^j(1-x)^{m+n+\ell-j+1}}{\prod_{k=0}^{m+\ell-j}(n+k+1)}$$

where $\beta(a,b)$ is the usual beta function. For small values of m, the approximation is obtained by substituting (2.2) in (2.1), expanding, neglecting terms of order $(1-x)^{2n}$ and higher [6]. This gives

$$(2.3)\qquad \Pr\{w_2 \le x;m,n\} \doteq C_1\Bigg[\beta(m+1,n+1)\beta_2 - \beta(m+3,n+1)\sum_{j=0}^{m}\frac{m!}{j!}\,\frac{x^j(1-x)^{m+n-j+1}}{\prod_{k=0}^{m-j}(n+k+1)}$$

$$- \beta(m+2,n+1)\beta_1 + \beta(m+2,n+1)\sum_{j=0}^{m+1}\frac{(m+1)!}{j!}\,\frac{x^j(1-x)^{m+n-j+2}}{\prod_{k=0}^{m-j+1}(n+k+1)}\Bigg].$$

Adding and subtracting $\beta(m+3,n+1)\beta(m+1,n+1)$ and $\beta(m+2,n+1)^2$ in (2.3) we find

$$(2.4) \qquad \Pr\{w_2 \le x;m,n\} \doteq C_1[\beta(m+1,n+1)\beta_2 + \beta(m+3,n+1)\beta_o - 2\beta(m+2,n+1)\beta_1] - 1.$$

Noting from (2.1) with $x=1$ that

$$(2.5) \qquad C_1 = [\beta(m+3,n+1)\beta(m+1,n+1) - \beta(m+2,n+1)^2]^{-1},$$

and simplifying in (2.4) we find

$$(2.6) \qquad \Pr\{w_2 \le x;m,n\} \doteq \frac{m+n+2}{\beta(m+2,n+2)}\left[\beta_2 + \frac{(m+1)_2}{(m+n+2)_2}\,\beta_o - \frac{2m}{m+n+2}\,\beta_1\right] - 1$$

where

$$(2.7) \qquad (a)_k = a(a+1)\ \dots\ (a+k-1).$$

This approximation is very simple for computational use and no products of incomplete beta functions are involved. Upper 1% and 5% points using (2.6) are given in Tables 2,3. The error involved in using this approximation has been computed and the difference between the exact and approximate percentage points occurs in the seventh place. (See Table 1).

3. Case For Three Roots

When there are 3 roots, we have

(3.1) $$\Pr\{w_3 \le x;m,n\} = C_1\{\beta_o\beta_2\beta_4 + 2\beta_1\beta_2\beta_3 - \beta_2^3 - \beta_o\beta_3^2 - \beta_1^2\beta_4\}$$

where β_ℓ is defined in (1.4) and C_1 in (1.2). As in the two-roots case we write the β_ℓ's as finite sums, expand, neglect terms involving $(1-x)^{2n}$ and higher powers and find

(3.2) $$\Pr\{w_3 \le x;m,n\} \doteq C_1\{[\beta(3)\beta(5)-\beta(4)^2]\beta_o + 2[\beta(3)\beta(4)-\beta(2)\beta(5)]\beta_1$$

$$+[2\beta(2)\beta(4)+\beta(1)\beta(5)-3\beta(3)^2]\beta_2 + 2[\beta(2)\beta(3)-\beta(1)\beta(4)]\beta_3 + [\beta(1)\beta(3)-\beta(2)^2]\beta_4\}-2$$

where $\beta(j) = \beta(m+j,n+1)$. Simplifying in (3.2) we find

(3.3) $$\Pr\{w_3 \le x;m,n\} \doteq [\beta(m+1,n+3)]^{-1}\left\{\frac{(m+2)_2}{2}\beta_o - 2(m+3)(m+n+4)\beta_1\right.$$

$$\left. + \frac{3\{m(m+n+6)+2n+7\}(m+n+4)}{m+1}\beta_2 - \frac{2(m+n+3)_3}{m+1}\beta_3 + \frac{(m+n+4)(m+n+3)_3}{2(m+1)_2}\right\} - 2.$$

As in the two-roots case, no products of incomplete beta functions are involved. Upper 1% and 5% points using (3.3) are given in Tables 4 and 5. Comparison of some exact and approximate values is made in Table 1.

4. Case For q Roots

Denote the product $\beta_{i_1}\beta_{i_2}\cdots\beta_{i_q}$ occuring in $|(\beta_\ell)|$ by $\beta_{i_1 i_2 \ldots i_q}$ where i_j are any of the integers $0,1,\ldots,2q-2$. It can be seen from cases with $q=2$ and 3 that $\beta_{i_1 i_2 \ldots i_q}$ is approximated by $\beta'_{i_1 i_2 \ldots i_q}$

(4.1) $$\beta'_{i_1 i_2 \ldots i_q} = \sum_{j=1}^{q} \prod_{\substack{k=1 \\ k\neq j}}^{q} \beta(m+i_k+1, n+1)\beta_{i_j}$$

where β_{i_j} is defined in (1.4). Thus the distribution of w_q can be approximated by

(4.2) $$\Pr\{w_q \leq x; m,n\} \doteq C_1 \, |(\beta_\ell)|' - (q-1)$$

where $|(\beta_\ell)|'$ is obtained by replacing $\beta_{i_1 i_2 \ldots i_q}$ in $|(\beta_\ell)|$ by $\beta'_{i_1 i_2 \ldots i_q}$. By collecting the coefficients of incomplete beta functions we get the following form which is simpler for computer computations

(4.3) $$\Pr\{w_q \leq x; m,n\} = C_1 \sum_{k=0}^{2q-2} D'_k \beta_k - (q-1),$$

where D'_k is the sum of the cofactors of $\beta(k+1)$ in the qxq matrix

$$\begin{vmatrix} \beta(1) & \beta(2) & \cdots & \beta(q) \\ \beta(2) & \beta(3) & \cdots & \beta(q+1) \\ \cdot & \cdot & \cdots & \cdot \\ \beta(q) & \beta(q+1) & \cdots & \beta(2q-1) \end{vmatrix}$$

where $\beta(j)$ is defined as $\beta(m+j,n+1)$, the usual beta function. Letting $q=2$ and 3 in (4.3) we get (2.4) and (3.2) respectively.

5. Computation of Percentage Points

Based on the results of the preceeding sections upper 1% and 5% points were computed for $q=2,3,4,5,6$ on the CDC 6500 computer. Results are given to five significant figures for the arguments $m=0(1)5,7,10,15$ and $n=5(5)30(10)40(20)120(40)200,300,500,1000$. As can be seen from the comparisons below the percentage points from the exact and approximate cdf's agree through five figures and generally six.

Table 1. Comparison of Percentage Points From the Exact and Approximate CDF's

			1%		5%	
q	m	n	exact	approximate	exact	approximate
2	0	30	.246078	.246078	.194089	.194089
2	10	160	.142231	.142231	.124867	.124867
3	0	30	.332512	.332512	.280221	.280222
3	10	160	.166031	.166031	.148441	.148441
4	0	30	.353382	.353384	.404003	.404003
4	15	200	.183258	.183258	.167724	.167725
5	0	5	.906746	.906746	.867886	.867887
6	5	100	.278743	.278744	.254438	.254439

In addition to the above, the approximate expression is attractive for two reasons; first, computation time is less for the approximation because we don't evaluate a determinant at each step in the iteration scheme, as we do for the exact case; second, round off error is less troublesome in the approximate expression.

Table 2. Upper 1% Points of the Largest Root for q=2

n \ m	0	1	2	3	4	5	7	10	15
5	0.73163	0.78184	0.81537	0.83962	0.85809	0.87266	0.89424	0.91559	0.93676
10	.53116	.59150	.63618	.67120	.69962	.72326	.76050	.80033	.84333
15	.41313	.47060	.51546	.55216	.58304	.60955	.65296	.70184	.75786
20	.33720	.38956	.43175	.46720	.49774	.52449	.56946	.62199	.68488
25	.28459	.33194	.37091	.40426	.43346	.45942	.50389	.55722	.62327
30	0.24608	0.28901	0.32489	0.35600	0.38357	0.40834	0.45138	0.50409	0.57110
40	.19356	.22946	.26007	.28710	.31144	.33364	.37300	.42265	.48826
60	.13558	.16236	.18570	.20672	.22599	.24387	.27631	.31867	.37738
80	.10430	.12558	.14434	.16141	.17722	.19203	.21924	.25548	.30712
100	.084737	.10237	.11803	.13237	.14574	.15832	.18165	.21311	.25878
120	0.071353	0.086398	0.099823	0.11218	0.12374	0.13467	0.15505	0.18277	0.22353
160	.054221	.065846	.076284	.085945	.095034	.10367	.11900	.14223	.17563
200	.043723	.053191	.061725	.069652	.077135	.084271	.097732	.11640	.14461
300	.029460	.035927	.041786	.047253	.052438	.057405	.066832	.080034	.10029
500	.017829	.021785	.025384	.028756	.031966	.035051	.040938	.049252	.062166
1000	.0089721	.010980	.012811	.014533	.016176	.017760	.020794	.025107	.031871

Table 3. Upper 5% Points of the Largest Root for q=2

n \ m	0	1	2	3	4	5	7	10	15
5	0.63265	0.69818	0.74271	0.77533	0.80039	0.82031	0.85006	0.87975	0.90950
10	.43902	.50675	.55776	.59825	.63143	.65925	.70356	.75138	.80371
15	.33433	.39500	.44309	.48290	.51673	.54598	.59435	.64947	.71347
20	.26957	.32303	.36671	.40383	.43608	.46455	.51286	.56993	.63918
25	.22572	.27305	.31252	.34665	.37678	.40376	.45038	.50695	.57791
30	0.19409	0.23638	0.27217	0.30350	0.33149	0.35681	0.40118	0.45612	0.52687
40	.15156	.18626	.21619	.24286	.26705	.28925	.32893	.37949	.44714
60	.10534	.13073	.15308	.17337	.19210	.20957	.24150	.28358	.34255
80	.080711	.10068	.11846	.13475	.14994	.16423	.19067	.22619	.27734
100	.065413	.081862	.096599	.11019	.12294	.13500	.15749	.18808	.23290
120	0.054990	0.068967	0.081547	0.093203	0.10417	0.11459	0.13413	0.16093	0.20070
160	.041699	.052443	.062169	.071230	.079800	.087984	.10344	.12487	.15720
200	.033582	.042306	.050231	.057639	.064668	.071400	.084170	.10200	.12918
300	.022589	.028522	.033937	.039022	.043868	.048529	.057421	.069958	.089343
500	.013651	.017268	.020583	.023707	.026695	.029578	.035105	.042961	.055254
1000	.0068625	.0086933	.010376	.011966	.013491	.014966	.017805	.021864	.028277

Table 4. Upper 1% Points of the Largest Root for q=3

n \ m	0	1	2	3	4	5	7	10	15
5	0.82375	0.85143	0.87131	0.88637	0.89821	0.90777	0.92231	0.93711	0.95222
10	.64650	.68669	.71788	.74301	.76379	.78132	.80934	.83983	.87331
15	.52543	.56765	.60185	.63041	.65476	.67586	.71074	.75044	.79641
20	.44092	.48177	.51575	.54480	.57008	.59238	.63013	.67453	.72803
25	.37927	.41773	.45033	.47865	.50366	.52602	.56453	.61094	.66864
30	0.33251	0.36841	0.39925	0.42638	0.45059	0.47246	0.51062	0.55750	0.61724
40	.26649	.29770	.32502	.34944	.37158	.39185	.42791	.47348	.53372
60	.19053	.21482	.23651	.25628	.27451	.29148	.32234	.36267	.41848
80	.14819	.16793	.18576	.20217	.21745	.23180	.25824	.29346	.34355
100	.12123	.13782	.15290	.16688	.17997	.19234	.21531	.24630	.29118
120	0.10256	0.11685	0.12991	0.14206	0.15349	0.16433	0.18458	0.21214	0.25258
160	.078406	.089581	.099855	.10947	.11857	.12724	.14356	.16603	.19957
200	.063457	.072627	.081090	.089040	.096582	.10380	.11744	.13636	.16491
300	.042971	.049297	.055165	.060702	.065983	.071057	.080708	.094234	.11495
500	.026110	.030012	.033647	.037089	.040386	.043565	.049643	.058234	.071564
1000	.013180	.015173	.017034	.018803	.020501	.022143	.025297	.029783	.036813

Table 5. Upper 5% Points of the Largest Root for q=3

n \ m	0	1	2	3	4	5	7	10	15
5	0.75420	0.79166	0.81882	0.83952	0.85589	0.86916	0.88944	0.91022	0.93158
10	.57006	.61690	.65359	.68337	.70816	.72917	.76298	.80005	.84112
15	.45433	.50053	.53828	.57004	.59728	.62101	.66049	.70582	.75881
20	.37674	.41989	.45608	.48723	.51450	.53868	.57988	.62874	.68820
25	.32148	.36119	.39512	.42479	.45114	.47481	.51582	.56568	.62826
30	0.28022	0.31672	0.34830	0.37625	0.40134	0.42410	0.46405	0.51355	0.57723
40	.22286	.25394	.28133	.30595	.32839	.34903	.38593	.43292	.49563
60	.15800	.18168	.20295	.22243	.24048	.25734	.28815	.32870	.38530
80	.12235	.14138	.15866	.17463	.18957	.20365	.22970	.26463	.31471
100	.099816	.11569	.13021	.14371	.15641	.16845	.19090	.22138	.26585
120	0.084286	0.097902	0.11040	0.12208	0.13311	0.14361	0.16329	0.19024	0.23006
160	.064281	.074868	.084645	.093829	.10255	.11089	.12664	.14844	.18120
200	.051949	.060606	.068628	.076190	.083393	.090302	.10341	.12169	.14944
300	.035108	.041052	.046587	.051828	.056842	.061672	.070891	.083873	.10388
500	.021298	.024951	.028365	.031610	.034725	.037737	.043516	.051719	.064517
1000	.010738	.012597	.014341	.016002	.017602	.019153	.022140	.026407	.033128

Table 4. Upper 1% Points of the Largest Root for q=4

n \ m	0	1	2	3	4	5	7	10	15
5	0.87509	0.89200	0.90477	0.91478	0.92285	0.92951	0.93985	0.95067	0.96202
10	.72325	.75144	.77404	.79266	.80830	.82167	.84335	.86735	.89420
15	.60746	.63947	.66606	.68862	.70810	.72512	.75355	.78631	.82470
20	.52116	.55371	.58136	.60532	.62638	.64508	.67699	.71484	.76085
25	.45542	.48714	.51454	.53863	.56008	.57936	.61278	.65332	.70404
30	0.40400	0.43437	0.46093	0.48453	0.50576	0.52502	0.55881	0.60055	0.65398
40	.32914	.35648	.38080	.40274	.42276	.44116	.47403	.51571	.57097
60	.23973	.26184	.28189	.30031	.31739	.33335	.36244	.40055	.45336
80	.18837	.20673	.22355	.23915	.25374	.26751	.29292	.32684	.37510
100	.15510	.17073	.18514	.19860	.21128	.22328	.24564	.27585	.31961
120	0.13180	0.14539	0.15798	0.16978	0.18094	0.19156	0.21144	0.23854	0.27830
160	.10134	.11209	.12211	.13156	.14054	.14914	.16534	.18768	.22103
200	.082305	.091193	.099506	.10737	.11488	.12208	.13571	.15467	.18326
300	.056004	.062195	.068016	.073551	.078854	.083965	.093709	.10739	.12835
500	.034164	.038014	.041648	.045118	.048455	.051682	.057869	.066630	.08023
1000	.017298	.019276	.021149	.022942	.024672	.026350	.029579	.034183	.041402

Table 7. Upper 5% Points of the Largest Root for q=4

n \ m	0	1	2	3	4	5	7	10	15
5	0.82407	0.84740	0.86511	0.87905	0.89033	0.89965	0.91420	0.92947	0.94557
10	.66010	.69366	.72072	.74312	.76203	.77824	.80465	.83405	.86715
15	.54470	.58047	.61034	.63583	.65792	.67730	.70983	.74754	.79206
20	.46207	.49715	.52713	.55323	.57625	.59679	.63198	.67400	.72548
25	.40064	.43405	.46306	.48868	.51158	.53225	.56823	.61217	.66756
30	0.35338	0.38485	0.41250	0.43718	0.45947	0.47976	0.51552	0.55997	0.61732
40	.28566	.31338	.33815	.36059	.38114	.40009	.43408	.47745	.53537
60	.20626	.22818	.24813	.26652	.28363	.29966	.32900	.36765	.42156
80	.16132	.17929	.19582	.21121	.22565	.23929	.26457	.29848	.34706
100	.13243	.14763	.16169	.17485	.18728	.19909	.22114	.25109	.29473
120	0.11231	0.12545	0.13767	0.14916	0.16004	0.17043	0.18992	0.21663	0.25604
160	.086129	.096463	.10612	.11526	.12396	.13230	.14807	.16992	.20271
200	.069843	.078351	.086332	.093904	.10114	.10810	.12132	.13976	.16772
300	.047422	.053318	.058876	.064172	.069258	.074168	.083555	.096781	.11714
500	.028878	.032528	.035983	.039288	.042473	.045559	.051488	.059913	.073044
1000	.014602	.016471	.018245	.019948	.021593	.023191	.026275	.030684	.037625

Table 8 . Upper 1% Points of the Largest Root for q=5

n \ m	0	1	2	3	4	5	7	10	15
5	0.90675	0.91785	0.92654	0.93354	0.93930	0.94414	0.95179	0.96006	0.96964
10	.77719	.79776	.81465	.82882	.84089	.85132	.86846	.88775	.90963
15	.66956	.69444	.71549	.73360	.74940	.76332	.78679	.81416	.84656
20	.58489	.61127	.63404	.65399	.67167	.68747	.71464	.74716	.78704
25	.51795	.54445	.56766	.58825	.60673	.62343	.65254	.68812	.73298
30	0.46418	0.49012	0.51310	0.53369	0.55234	0.56934	0.59930	0.63654	0.68450
40	.38370	.40780	.42946	.44919	.46727	.48395	.51387	.55198	.60273
60	.28423	.30444	.32294	.34005	.35600	.37092	.39824	.43413	.48396
80	.22548	.24259	.25841	.27319	.28707	.30019	.32447	.35700	.40329
100	.18678	.20154	.21527	.22817	.24037	.25195	.27357	.30285	.34532
120	0.15940	0.17234	0.18444	0.19585	0.20668	0.21701	0.23639	0.26288	0.30177
160	.12324	.13359	.14333	.15256	.16138	.16982	.18579	.20787	.24084
200	.10043	.10905	.11719	.12493	.13234	.13947	.15300	.17185	.20031
300	.068662	.074724	.080474	.085972	.091260	.096367	.10613	.11986	.14092
500	.042050	.045848	.049467	.052940	.056293	.059544	.065792	.074657	.088430
1000	.021337	.023318	.025194	.027000	.028748	.030449	.033730	.038418	.045776

Table [illegible]. Upper 5% Points of the Largest Root for q=5

n \ m	0	1	2	3	4	5	7	10	15
5	0.86789	0.88338	0.89554	0.90537	0.91348	0.92029	0.93111	0.94276	0.95554
10	.72458	.74939	.76985	.78708	.80180	.81455	.83558	.85934	.88648
15	.61439	.64258	.66652	.68720	.70529	.72128	.74834	.78004	.81787
20	.53107	.55989	.58488	.60684	.62637	.64388	.67408	.71042	.75528
25	.46679	.49507	.51993	.54208	.56199	.58005	.61165	.65046	.69969
30	0.41600	0.44323	0.46744	0.48921	0.50897	0.52703	0.55900	0.59893	0.65067
40	.34122	.36596	.38829	.40865	.42738	.44471	.47589	.51580	.56926
60	.25055	.27081	.28941	.30666	.32277	.33789	.36565	.40227	.45343
80	.19780	.21474	.23044	.24514	.25899	.27209	.29643	.32913	.37599
100	.16335	.17785	.19137	.20410	.21616	.22763	.24911	.27831	.32088
120	0.13911	0.15175	0.16360	0.17480	0.18544	0.19562	0.21476	0.24101	0.27975
160	.10726	.11730	.12677	.13577	.14438	.15264	.16829	.18999	.22256
200	.087264	.094493	.10347	.11098	.11818	.12512	.13832	.15676	.18473
300	.059521	.065347	.070882	.076184	.081291	.086232	.095694	.10904	.12959
500	.036382	.040017	.043484	.046817	.050039	.053168	.059191	.067761	.081120
1000	.018450	.020325	.022114	.023839	.025514	.027144	.030294	.034807	.041912

Table 10. Upper 1% Points of the Largest Root for q=6

n \ m	0	1	2	3	4	5	7	10	15
5	0.92768	0.93537	0.94156	0.94665	0.95091	0.95456	0.95924	0.97249	0.98011
10	.81665	.83213	.84508	.85611	.86561	.87389	.88751	.90160	.93327
15	.71783	.73757	.75451	.76926	.78223	.79376	.81335	.83734	.87568
20	.63648	.65818	.67714	.69391	.70887	.72234	.74560	.77358	.80940
25	.57007	.59245	.61226	.62999	.64598	.66052	.68603	.71760	.76176
30	0.51545	0.53781	0.55780	0.57585	0.59227	0.60731	0.63398	0.66750	0.71631
40	.43163	.45300	.47238	.49012	.50646	.52159	.54884	.58380	.62935
60	.32472	.34325	.36034	.37622	.39107	.40502	.43062	.46433	.51201
80	.25991	.27589	.29078	.30474	.31790	.33037	.35352	.38460	.42872
100	.21656	.23051	.24358	.25591	.26761	.27874	.29958	.32785	.36905
120	0.18556	0.19790	0.20951	0.22051	0.23098	0.24099	0.25982	0.28558	0.32356
160	.14423	.15421	.16365	.17265	.18126	.18953	.20521	.22692	.25928
200	.11793	.12630	.13424	.14183	.14912	.15615	.16952	.18818	.21628
300	.080997	.086935	.092602	.098044	.10329	.10837	.11811	.13183	.15284
500	.049794	.053543	.057137	.060602	.063956	.067216	.073494	.082419	.096317
1000	.025362	.027312	.029186	.030998	.032758	.034473	.037790	.042539	.049989

Table 15. Upper 5% Points of the Largest Root for q=6

n \ m	0	1	2	3	4	5	7	10	15
5	0.89716	0.90796	0.91667	0.92385	0.92988	0.93502	0.94291	0.95382	0.96990
10	.77232	.79117	.80699	.82049	.83217	.84237	.85933	.87831	.92571
15	.66925	.69183	.71127	.72824	.74321	.75654	.77926	.80640	.84101
20	.58762	.61157	.63256	.65117	.66782	.68283	.70888	.74044	.78006
25	.52259	.54671	.56812	.58733	.60471	.62053	.64836	.68281	.72777
30	0.46998	0.49367	0.51492	0.53415	0.55169	0.56778	0.59639	0.63236	0.68045
40	.39058	.41271	.43285	.45131	.46836	.48418	.51275	.54948	.59853
60	.29127	.30999	.32731	.34344	.35855	.37277	.39893	.43353	.48214
80	.23199	.24793	.26282	.27680	.29001	.30255	.32588	.35729	.40229
100	.19268	.20649	.21945	.23170	.24334	.25444	.27525	.30359	.34498
120	0.16474	0.17688	0.18833	0.19919	0.20955	0.21947	0.23816	0.26383	0.30178
160	.12768	.13744	.14668	.15550	.16396	.17209	.18753	.20898	.24115
200	.10422	.11236	.12010	.12751	.13463	.14151	.15462	.17296	.20076
300	.071406	.077150	.082639	.087917	.093015	.097956	.10743	.12083	.14145
500	.043810	.047420	.050884	.054227	.057468	.060620	.066699	.07536	.088882
1000	.022280	.024150	.025950	.027692	.029385	.031037	.034236	.038825	.046054

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