ON CLASSICAL AND COMPLEX MULTIVARIATE NORMAL DISTRIBUTION PROBLEMS[*]

by

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Page	Line	Change	To
1	19	$\underset{\sim}{Y}\underset{\sim}{Y}'$	$\underset{\sim}{Y}'\underset{\sim}{Y}$
6	17	$K(p,f)$	$K(p,f)$
23	8	$\prod_{\ell=1}^{p-3} \frac{\Gamma(g_\ell + j_\ell)}{\Gamma(h_\ell)}$	$\prod_{\ell=1}^{n-3} \frac{\Gamma(g_\ell + j_\ell)}{\Gamma(h_\ell)}$
23	9	f_{p-2}	f_{n-2}
23	9	g_{p-2}	g_{n-2}
31	6	$\tilde{\underset{\sim}{\Sigma}}'_{12}$	$\underset{\sim}{\Sigma}'_{12}$
31	10	$\underset{\sim}{\Sigma}_{12}\tilde{\underset{\sim}{\Sigma}}_{22}^{-1}\underset{\sim}{\Sigma}'_{12}$	$\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}'_{12}$
36	1		include $\binom{b}{\ell}$ on r.h.s.
39	8	$n-\tilde{p}_3$	$n-\tilde{p}_3$
39	12	$\log S_1$	$-\log S_1$
40	4	$\binom{b'}{\ell}$	$\binom{b'}{m}$
42	10	Z_1	Z'_1
42	14	$\sum_{m=0}^{f_2-2}$	$\sum_{m=0}^{f_2/2}$

Page	Line	Change	To
42	16	Z_1	Z_1'
44	1,2	$f_1 + 2\ell - 4$	$f_1 + 2m - 4$
44	1	$2\beta_2$	$\frac{1}{2}\beta_2$
45	14	$f_1 + m - 1$	$f_1' + m - 1$
45	16	$\binom{f_2 - 1}{m}$	$\binom{f_2 + 1}{m}$
57,58	28	$(p^2 - \Sigma p_i^2)^{-1} \log V$	$(p^2 - \Sigma p_i^2)^{-1}\} \log V$
63	16	$N_e(\underset{\sim}{\nu}, \underset{\sim}{\Sigma}_D)$	$N_e(\underset{\sim}{\mu}\ \underset{\sim}{M}, \underset{\sim}{\Sigma})$
65	21	${}_1F_0$	${}_1\tilde{F}_0$
70	3	$\bar{\underset{\sim}{\mu}}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}.$	$\underset{\sim}{\mu}\,\bar{\underset{\sim}{\mu}}'\underset{\sim}{\Sigma}^{-1}.$
71	3	$\underset{\sim}{\Sigma}_{12}'$	$\bar{\underset{\sim}{\Sigma}}_{12}$
71	10	$\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{21}\underset{\sim}{\Sigma}_{11}^{-1}$	$\underset{\sim}{\Sigma}_{11}^{-1}\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{12}'$
88	3	$\vert(\beta_{i+j-2})\vert$	$c_1\vert(\beta_{i+j-2})\vert$

CHAPTER I

ON THE MOMENTS OF ELEMENTARY SYMMETRIC FUNCTIONS OF THE ROOTS OF TWO MATRICES

1. Introduction and Summary

Let $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ be two symmetric matrices of order p, $\underset{\sim}{A}_1$, positive definite and having a Wishart distribution [3], [33], with f_1 degrees of freedom and $\underset{\sim}{A}_2$, at least positive semi-definite and having a non-central (linear) Wishart distribution [2], [4], [14], [33], [34] with f_2 degrees of freedom. Now let

$$\underset{\sim}{A}_2 = \underset{\sim}{C}\,\underset{\sim}{Y}\,\underset{\sim}{Y}'\,\underset{\sim}{C}'$$

where $\underset{\sim}{Y}$ is $p \times f_2$ and $\underset{\sim}{C}$ is a lower triangular matrix such that

$$\underset{\sim}{A}_1 + \underset{\sim}{A}_2 = \underset{\sim}{C}\,\underset{\sim}{C}' \quad .$$

Now consider the $s(=\text{minimum}\ (f_2,p))$ non-zero characteristic roots of the matrix $\underset{\sim}{Y}\,\underset{\sim}{Y}'$. It can be shown that the density function of the characteristic roots of $\underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \leq p$ can be obtained from that of the characteristic roots of $\underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \geq p$ if in the latter case the following changes are made [16], [33];

$$(1.1) \qquad (f_1,f_2,p) \rightarrow (f_1+f_2-p,\ p,\ f_2) \quad .$$

Now define $U_i^{(p)} = \mathrm{tr}_i(\underset{\sim}{I}_p - \underset{\sim}{Y}\,\underset{\sim}{Y}')^{-1} - p = \mathrm{tr}_i(\underset{\sim}{I}_{f_2} - \underset{\sim}{Y}'\,\underset{\sim}{Y})^{-1} - f_2$, where $\mathrm{tr}_i\underset{\sim}{A}$ denotes the ith elementary symmetric function (esf) of the characteristic roots of $\underset{\sim}{A}$. In view of (1.1) we only consider $U_i^{(s)}$ when $s = p$, i.e. $U_i^{(p)}$ based on the density function [24] of $\underset{\sim}{L} = \underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \geq p$. Now define $V_i^{(p)} = \mathrm{tr}_i\underset{\sim}{L}$ and further $\underset{\sim}{U} = (\underset{\sim}{I}_p - \underset{\sim}{Y}\,\underset{\sim}{Y}')^{-1} - p$. Khatri and Pillai [23] have obtained some results towards finding the moments of $U_i^{(p)}$ and $V_i^{(p)}$ and in this paper an attempt is made to give general expressions of the first three moments of $U_i^{(p)}$ and the first two moments of $V_i^{(p)}$. Further, the moments of the second esf of a matrix in the non-central means case (James [12]) have been considered and tabulation of certain constants made which arose in this context.

2. Results on the ith esf of the Roots of a Matrix

The lemma below is proved by Khatri and Pillai [23] and is used to obtain the results of Section 3.

Lemma: Let $\underset{\sim}{L} = \begin{array}{cc} \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \ell & \underset{\sim}{L}_{11} \end{pmatrix} & \begin{array}{c} 1 \\ p-1 \end{array} \\ \begin{array}{cc} 1 & p-1 \end{array} & \end{array}$ be a symmetric matrix of order p, $\underset{\sim}{L}_{22} = \underset{\sim}{L}_{11} - \underset{\sim}{\ell}\,\underset{\sim}{\ell}' / \ell_{11}$, $\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22}$ be positive definite and $\underset{\sim}{u} = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-\frac{1}{2}}\underset{\sim}{\ell} / \{\ell_{11}(1-\ell_{11})\}^{\frac{1}{2}}$. Further let $\underset{\sim}{U} = (\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} - \underset{\sim}{I}_p$ and $\underset{\sim}{M} = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-1} - \underset{\sim}{I}_{p-1}$. Then

$$\mathrm{tr}_i \underset{\sim}{U} = \ell_{11}\{(1-\ell_{11})(1-\underset{\sim}{u}'\underset{\sim}{u})\}^{-1}\ \mathrm{tr}_{i-1}\ \underset{\sim}{M} + \mathrm{tr}_i\ \underset{\sim}{M}$$

$$+ (1-\underset{\sim}{u}'\underset{\sim}{u})^{-1} \sum_{j=0}^{i-1} (-1)^j\ \underset{\sim}{u}'(\underset{\sim}{M}^j+\underset{\sim}{M}^{j+1})\ \underset{\sim}{u}(\mathrm{tr}_{i-1-j}\underset{\sim}{M}) \quad \text{for} \quad i < p$$

$$= \ell_{11}\{(1-\ell_{11})(1-\underset{\sim}{u}'\underset{\sim}{u})\}^{-1}\ |\underset{\sim}{M}| \qquad \text{for} \quad i = p \ .$$

Notice that the distributions of ℓ_{11}, $\underset{\sim}{u}$ and $\underset{\sim}{L}_{22}$ are available in [21], [22] except that the non-centrality parameter, which is involved in the density of ℓ_{11} above, will be denoted here by λ in place of $2\lambda^2$ given there.

3. Moments of $\mathrm{tr}_i \underset{\sim}{U}$

Let $\underset{\sim}{U}_o$ be a $\underset{\sim}{U}$ matrix when $\lambda = 0$, let $\ell_{11,0}$ be the top left corner element of $\underset{\sim}{L}_o$, ($\underset{\sim}{L}$ matrix where $\lambda = 0$) and let

$$(3.1) \qquad y_1 = E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}[E\{\ell_{11}/(1-\ell_{11})\} - E\{\ell_{11,0}/(1-\ell_{11,0})\}] = \lambda/(a-1),$$

$$(3.2) \qquad y_2 = E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-2}[E\{\ell_{11}/(1-\ell_{11})\}^2 - E\{\ell_{11,0}/(1-\ell_{11,0})\}^2]$$

$$= \{2(f_2+2)\lambda+\lambda^2\}/\{(a-1)(a-3)\} \ ,$$

$$(3.3) \qquad y_3 = E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-3}[E\{\ell_{11}/(1-\ell_{11})\}^3 - E\{\ell_{11,0}/(1-\ell_{11,0})\}^3]$$

$$= \{3(f_2+2)(f_2+4)\lambda + 3(f_2+4)\lambda^2+ \lambda^3\}/\{(a-1)(a-3)(a-5)\} \ ,$$

and

$$(3.4) \qquad y_4 = E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-4}[E\{\ell_{11}/(1-\ell_{11})\}^4 - E\{\ell_{11,0}/(1-\ell_{11,0})\}^4]$$

$$= \{4(f_2+2)(f_2+4)(f_2+6)\lambda + 6(f_2+4)(f_2+6)\lambda^2 + 4(f_2+6)\lambda^3 + \lambda^4\}$$

$$/ \{(a-1)(a-3)(a-5)(a-7)\}$$

where $a = f_1 - p$.

Now let

$$\beta_i = tr_{i-1}\underset{\sim}{M} \quad \text{and} \quad \alpha_i = tr_i\underset{\sim}{M} + \sum_{j=0}^{i-1} (-1)^j(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}\underset{\sim}{u}'(\underset{\sim}{M}^j+\underset{\sim}{M}^{j+1})\underset{\sim}{u}$$

$$\cdot\, tr_{i-1-j}\underset{\sim}{M} \quad .$$

Then

$$(3.5) \qquad E[tr_i\underset{\sim}{U}] = E[tr_i\underset{\sim}{U}_o] + y_1 E\, \beta_i \quad ,$$

$$(3.6) \qquad E[tr_i\underset{\sim}{U}]^2 = E[tr_i\underset{\sim}{U}_o]^2 + y_2 E\, \beta_i^2 + 2y_1 E\, \beta_i\, \alpha_i \quad ,$$

$$(3.7) \qquad E[tr_i\underset{\sim}{U}]^3 = E[tr_i\underset{\sim}{U}_o]^3 + y_3 E\, \beta_i^3 + 3y_2 E\, \beta_i^2\, \alpha_i + 3y_1 E\, \beta_i\, \alpha_i^2 \quad ,$$

and

$$(3.8) \qquad E[tr_i\underset{\sim}{U}]^4 = E[tr_i\underset{\sim}{U}_o]^4 + y_4 E\, \beta_i^4 + 4y_3 E\, \beta_i^3\, \alpha_i$$

$$+ 6y_2 E\, \beta_i^2\, \alpha_i^2 + 4y_1 E\, \beta_i\, \alpha_i^3 \quad .$$

In order to evaluate the right sides of (3.5) - (3.8), it appears that general results are obtainable in terms of functions of esf's of latent roots of $\underset{\sim}{M}$. Hence we suggest the following general form for $E\,\beta_i\,\alpha_i$

$$(3.9) \qquad E\,\beta_i\,\alpha_i = \frac{1}{a-1}\,E[\mathrm{tr}_{i-1}\underset{\sim}{M}\{(p-1)\mathrm{tr}_{i-1}\underset{\sim}{M} + (a+i-1)\mathrm{tr}_i\underset{\sim}{M}\}] \quad .$$

The above result as well as others in this section and the next have been suggested by computing special cases for $i = 1, 2, 3, 4$ and further checking the result for $i = 5$.

Similarly

$$(3.10) \qquad E\,\beta_i^2\,\alpha_i = \frac{1}{a-1}\,E[(\mathrm{tr}_{i-1}\underset{\sim}{M})^2\{(p-i)\mathrm{tr}_{i-1}\underset{\sim}{M} + (a+i-1)\mathrm{tr}_i\underset{\sim}{M}\}] \quad ,$$

and

$$(3.11) \qquad E\,\beta_i\,\alpha_i^2 = \frac{1}{(a-1)(a-3)}\,E[\mathrm{tr}_{i-1}\underset{\sim}{M}\{(p-i)(p-i+2)(\mathrm{tr}_{i-1}\underset{\sim}{M})^2$$

$$+ 2[(a+i-3)(p-i)+2]\ \mathrm{tr}_{i-1}\underset{\sim}{M}\ \mathrm{tr}_i\underset{\sim}{M} + (a+i-3)(a+i-1)$$

$$\cdot\,(\mathrm{tr}_i\underset{\sim}{M})^2 + \sum_{k=0}^{i-1}\ \sum_{j=2i-2-k}^{2i-k} a_{kj}\,\mathrm{tr}_k\underset{\sim}{M}\ \mathrm{tr}_j\underset{\sim}{M}\}] \quad ,$$

where

$$a_{kj} = \begin{cases} 0 & \text{if } j\text{-}k \leq 1 \\ -2(j\text{-}k) & \text{if } j\text{-}k > 1 \text{ and } j\text{-}k \text{ is even} \\ 4(j\text{-}k) & \text{if } j\text{-}k > 1 \text{ and } j\text{-}k \text{ is odd} \end{cases} .$$

Now noting that $E[tr_i \underset{\sim}{U}_o]$, $E[tr_i \underset{\sim}{U}_o]^2$ and $E[tr_i U_o]^3$ are available in Pillai [27], [28], using (3.9) - (3.11) in (3.5) - (3.7) and the fact that $E\,\beta_i^j = E(tr_{i-1}\underset{\sim}{M})^j$, we can obtain the first three moments of $U_i^{(p)} = tr_i\{(\underset{\sim}{I}-\underset{\sim}{L})^{-1}-\underset{\sim}{I}\}$ (which are suggested based on computations for i = 1, 2, 3, 4, 5). Expected values of functions of $tr_i\underset{\sim}{M}$ can be obtained in individual cases by use of zonal polynomials [13] or by Pillai's lemma on multiplication of a basic Vandermonde determinant by monomials of esf's [27].

4. Moments of $tr_i\underset{\sim}{L}$

Khatri and Pillai have shown [23] that

$$(4.1) \qquad E[tr_i\underset{\sim}{L}] = E[tr_i\underset{\sim}{L}_o] + x_1 E\,\beta_{1(i)} \quad ,$$

$$(4.2) \qquad E[tr_i\underset{\sim}{L}]^2 = E[tr_i\underset{\sim}{L}_o]^2 - x_2 E\,\beta^2_{1(i)} + 2x_1 E\,\alpha_{1(i)}\,\beta_{1(i)} \quad ,$$

$$(4.3) \qquad E[tr_i\underset{\sim}{L}]^3 = E[tr_i\underset{\sim}{L}_o]^3 + x_3\beta^3_{1(i)} - 3x_2 E\,\beta_{1(i)}\,\alpha_{1(i)} + 3x_1 E\,\beta_{1(i)}\,\alpha^2_{1(i)} \quad ,$$

and

$$(4.4) \qquad E[\mathrm{tr}_i \underset{\sim}{L}]^4 = E[\mathrm{tr}_i \underset{\sim}{L}_o]^4 - x_4\beta_{1(i)}^4 + 4x_3\beta_{1(i)}\alpha_{1(i)} - 6x_2\beta_{1(i)}^2\alpha_{1(i)}^2 + 4x_1\beta_{1(i)}\alpha_{1(i)}^3 \quad ,$$

where x_1, x_2, x_3, x_4, $\alpha_{1(i)}$, $\beta_{1(i)}$ and $\underset{\sim}{L}_o$ are defined in [23] and are functions similar to y_i's, α_i's and β_i's in the preceding section.

Using the results of Section 2 of [22] and further computing as in the previous section we get

$$(4.5) \qquad E[\beta_{1(i)}] = \frac{1}{f_1} E[(a+i)\ \mathrm{tr}_{i-1}\underset{\sim}{L}_{22} + i\ \mathrm{tr}_i\underset{\sim}{L}_{22}] \quad ,$$

where $a = f_1 - p$ and $\mathrm{tr}_o\underset{\sim}{L}_{22} = 1$.

Similarly

$$(4.6) \qquad E[\alpha_{1(i)}\beta_{1(i)}] = \frac{1}{f_1} E[(a+2i)\mathrm{tr}_{i-1}\underset{\sim}{L}_{22}\mathrm{tr}_i\underset{\sim}{L}_{22} + (a+i)(\mathrm{tr}_{i-1}\underset{\sim}{L}_{22})^2 + i(\mathrm{tr}_i\underset{\sim}{L}_{22})^2]$$

and

$$(4.7) \qquad E[\beta_{1(i)}^2] = \frac{1}{f_1(f_1+2)} E[(a+i)(a+i+2)(\mathrm{tr}_{i-1}\underset{\sim}{L}_{22})^2 + [2i(a+i+1) + 2(i-2)]\mathrm{tr}_{i-1}\underset{\sim}{L}_{22}\mathrm{tr}_i\underset{\sim}{L}_{22} + i(i\pm2)(\mathrm{tr}_i\underset{\sim}{L}_{22})^2 - \sum_{k=0}^{i-1} \sum_{j=2i-2-k}^{2i-k} a_{kj}\mathrm{tr}_k\underset{\sim}{L}_{22}\mathrm{tr}_j\underset{\sim}{L}_{22}] \quad ,$$

where

$$(4.4) \qquad E[\mathrm{tr}_i \underset{\sim}{L}]^4 = E[\mathrm{tr}_i \underset{\sim}{L}_o]^4 - x_4\beta^4_{1(i)} + 4x_3\beta_{1(i)}\alpha_{1(i)} - 6x_2\beta^2_{1(i)}\alpha^2_{1(i)} + 4x_1\beta_{1(i)}\alpha^3_{1(i)} \quad ,$$

where x_1, x_2, x_3, x_4, $\alpha_{1(i)}$, $\beta_{1(i)}$ and $\underset{\sim}{L}_o$ are defined in [23] and are functions similar to y_i's, α_i's and β_i's in the preceding section.

Using the results of Section 2 of [22] and further computing as in the previous section we get

$$(4.5) \qquad E[\beta_{1(i)}] = \frac{1}{f_1} E[(a+i)\ \mathrm{tr}_{i-1}\underset{\sim}{L}_{22} + i\ \mathrm{tr}_i\underset{\sim}{L}_{22}] \quad ,$$

where $\qquad a = f_1 - p \quad$ and $\quad \mathrm{tr}_o\underset{\sim}{L}_{22} = 1$.

Similarly

$$(4.6) \qquad E[\alpha_{1(i)}\beta_{1(i)}] = \frac{1}{f_1} E[(a+2i)\mathrm{tr}_{i-1}\underset{\sim}{L}_{22}\mathrm{tr}_i\underset{\sim}{L}_{22} + (a+i)(\mathrm{tr}_{i-1}\underset{\sim}{L}_{22})^2 + i(\mathrm{tr}_i\underset{\sim}{L}_{22})^2]$$

and

$$(4.7) \qquad E[\beta^2_{1(i)}] = \frac{1}{f_1(f_1+2)} E[(a+i)(a+i+2)(\mathrm{tr}_{i-1}\underset{\sim}{L}_{22})^2 + [2i(a+i+1) + 2(i-2)]\mathrm{tr}_{i-1}\underset{\sim}{L}_{22}\mathrm{tr}_i\underset{\sim}{L}_{22} + i(i\pm2)(\mathrm{tr}_i\underset{\sim}{L}_{22})^2 - \sum_{k=0}^{i-1}\ \sum_{j=2i-2-k}^{2i-k} a_{kj}\mathrm{tr}_k\underset{\sim}{L}_{22}\mathrm{tr}_j\underset{\sim}{L}_{22}] \quad ,$$

where

$$a_{kj} = \begin{cases} 0 & \text{if } j-k \leq 1 \\ 2(j-k) & \text{if } j-k > 1 \text{ and even} \\ 4(j-k) & \text{if } j-k > 1 \text{ and odd} \end{cases}$$

Now noting that $E[\mathrm{tr}_i \underset{\sim}{L}_o]$ and $E[\mathrm{tr}_i \underset{\sim}{L}_o]^2$ are available in Pillai [27], [28] and using the above results, we can obtain the first two moments of $V_i^{(p)} = \mathrm{tr}_i \underset{\sim}{L}$ ((4.5) - (4.7) being suggested based on computations for $i = 1, 2, 3, 4$ and 5). Further expected values of functions of $\mathrm{tr}_i \underset{\sim}{L}_{22}$ can be obtained by methods suggested at the end of the preceding section.

5. Moments of the Second esf of a Matrix

Let $\underset{\sim}{X}$: $p \times f$ be a matrix variate $(p \leq f)$ whose columns are independently normally distributed with $E(\underset{\sim}{X}) = \underset{\sim}{M}$ and covariance matrix $\underset{\sim}{\Sigma}$. Let $w_1, \ldots, w_p$ be the characteristic roots of $|\underset{\sim}{X}\,\underset{\sim}{X}' - w\,\underset{\sim}{\Sigma}| = 0$, then the distribution of $\underset{\sim}{W} = \mathrm{diag}\ (w_i)$ is given by James [12], [13]

$$(5.1) \qquad e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}} K(p,f)\ {}_0F_1(\tfrac{1}{2}f;\tfrac{1}{4}\underset{\sim}{\Omega},\underset{\sim}{W})\ e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{W}} |\underset{\sim}{W}|^{\frac{1}{2}(f-p-1)} \prod_{i>j} (w_i - w_j)$$

$$0 < w_1 \leq \cdots \leq w_p < \infty$$

where

$$(5.2) \qquad K(p,f) = \Pi^{\frac{1}{2}p^2} / \{2^{\frac{1}{2}pf}\ \Gamma_p(\tfrac{1}{2}f)\ \Gamma_p(\tfrac{1}{2}p)\} \quad ,$$

$\underset{\sim}{\Omega} = \mathrm{diag}\ (\omega_i)$ where ω_i, $i = 1, \ldots, p$ are the characteristic roots of $|\underset{\sim}{M}\,\underset{\sim}{M}' - \omega\,\underset{\sim}{\Sigma}| = 0$, ${}_0F_1$ is the hypergeometric function of matrix argument and $\Gamma_p(\cdot)$ is the multivariate gamma function defined in [13]. Now define $W_2^{(p)}$ as the second esf in $\frac{1}{2}w_1, \ldots, \frac{1}{2}w_p$. Then from Gupta [11] we have

(5.3) $$E[W_2^{(p)}]^3 = \frac{7}{64} L_{(3^2)} + \frac{1}{12} L_{(321)} + \frac{57}{320} L_{(2^2 1^2)} + \frac{3}{40} L_{(31^3)} + \frac{1}{8} L_{(2^3)} + \frac{9}{40} L_{(21^4)} + \frac{27}{64} L_{(1^6)}$$

where L_κ represents $L_\kappa^\gamma(-\frac{1}{2}\underset{\sim}{\Omega})$, which is the generalized Laguerre polynomial of the form [6], [13]

$$L_\kappa^\gamma(-\tfrac{1}{2}\underset{\sim}{\Omega}) = (\gamma + \tfrac{1}{2}(p+1))_\kappa C_\kappa(\underset{\sim}{I}) \sum_{n=0}^{k} \sum_{\nu} (-1)^n \left[\frac{a_{\kappa,\nu}}{(\gamma+\frac{1}{2}(p+1))_\nu}\right] \frac{C_\nu(-\frac{1}{2}\underset{\sim}{\Omega})}{C_\nu(\underset{\sim}{I})} \quad .$$

Pillai and Gupta [32] have evaluated the first two moments of $W_2^{(p)}$ using $a_{\kappa,\nu}$ coefficients for $k = 2, 4$ available in [6].

Here we evaluate the third moment in (5.3) using the table of $a_{\kappa,\eta}$ coefficients presented in the next section.

(5.4) $$E[W_2^{(p)}]^3 = \mu_3'^{(0)}\{W_2^{(p)}\} + \sum_{i=1}^{4} \sum_{\substack{j=1 \\ i \neq j}}^{3} \sum_{\substack{k=1 \\ k \neq \ell}}^{3} \sum_{\ell=0}^{2} a_{ijk\ell}\, b_i^k\, b_j^\ell$$

where

$$
\begin{aligned}
&a_{1210} = 12\mu_3'^{(0)}\{W_2^{(p)}\}/c_0 ,\ a_{1220}=c_{-1}[c_{-2}\{c_1(342c_4+70c_0)+c_{-3}(3d_{41}+175c_4)\} \\
&+4d_3]/13440, \quad a_{2110} = [c_{-2}\{4c_{-3}(175c_4d_{21}+27c_{-4}d_{52}-3c_{-1}d_{41} \\
&+627c_1c_2)+8c_1(35c_0d_{21}+9c_4\{13c_2-19c_1\})\}+d_3\{7c_2-16c_{-1}\}]/40320, \\
&a_{2120} = [c_{-2}\{4d_{40}+290c_{-3}-504c_1\}+7c_3\{7c_4-8c_1\}]/840, \quad a_{3110} = \\
&[c_{-3}\{5c_2(912c_1+245c_4)+135c_{-4}(39c_2+20c_{-5})-c_{-2}d_9+1120c_{-1}c_2\} \\
&+c_1\{6c_4(150c_2-666c_{-2}-49c_3)+14(16c_{-1}d_{32}+25c_0d_{22})\}]/16800, \\
&a_{1230}=c_{-1}[c_{-2}d_{13}+c_1c_3]/120, \quad a_{2130}= 1, \quad a_{4110}= [10c_{-2}\{2d_{40} \\
&+397c_{-3}\}+c_1\{1120d_{32}-400d_{05}+23712c_{-3}+7182c_2\}+35c_{-4}\{7c_3-184c_{-3} \\
&+54c_2\}+243c_{-4}\{66c_2+25c_{-5}+56c_{-3}\}/12600, \quad a_{2311}= 18, \quad a_{1212} = \\
&(3/2)c_1+6, \quad a_{1221}= a_{1230}/(6c_{-1}), \quad a_{1211}= [c_{-2}\{c_{-3}(d_9-2520c_{-1}) \\
&+6c_1(666c_4-756c_{-1}+175c_0)\}+42c_1c_3\{7c_4-12c_1\}]/25200, \quad a_{1311}= \\
&[c_{-3}\{805c_4+1701c_{-4}+2964c_1-2980c_{-2}\}+10c_1\{5d_{05}-378c_{-2}-14c_3\} \\
&-16c_{-2}d_{40}]/2520,
\end{aligned}
$$

and all other $a_{ijk\ell} = 0$, $c_\alpha = (f+\alpha)(p+\alpha)$, $\mu_3'^{(0)}\{W_2^{(p)}\}$ is the third moment in the central case [5] with $2m = f-p-1$ and

$$
\begin{aligned}
&d_3=7c_1c_3c_4, \quad d_{52}=19c_2-7c_{-5}, \quad d_{21}=c_2-c_{-1}, \quad d_{40}=35c_0+99c_4, \quad d_{32}=c_3-9c_{-2} \\
&d_9=6840c_1+1995c_4+2835c_{-4}, \quad d_{22}=c_2-3c_{-2}, \quad d_{05}=7c_0+18c_4, \quad d_{41}=152c_1+63c_4 .
\end{aligned}
$$

6. Results for $a_{\kappa,\tau}$

The $a_{\kappa,\tau}$'s are constants [6] satisfying the equality

$$C_\kappa(\underset{\sim}{A} + \underset{\sim}{I}) \,/\, C_\kappa(\underset{\sim}{I}) = \sum_{t=0}^{k} \sum_{\tau} a_{\kappa,\tau} C_\tau(\underset{\sim}{A})/C_\tau(\underset{\sim}{I}) \tag{6.1}$$

where τ is a partition of t. The following are suggested based on the available results. For $\kappa = k\text{-}j,1^j$

$$a_{\kappa,\tau} = \begin{cases} j(2k-(j+1))/(2k-(j+2)) & \text{if } \tau = k\text{-}j,1^{j-1} \\ (2k-j)(k-(j+1))/(2k-(j+2)) & \text{if } \tau = k\text{-}j\text{-}1,1^j \end{cases} . \tag{6.2}$$

Also for $\kappa = k\text{-}j,j$, $k \geq 2j$

$$a_{\kappa,\tau} = \begin{cases} j(2k-(4j-2))/(2k-(4j-1)) & \text{if } \tau = k\text{-}j,j\text{-}1 \\ (k-2j)(2k-(2j-1))/(2k-(4j-1)) & \text{if } \tau = k\text{-}j\text{-}1,j \end{cases} . \tag{6.3}$$

For $\kappa = k$, $\tau = k\text{-}j$

$$a_{\kappa,\tau} = k! \,/\, (j!(k-j)!) \; . \tag{6.4}$$

As previously stated the $a_{\kappa,\tau}$ for $k = 1, 2, 3, 4$ are available in [6] and for $k = 5, 6, 7, 8$ now follow.

Table 1. $a_{\kappa,\tau}$ Coefficients for $k = 5$

$\kappa \backslash \tau$	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5
5	1	5	10		10			5					1						
41	1	5	7	3	$\frac{23}{5}$	$\frac{27}{5}$		$\frac{8}{7}$	$\frac{27}{7}$					1					
32	1	5	$\frac{16}{3}$	$\frac{14}{3}$	$\frac{8}{5}$	$\frac{42}{5}$			$\frac{8}{3}$	$\frac{7}{3}$					1				
31^2	1	5	$\frac{13}{3}$	$\frac{17}{3}$	$\frac{7}{5}$	$\frac{33}{5}$	2		$\frac{7}{3}$		$\frac{8}{3}$					1			
2^21	1	5	$\frac{10}{3}$	$\frac{20}{3}$		$\frac{15}{2}$	$\frac{5}{2}$			$\frac{5}{3}$	$\frac{10}{3}$						1		
21^3	1	5	2	8		$\frac{9}{2}$	$\frac{11}{2}$				$\frac{18}{5}$	$\frac{7}{5}$						1	
1^5	1	5		10			10					5							1

Table 2. $a_{\kappa,\tau}$ Coefficients for $k = 6$

κ \ τ	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6
6	1	6	15		20			15					6							1										
51	1	6	$\frac{34}{3}$	$\frac{11}{3}$	$\frac{56}{5}$	$\frac{44}{5}$		$\frac{39}{7}$	$\frac{66}{7}$				$\frac{10}{9}$	$\frac{44}{9}$							1									
42	1	6	9	6	$\frac{28}{5}$	$\frac{72}{5}$		$\frac{48}{5}$	$\frac{66}{7}$	$\frac{147}{35}$				$\frac{12}{5}$	$\frac{18}{5}$							1								
41^2	1	6	8	7	$\frac{26}{5}$	$\frac{123}{10}$	$\frac{5}{2}$	$\frac{9}{7}$	$\frac{61}{7}$		5			$\frac{9}{4}$		$\frac{15}{4}$							1							
3^2	1	6	8	7	$\frac{16}{5}$	$\frac{84}{5}$			8	7					6									1						
321	1	6	$\frac{19}{3}$	$\frac{26}{3}$	$\frac{28}{15}$	$\frac{74}{5}$	$\frac{10}{3}$		$\frac{14}{3}$	$\frac{11}{3}$	$\frac{20}{3}$				$\frac{14}{9}$	$\frac{20}{9}$	$\frac{20}{9}$								1					
31^3	1	6	5	10	$\frac{8}{5}$	$\frac{57}{5}$	7		4		$\frac{46}{5}$	$\frac{9}{5}$				$\frac{24}{7}$		$\frac{18}{7}$								1				
2^3	1	6	5	10		15	5			5	10						6										1			
2^21^2	1	6	4	11		12	8			$\frac{5}{2}$	$\frac{52}{5}$	$\frac{21}{10}$					3	3										1		
21^4	1	6	$\frac{7}{3}$	$\frac{38}{3}$		7	13				$\frac{42}{5}$	$\frac{33}{5}$						$\frac{14}{3}$	$\frac{4}{3}$										1	
1^6	1	6		15			20					15							6											1

Table 3. $a_{\kappa,\tau}$ Coefficients* for k = 7

κ \ τ	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6
7	1	7	21		35			35					21							7										
61	1	7	$\frac{50}{3}$	$\frac{13}{3}$	22	13		$\frac{115}{7}$	$\frac{130}{7}$				$\frac{59}{9}$	$\frac{130}{9}$						$\frac{12}{11}$	$\frac{65}{11}$									
52	1	7	$\frac{41}{3}$	$\frac{22}{3}$	13	22		$\frac{32}{5}$	22	$\frac{33}{5}$			$\frac{80}{63}$	$\frac{506}{45}$	$\frac{297}{35}$						$\frac{16}{7}$	$\frac{33}{7}$								
51^2	1	7	$\frac{38}{3}$	$\frac{25}{3}$	$\frac{62}{5}$	$\frac{98}{5}$	3	$\frac{43}{7}$	$\frac{146}{7}$		8		$\frac{11}{9}$	$\frac{97}{9}$		9					$\frac{11}{5}$		$\frac{24}{5}$							
43	1	7	12	9	8	27		$\frac{64}{35}$	$\frac{144}{7}$	$\frac{63}{5}$				$\frac{24}{5}$	$\frac{81}{5}$							4		3						
421	1	7	$\frac{31}{3}$	$\frac{32}{3}$	$\frac{19}{3}$	$\frac{49}{2}$	$\frac{25}{6}$	$\frac{54}{35}$	$\frac{1006}{63}$	$\frac{287}{45}$	$\frac{100}{9}$			$\frac{81}{20}$	$\frac{244}{45}$	$\frac{275}{36}$	$\frac{35}{9}$					$\frac{3}{2}$	2		$\frac{7}{2}$					
41^3	1	7	9	12	$\frac{29}{5}$	$\frac{207}{10}$	$\frac{17}{2}$	$\frac{10}{7}$	$\frac{102}{7}$		$\frac{84}{5}$	$\frac{11}{5}$		$\frac{15}{4}$		$\frac{351}{28}$		$\frac{33}{7}$					$\frac{10}{3}$			$\frac{11}{3}$				
3^21	1	7	$\frac{28}{3}$	$\frac{35}{3}$	$\frac{56}{15}$	$\frac{133}{5}$	$\frac{14}{3}$		$\frac{112}{9}$	$\frac{91}{9}$	$\frac{112}{9}$				$\frac{77}{9}$	$\frac{56}{9}$	$\frac{56}{9}$							$\frac{7}{5}$	$\frac{28}{5}$					
32^2	1	7	$\frac{25}{3}$	$\frac{38}{3}$	$\frac{7}{3}$	26	$\frac{20}{3}$		$\frac{70}{9}$	$\frac{85}{9}$	$\frac{160}{9}$				$\frac{35}{9}$	$\frac{50}{9}$	$\frac{104}{9}$								5		2			
321^2	1	7	$\frac{22}{3}$	$\frac{41}{3}$	$\frac{32}{15}$	$\frac{227}{10}$	$\frac{61}{6}$		$\frac{64}{9}$	$\frac{95}{18}$	$\frac{896}{45}$	$\frac{27}{10}$			$\frac{20}{9}$	$\frac{416}{63}$	$\frac{115}{18}$	$\frac{81}{14}$							$\frac{20}{7}$	2		$\frac{15}{7}$		
31^4	1	7	$\frac{17}{3}$	$\frac{46}{3}$	$\frac{9}{5}$	$\frac{86}{5}$	16		6		$\frac{104}{5}$	$\frac{41}{5}$				$\frac{54}{7}$		$\frac{244}{21}$	$\frac{5}{3}$							$\frac{9}{2}$			$\frac{5}{2}$	
2^31	1	7	6	15		$\frac{45}{2}$	$\frac{25}{2}$			$\frac{15}{2}$	24	$\frac{7}{2}$					$\frac{27}{2}$	$\frac{15}{2}$									2	5		
2^21^3	1	7	$\frac{14}{3}$	$\frac{49}{3}$		$\frac{35}{2}$	$\frac{35}{2}$			$\frac{7}{2}$	$\frac{112}{5}$	$\frac{91}{10}$					$\frac{63}{10}$	$\frac{77}{6}$	$\frac{28}{15}$									$\frac{21}{5}$	$\frac{14}{5}$	
21^5	1	7	$\frac{8}{3}$	$\frac{55}{3}$		10	25				16	19						$\frac{40}{3}$	$\frac{23}{3}$										$\frac{40}{7}$	$\frac{9}{7}$
1^7	1	7		21			35					35							21											7

* $a_{\kappa,\tau} = 1$ when $\tau = \kappa$

Table 4. $a_{\kappa,\tau}$ Coefficients* for k=8

$\kappa \backslash \tau$	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5
8	1	8	28		56			70					56						
71	1	8	23	5	38	18		$\frac{265}{7}$	$\frac{225}{7}$				$\frac{68}{3}$	$\frac{100}{3}$					
62	1	8	$\frac{58}{3}$	$\frac{26}{3}$	$\frac{124}{5}$	$\frac{156}{5}$		$\frac{643}{35}$	$\frac{884}{21}$	$\frac{143}{15}$			$\frac{460}{63}$	$\frac{1456}{45}$	$\frac{572}{35}$				
61^2	1	8	$\frac{55}{3}$	$\frac{29}{3}$	24	$\frac{57}{2}$	$\frac{7}{2}$	$\frac{125}{7}$	$\frac{850}{21}$		$\frac{35}{3}$		$\frac{64}{9}$	$\frac{565}{18}$		$\frac{35}{2}$			
53	1	8	17	11	$\frac{82}{5}$	$\frac{198}{5}$		$\frac{272}{35}$	$\frac{297}{7}$	$\frac{99}{5}$			$\frac{32}{21}$	$\frac{308}{15}$	$\frac{1188}{35}$				
521	1	8	$\frac{46}{3}$	$\frac{38}{3}$	$\frac{72}{5}$	$\frac{183}{5}$	5	$\frac{247}{35}$	$\frac{766}{21}$	$\frac{49}{5}$	$\frac{50}{3}$		$\frac{88}{63}$	$\frac{1673}{90}$	$\frac{438}{35}$	$\frac{35}{2}$	6		
51^3	1	8	14	14	$\frac{68}{5}$	$\frac{162}{5}$	10	$\frac{47}{7}$	$\frac{240}{7}$		$\frac{132}{5}$	$\frac{13}{5}$	$\frac{4}{3}$	$\frac{53}{3}$		$\frac{207}{7}$		$\frac{52}{7}$	
4^2	1	8	16	12	$\frac{64}{5}$	$\frac{216}{5}$		$\frac{128}{35}$	$\frac{288}{7}$	$\frac{126}{5}$				$\frac{64}{5}$	$\frac{216}{5}$				
431	1	8	$\frac{41}{3}$	$\frac{43}{3}$	$\frac{136}{15}$	$\frac{411}{10}$	$\frac{35}{6}$	$\frac{72}{35}$	$\frac{1948}{63}$	$\frac{791}{45}$	$\frac{175}{9}$			$\frac{36}{5}$	$\frac{1006}{45}$	$\frac{140}{9}$	$\frac{98}{9}$		
42^2	1	8	$\frac{38}{3}$	$\frac{46}{3}$	$\frac{112}{15}$	$\frac{201}{5}$	$\frac{25}{3}$	$\frac{9}{5}$	$\frac{226}{9}$	$\frac{689}{45}$	$\frac{250}{9}$			$\frac{63}{10}$	$\frac{574}{45}$	$\frac{325}{18}$	$\frac{170}{9}$		
421^2	1	8	$\frac{35}{3}$	$\frac{49}{3}$	$\frac{106}{15}$	$\frac{183}{5}$	$\frac{37}{3}$	$\frac{12}{7}$	$\frac{1495}{63}$	$\frac{161}{18}$	$\frac{1454}{45}$	$\frac{33}{10}$		6	$\frac{68}{9}$	$\frac{1394}{63}$	$\frac{98}{9}$	$\frac{66}{7}$	
41^4	1	8	10	18	$\frac{32}{5}$	$\frac{153}{5}$	19	$\frac{11}{7}$	$\frac{150}{7}$		$\frac{186}{5}$	$\frac{49}{5}$		$\frac{11}{2}$		$\frac{387}{14}$		$\frac{146}{7}$	2
3^22	1	8	$\frac{35}{3}$	$\frac{49}{3}$	$\frac{14}{3}$	42	$\frac{28}{3}$		$\frac{175}{9}$	$\frac{175}{9}$	$\frac{280}{9}$				$\frac{140}{9}$	$\frac{140}{9}$	$\frac{224}{9}$		
3^21^2	1	8	$\frac{32}{3}$	$\frac{52}{3}$	$\frac{64}{15}$	$\frac{192}{5}$	$\frac{40}{3}$		$\frac{160}{9}$	$\frac{124}{9}$	$\frac{1568}{45}$	$\frac{18}{5}$			$\frac{104}{9}$	$\frac{1088}{63}$	$\frac{152}{9}$	$\frac{72}{7}$	
32^21	1	8	$\frac{29}{3}$	$\frac{55}{3}$	$\frac{8}{3}$	$\frac{75}{2}$	$\frac{95}{6}$		$\frac{100}{9}$	$\frac{245}{18}$	$\frac{367}{9}$	$\frac{9}{2}$			$\frac{50}{9}$	$\frac{800}{63}$	$\frac{224}{9}$	$\frac{90}{7}$	
321^3	1	8	$\frac{25}{3}$	$\frac{59}{3}$	$\frac{12}{5}$	$\frac{321}{10}$	$\frac{43}{2}$		10	$\frac{43}{6}$	$\frac{623}{15}$	$\frac{113}{10}$			3	$\frac{96}{7}$	13	$\frac{503}{21}$	$\frac{7}{3}$
31^5	1	8	$\frac{19}{3}$	$\frac{65}{3}$	2	24	30		$\frac{25}{3}$		$\frac{116}{3}$	23				$\frac{100}{7}$		$\frac{680}{21}$	$\frac{28}{3}$
2^4	1	8	8	20		36	20			15	48	7					36	20	
2^31^2	1	8	7	21		$\frac{63}{2}$	$\frac{49}{2}$			$\frac{21}{2}$	$\frac{231}{5}$	$\frac{133}{10}$					$\frac{126}{5}$	28	$\frac{14}{5}$
2^21^4	1	8	$\frac{16}{3}$	$\frac{68}{3}$		24	32			$\frac{14}{3}$	$\frac{608}{15}$	$\frac{124}{5}$					$\frac{56}{5}$	$\frac{104}{3}$	$\frac{152}{15}$
21^6	1	8	3	25		$\frac{27}{2}$	$\frac{85}{2}$				27	43						30	26
1^8	1	8		28			56					70							56

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* $a_{\kappa,\tau} = 1$ when $\tau = \kappa$

Table 4. $a_{\kappa,\tau}$ Coefficients* for k=8 (cont'd.)

$\kappa \backslash \tau$	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6	7	61	52	51^2	43	421	41^3	3^21	32^2	321^2	31^4	2^31	2^21^3	21^5	1^7
8	28											8														
71	$\frac{83}{11}$	$\frac{225}{11}$										$\frac{14}{13}$	$\frac{90}{13}$													
62	$\frac{40}{33}$	$\frac{1014}{77}$	$\frac{286}{21}$										$\frac{20}{9}$	$\frac{52}{9}$												
61^2	$\frac{13}{11}$	$\frac{141}{11}$		14									$\frac{13}{6}$		$\frac{35}{6}$											
53	$\frac{144}{35}$	$\frac{121}{7}$		$\frac{33}{5}$										$\frac{18}{5}$		$\frac{22}{5}$										
521		$\frac{132}{35}$	$\frac{97}{14}$	$\frac{46}{5}$		$\frac{81}{10}$								$\frac{22}{15}$	$\frac{40}{21}$		$\frac{162}{35}$									
51^3		$\frac{18}{5}$		$\frac{236}{15}$		$\frac{26}{3}$									$\frac{36}{11}$			$\frac{52}{11}$								
4^2			16		12											8										
431			$\frac{11}{2}$	$\frac{56}{15}$	$\frac{61}{15}$	$\frac{147}{10}$										$\frac{27}{20}$	$\frac{56}{15}$		$\frac{35}{12}$							
42^2			$\frac{7}{2}$	$\frac{14}{3}$		$\frac{33}{2}$		$\frac{10}{3}$									$\frac{14}{3}$			$\frac{10}{3}$						
421^2			$\frac{25}{12}$	$\frac{52}{9}$		$\frac{39}{4}$	$\frac{121}{18}$		$\frac{11}{3}$								$\frac{25}{9}$	$\frac{9}{5}$			$\frac{154}{45}$					
41^4				$\frac{22}{3}$			$\frac{97}{6}$			$\frac{9}{2}$								$\frac{22}{5}$				$\frac{18}{5}$				
3^22					$\frac{7}{3}$	21		$\frac{14}{3}$											$\frac{10}{3}$	$\frac{14}{3}$						
3^21^2					$\frac{28}{15}$	$\frac{528}{35}$	$\frac{16}{3}$		$\frac{40}{7}$										$\frac{8}{3}$		$\frac{16}{3}$					
32^21						$\frac{75}{7}$	$\frac{25}{6}$	$\frac{23}{6}$	$\frac{65}{7}$											$\frac{40}{21}$	$\frac{25}{6}$		$\frac{27}{14}$			
321^3						$\frac{81}{14}$	$\frac{33}{4}$		$\frac{61}{7}$	$\frac{21}{4}$											$\frac{81}{20}$	$\frac{28}{15}$		$\frac{25}{12}$		
31^5							$\frac{25}{2}$			$\frac{195}{14}$	$\frac{11}{7}$											$\frac{50}{9}$			$\frac{22}{9}$	
2^4								8	20														8			
2^31^2								$\frac{7}{2}$	$\frac{91}{5}$	$\frac{63}{10}$													$\frac{7}{2}$	$\frac{9}{2}$		
2^21^4									$\frac{56}{5}$	$\frac{528}{35}$	$\frac{12}{7}$													$\frac{16}{3}$	$\frac{8}{3}$	
21^6										$\frac{135}{7}$	$\frac{61}{7}$														$\frac{27}{4}$	$\frac{5}{4}$
1^8											28															8

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* $a_{\kappa,\tau} = 1$ when $\tau = \kappa$

7. Further Uses of $a_{\kappa,\tau}$

Pillai [29] has shown that

$$(7.1)\qquad E[e^{t\,\mathrm{tr}\underset{\sim}{L}}] = e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa(\frac{1}{2}\nu)_\eta\; a_{\kappa,\eta} t^{k-n} C_\kappa(\underset{\sim}{I}) C_\eta(\frac{1}{2}\underset{\sim}{\Omega})}{(\frac{1}{2}\nu)_\kappa\; (\frac{1}{2}f_2)_\eta\; k!\; C_\eta(\underset{\sim}{I})}$$

where $\underset{\sim}{\Omega}$, ν, f_1 and f_2 are defined in [29].

From (7.1) we get the moments of $\mathrm{tr}\underset{\sim}{L}$ by differentiation with respect to t and letting $t = 0$. Thus

$$\frac{\partial^r}{\partial t^r} E[e^{t\,\mathrm{tr}\underset{\sim}{L}}] = e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \Big((\tfrac{1}{2}f_2)_\kappa(\tfrac{1}{2}\nu)_\eta\; a_{\kappa,\eta} \quad \cdot$$

$$\frac{(k-n)(k-n-1)\ldots(k-n-r+1)t^{k-n-r} C_\kappa(\underset{\sim}{I}) C_\eta(\frac{1}{2}\underset{\sim}{\Omega})}{(\frac{1}{2}\nu)_\kappa(\frac{1}{2}f_2)_\eta k!\; C_\eta(\underset{\sim}{I})} \Big)$$

and hence

$$(7.2)\qquad E[\mathrm{tr}\underset{\sim}{L}]^r = e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa(\frac{1}{2}\nu)_\eta\; a_{\kappa,\eta}\; r!\; C_\kappa(\underset{\sim}{I})\; C_\eta(\frac{1}{2}\underset{\sim}{\Omega})}{(\frac{1}{2}\nu)_\kappa\; (\frac{1}{2}f_2)_\eta\; k!\; C_\eta(\underset{\sim}{I})}$$

where η is a partition of $n = k-r$.

Pillai [29] also gives in the case of canonical correlation

$$(7.3)\qquad E[e^{t\,\mathrm{tr}\underset{\sim}{R}^2}] = |\underset{\sim}{I}-\underset{\sim}{P}^2|^{\frac{\nu}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa((\frac{1}{2}\nu)_\eta)^2 a_{\kappa,\eta} t^{k-n} C_\kappa(\underset{\sim}{I}) C_\eta(\underset{\sim}{P}^2)}{(\frac{1}{2}\nu)_\kappa(\frac{1}{2}f_2)_\eta\; k!\; C_\eta(\underset{\sim}{I})},$$

from which, as before, we obtain the rth moment,

$$(7.4)\qquad E[\mathrm{tr}\underset{\sim}{R}^2]^r = |\underset{\sim}{I}-\underset{\sim}{P}^2|^{\frac{\nu}{2}} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\eta} \frac{(\frac{1}{2}f_2)_\kappa((\frac{1}{2}\nu)_\eta)^2 a_{\kappa,\eta} r!\; C_\kappa(\underset{\sim}{I})\; C_\eta(\underset{\sim}{P}^2)}{(\frac{1}{2}\nu)_\kappa(\frac{1}{2}f_2)_\eta k!\; C_\eta(\underset{\sim}{I})} ,$$

where η is as above, and $\underset{\sim}{R}^2$ and $\underset{\sim}{P}^2$ are defined in [29].

Further, Khatri [18] has obtained the moment generating function of $V^{(p)}$ associated with the test $\lambda \underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ as

$$(7.5) \quad E[e^{tV^{(p)}}] = |\lambda \underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{n=0}^{k} \sum_{\eta} \frac{(\frac{1}{2}f_1)_\kappa (\frac{1}{2}\nu)_\eta a_{\kappa,\eta} t^{k-n} C_\kappa(\underset{\sim}{I}) C_\eta(\underset{\sim}{I}-(\lambda \underset{\sim}{\Lambda})^{-1})}{(\frac{1}{2}\nu)_\kappa \; k! \; C_\eta(\underset{\sim}{I})} .$$

We get the rth moment of $V^{(p)}$ in this case as

$$(7.6) \quad E[V^{(p)}]^r = |\lambda \underset{\sim}{\Lambda}|^{-\frac{1}{2}f_1} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\eta} \frac{(\frac{1}{2}f_1)_\kappa (\frac{1}{2}\nu)_\eta a_{\kappa,\eta} r! \; C_\kappa(\underset{\sim}{I}) C_\eta(\underset{\sim}{I}-(\lambda \underset{\sim}{\Lambda})^{-1})}{(\frac{1}{2}\nu)_\kappa \; k! \; C_\eta(\underset{\sim}{I})} ,$$

where η is a partition of $n = k-p$ and $\underset{\sim}{\Lambda}$ is defined in [18], [29] .

CHAPTER II

ON THE NON-CENTRAL DISTRIBUTIONS OF WILKS' - Λ FOR TESTS OF THREE HYPOTHESES

1. Introduction and Summary

In multivariate analysis we are interested in testing three hypotheses, namely

1) that of equality of the dispersion matrices of two p-variate normal populations,

2) that of equality of the p-dimensional mean vectors for k p-variate normal populations having a common covariance matrix and

3) that of independence between a p-set and a q-set of variates in a $(p+q)$-variate normal population, with $p \leq q$. We obtain the non-central distribution of Wilks' criterion $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1-c_i)$ in each of the above cases, where the c_i's are functions of the characteristic roots of the appropriate matrices. The density functions for Case 2 have been obtained by Pillai and Al-Ani [30] for $p = 2, 3, 4$ and here we obtain the density functions for all three cases for general p in terms of Meijer's G-function [25] with special cases being explicitly evaluated. In this connection a theorem has been proved using some results on Mellin transforms [7], [8], [9]. Also the cumulative distribution function (or cdf) of $W^{(p)}$ is obtained for $p = 2$ in the above three cases. The densities in all cases may be put in a single general form given by

$$(1.1) \qquad f(W^{(p)}) = \frac{\Gamma_p(\delta)}{\Gamma_p(\frac{1}{2}\gamma)} \, \alpha\{W^{(p)}\}^{\frac{1}{2}(\gamma-p-1)}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\delta)_\kappa \, \beta}{k!} \, C_\kappa(\underset{\sim}{M}) \, G^{p,0}_{p,p}\left(W^{(p)}\Big|^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots,b_p}\right)$$

where

$$a_i = \tfrac{1}{2}(2\delta-\gamma) + k_{p-i+1} + b_i \quad \text{and} \quad b_i = (i-1)/2$$

and

Case 1	Case 2	Case 3
$\gamma = n_2$	t	$n-q$
$\delta = \frac{1}{2}n$	ν	$\frac{1}{2}n$
$\beta = (\frac{1}{2}n_1)_\kappa$	1	$(\frac{1}{2}n)_\kappa$
$\alpha = \lvert\lambda \, \underset{\sim}{\Lambda}\rvert^{-\frac{1}{2}n_1}$	$e^{-\mathrm{tr}\underset{\sim}{\Omega}}$	$\lvert\underset{\sim}{I}_p - \underset{\sim}{P}^2\rvert^{\frac{1}{2}n}$
$\underset{\sim}{M} = \underset{\sim}{I}_p - (\lambda \, \underset{\sim}{\Lambda})^{-1}$	$\underset{\sim}{\Omega}$	$\underset{\sim}{P}^2$

See the following sections for definitions of the parameters as well as the G-function.

2. Preliminary Results

Some results on Mellin transforms [7], [8], [9] and Meijer's G-function [25] useful in proving the theorem below will now be given.

Lemma 1. If s is any complex variate and $f(x)$ is a function of a real variable x, such that

$$(2.1) \qquad F(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

exists, then under certain regularity conditions

$$(2.2) \qquad f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\, ds \; .$$

$F(s)$ is called the Mellin transform of $f(x)$ and $f(x)$ is the inverse Mellin transform of $F(s)$.

Lemma 2. If $f_1(x)$ and $f_2(x)$ are the inverse Mellin transforms of $F_1(s)$ and $F_2(s)$ respectively, then the inverse Mellin transform of $F_1(s)\, F_2(s)$ is

$$(2.3) \qquad (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F_1(s) F_2(s)\, ds = \int_0^\infty f_1(u) f_2(x/u)\, du/u \; .$$

Meijer [25] defined the G-function by

$$(2.4) \qquad G_{p,q}^{m,n}\left(x \middle| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_p \end{matrix}\right) =$$

$$(2\pi i)^{-1} \int_C \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \;\; \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \;\; \prod_{j=n+1}^{p} \Gamma(a_j - s)} \; x^s\, ds \; ,$$

where an empty product is interpreted as unity and C is a curve separating the singularities of $\prod_{j=1}^{m} \Gamma(b_j - s)$ from those of $\prod_{j=1}^{n} \Gamma(1 - a_j + s)$, $q \geq 1$, $0 \leq n \leq p \leq q$, $0 \leq m \leq q$; $x \neq 0$ and $|x| < 1$ if $q = p$;

$x \neq 0$ if $q > p$. It is easily verified that

$$(2.5) \qquad G^{2,0}_{2,2}\left(x\,\Big|\,\begin{matrix}a_1,a_2\\ b_1,b_2\end{matrix}\right) = \frac{x^{b_1}(1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)}$$

$${}_2F_1(a_2-b_2,a_1-b_2;\ a_1+a_2-b_1-b_2;\ 1-x) \quad 0 < x < 1$$

where the generalized hypergeometric function ${}_2F_1$ is given by James [13]. The G-function of (2.4) can be expressed as a finite number of generalized hypergeometric functions as follows [26],

$$G^{m,n}_{p,q}\left(x\,\Big|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = \sum_{h=1}^{m} \frac{\prod\limits_{\substack{j=1\\ j\neq h}}^{m} \Gamma(b_j-b_h) \prod\limits_{j=1}^{n} \Gamma(1+b_h-a_j)}{\prod\limits_{j=m+1}^{q} \Gamma(1+b_h-b_j) \prod\limits_{j=n+1}^{p} \Gamma(a_j-b_h)} \quad x^{b_h}$$

$$\cdot\ {}_pF_{q-1}(1+b_h-a_1,\ldots,1+b_h-a_p;1+b_h-b_1,\ldots*\ldots1+b_h-b_q;(-1)^{p-m-n}x)$$

where the asterisk denotes that the number $1+b_h-b_h$ is omitted in the sequence $1+b_h-b_1,\ldots,1+b_h-b_q$. Although the following theorem gives a more complicated form for expressing the G-function, it is useful in that expression (2.4) of Consul [9] and Lemma 1 of Pillai and Al-Ani [30] are special cases.

Theorem 1. If s is a complex variate, a_i,b_i, $i = 1,2,\ldots,p$ are reals, then for $p \geq 3$

$$(2.6) \qquad G^{p,0}_{p,p}\left(x\middle|\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_p\end{matrix}\right) =$$

$$\frac{x^{b_p}(1-x)^{c-1}}{\Gamma(c_1+c_2+c_3)} \prod_{i=1}^{p-3} \left(\sum_{j_i=0}^{\infty} \frac{(b_{p-i+1}+c_{p-i+1}-b_{p-i})_{j_i}}{(j_i)!} \right)$$

$$\cdot \sum_{j=0}^{\infty} \frac{(c_1)_j(b_2+c_2-b_1)_j}{(c_1+c_2+c_3)_j\, j!} \quad (1-x)^{j+\sum_{i=1}^{p-3} j_i} \prod_{\ell=1}^{p-3} \left[\frac{\Gamma(g_\ell+j_\ell)}{\Gamma(h_\ell)}\right]$$

$$\cdot \; {}_{p-1}F_{p-2}(b_3+c_3-b_2, f_1, f_2, \ldots, f_{p-2};\; g_1, g_2, \ldots, g_{p-2}; 1-x)$$

$$0 < x < 1$$

where for notational convenience $c_i = a_i - b_i$, $c = \sum_{i=1}^{p} c_i$,

$f_\ell = \sum_{i=1}^{\ell+1} c_i + \sum_{i=1}^{\ell-1} j_i + j$, $g_\ell = \sum_{i=1}^{\ell+2} c_i + \sum_{i=1}^{\ell-1} j_i + j$, $h_\ell = \sum_{i=1}^{\ell+3} c_i + \sum_{i=1}^{\ell} j_i + j$ and

$(a)_k = a(a+1)\ldots(a+k-1)$.

Proof. Using mathematical induction starting with $p = 3$, we see making the substitution $(a,b,c,m,n,p) \rightarrow (b_3,b_2,b_1,c_3,c_2,c_1)$ in (2.4) of Consul [8] that

$$(2.7) \qquad G^{3,0}_{3,3}\left(x\middle|\begin{matrix}a_1,a_2,a_3\\ b_1,b_2,b_3\end{matrix}\right) = \frac{x^{b_3}(1-x)^{c_1+c_2+c_3-1}}{\Gamma(c_1+c_2+c_3)} \sum_{j=0}^{\infty} \frac{(c_1)_j(b_2+c_2-b_1)_j}{j!(c_1+c_2+c_3)_j}$$

$$\cdot \; (1-x)^j \; {}_2F_1(b_3+c_3-b_2,\; c_1+c_2+j;\; c_1+c_2+c_3+j;\; 1-x)$$

$$0 < x < 1$$

which is (2.6) with $p = 3$. Now assuming (2.6) is true for $p = n$, we show it holds for $p = n+1$. Applying Lemma 2 with

$$F_1(s) = \frac{\prod_{i=1}^{n} \Gamma(s+b_i)}{\prod_{i=1}^{n} \Gamma(s+a_i)} \quad \text{and} \quad F_2(s) = \frac{\Gamma(s+b_{n+1})}{\Gamma(s+a_{n+1})}$$

we have $f_1(x)$ is (2.6) with $p = n$ and $f_2(x) = \dfrac{x^{b_{n+1}}(1-x)^{c_{n+1}-1}}{\Gamma(c_{n+1})}$

and it follows that

$$(2.8) \qquad G^{n+1,0}_{n+1,n+1}\left(x\middle|{a_1,a_2,\dots,a_{n+1} \atop b_1,b_2,\dots,b_{n+1}}\right) = \frac{x^{b_{n+1}}}{\Gamma(c_1+c_2+c_3)\Gamma(c_{n+1})} \int_x^1 u^{b_n-b_{n+1}-c_{n+1}}$$

$$\cdot\ (1-u)^{\sum_{i=1}^{n} c_i+1} \prod_{i=1}^{n-3} \left(\sum_{j_i=0}^{\infty} \frac{(b_{n-i+1}+c_{n-i+1}-b_{n-i})_{j_i}}{(j_i)!} \right)$$

$$\cdot\ \sum_{j=0}^{\infty} \frac{(c_1)_j(b_2+c_2-b_1)_j}{(c_1+c_2+c_3)_j\ j!} (1-u)^{j+\sum_{i=1}^{n-3} j_i} \prod_{\ell=1}^{p-3} \frac{\Gamma(q_\ell+j_\ell)}{\Gamma(h_\ell)}$$

$$\cdot\ {}_{n-1}F_{n-2}(b_3+c_3-b_2,f_1,f_2,\dots,f_{p-2};\ g_1,g_2,\dots,g_{p-2};\ 1-u)$$

$$(u-x)^{c_{n+1}-1}\,du \ .$$

Expanding $u^{b_n-b_{n+1}-c_{n+1}}$ in powers of $1-u$ when $b_{n+1}+c_{n+1} > b_n$, letting $u = x+(1-x)t$ and integrating with respect to t, the result is the same as (2.6) with $p = n+1$.

It is easily verified that Lemma 1 of Pillai and Al-Ani [30] is a special case of (2.6) with $p = 4$ by making the following substitution

$$(b_1,b_2,b_3,b_4,c_1,c_2,c_3,c_4) \rightarrow (c,b,a,d,p,n,m,\ell) .$$

It should be mentioned that this theorem doesn't apply when $p = 1,2$. This is due to the fact that a simplification in the form of the G-function for $p = 3$ reduces the hypergeometric function involved from ${}_3F_2$ to ${}_2F_1$. A general form for all p can be given as below, but we see it is more cumbersome to use because we have ${}_pF_{p-1}$ rather than ${}_{p-1}F_{p-2}$ as in (2.6)

$$(2.9) \qquad G^{p,0}_{p,p}\left(x\Big|{a_1,a_2,\ldots,a_p \atop b_1,b_2,\ldots,b_p}\right) = \frac{x^{b_1}(1-x)^{c-1}}{\Gamma(c)} \prod_{i=1}^{p-3}\left[\sum_{\ell_{p-i-2}}^{\infty} \frac{(b_i+c_i-b_{i+1})_{\ell_{p-i-2}}}{(\ell_{p-i-2})!}\right]$$

$$\cdot \sum_{r=0}^{\infty} \frac{(b_{p-2}+c_{p-2}-b_{p-1})_r(c_{p-1}+c_p)_r \prod_{i=1}^{p-3} (\sum_{j=i+2}^{p} c_j + \sum_{j=1}^{p-4} \ell_j+r)_{\ell_i}}{r!\ (c)_{\ell+r}}$$

$$\cdot {}_pF_{p-1}(c_p,b_{p-1}+c_{p-1}-b_p,f_1,\ldots,f_{p-2};c_{p-1}+c_p,g_1,\ldots g_{p-2};1-x)$$

$$\cdot (1-x)^{\ell+r} \qquad\qquad 0 < x < 1$$

where

$$\ell = \sum_{i=1}^{p-3} \ell_i,\ f_1 = \sum_{j=p-i}^{p} c_i + \sum_{j=1}^{p+i-6} \ell_j+r,\ g_i = \sum_{j=p-i-1}^{p} c_j + \sum_{j=1}^{p+i-6} \ell_j+r, c = \sum_{i=1}^{p} c_i .$$

It follows that letting $p = 2$ we get (2.5) and $p = 1$ gives

$$G_{1,1}^{1,0}(x|{}^{a_1}_{b_1}) = x^b(1-x)^{c_1-1}/\Gamma(c_1) \quad .$$

3. The Non-Central Distribution of $W^{(p)}$ in Case 1

Let $\underset{\sim}{X}$: $p \times n_1$ and $\underset{\sim}{Y}$: $p \times n_2$, $p \leq n_i$, $i = 1,2$, be independent matrix variates with the columns of $\underset{\sim}{X}$ independently distributed as $N(0,\underset{\sim}{\Sigma_1})$ and those of $\underset{\sim}{Y}$ independently distributed as $N(0,\underset{\sim}{\Sigma_2})$. Hence $\underset{\sim}{S_1} = \underset{\sim}{XX'}$ and $\underset{\sim}{S_2} = \underset{\sim}{YY'}$ are independently distributed as Wishart $(n_i,p,\underset{\sim}{\Sigma_i})$, $i = 1,2$. Let $0 < f_1 < f_2 < \ldots < f_p < \infty$ be the characteristic (ch.) roots of the determinantal equation

$$(3.1) \qquad |\underset{\sim}{S_1} - f\, \underset{\sim}{S_2}| = 0$$

and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p < \infty$ be the ch. roots of

$$(3.2) \qquad |\underset{\sim}{\Sigma_1} - \gamma\, \underset{\sim}{\Sigma_2}| = 0 \quad .$$

For testing the hypothesis H_o: $\lambda\, \underset{\sim}{\Lambda} = \underset{\sim}{I_p}$, $\lambda > 0$ being given, we will use

$$(3.3) \qquad W^{(p)} = \prod_{i=1}^{p} (1-w_i)$$

where

$$\underset{\sim}{\Lambda} = \text{diag}\,(\lambda_1,\lambda_2,\ldots,\lambda_p), \quad w_i = \lambda f_i/(1+\lambda f_i) \qquad i = 1,2,\ldots, p \quad .$$

Khatri [18] has shown that

$$(3.4)\qquad f(w_1,w_2,\ldots,w_p) = C|\underset{\sim}{\lambda\Lambda}|^{-\frac{1}{2}n_1}|\underset{\sim}{W}|^{\frac{1}{2}(n_1-p-1)}|\underset{\sim}{I_p}-\underset{\sim}{W}|^{\frac{1}{2}(n_2-p-1)}\prod_{i<j}(w_i-w_j)$$

$$\cdot\quad {}_1F_0(\tfrac{1}{2}n;\underset{\sim}{I_p}-(\lambda\underset{\sim}{\Lambda})^{-1},\ \underset{\sim}{W})$$

where

$$\underset{\sim}{W} = \text{diag}(w_1,w_2,\ldots,w_p),\ n=n_1+n_2,\ \Gamma_p(t) = \pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma(t-\tfrac{1}{2}j+\tfrac{1}{2})\ ,$$

$$C = \pi^{\frac{1}{2}p^2}\ \Gamma_p(\tfrac{1}{2}n)\ [\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)]^{-1}\ .$$

To find $E[W^{(p)}]^h$ we multiply (3.4) by $|\underset{\sim}{I_p}-\underset{\sim}{W}|^h = [\prod_{i=1}^{p}(1-w_i)]^h$, transform $\underset{\sim}{W}\to\underset{\sim}{H}\,\underset{\sim}{V}\,\underset{\sim}{H}'$, where $\underset{\sim}{H}$ is an orthogonal and $\underset{\sim}{V}$ is a symmetric matrix, integrate out $\underset{\sim}{H}$ and $\underset{\sim}{V}$ using (44) and (22) of Constantine [5] and we find

$$(3.5)\qquad E[W^{(p)}]^h = \frac{\Gamma_p(\frac{1}{2}n)\Gamma(\frac{1}{2}n_2+h)}{\Gamma(\frac{1}{2}n_2)\Gamma_p(\frac{1}{2}n+h)}\ |\underset{\sim}{\lambda\Lambda}|^{-\frac{1}{2}n_1}\ {}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n_1;\tfrac{1}{2}n+h;\underset{\sim}{I_p}-(\lambda\underset{\sim}{\Lambda})^{-1})\ .$$

Using Lemma 1, the density of $f(W^{(p)})$ has the form

$$(3.6)\qquad f(W^{(p)}) = C_p\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa}{k!}\ C_\kappa(\underset{\sim}{I_p}-(\lambda\underset{\sim}{\Lambda})^{-1})\ \{W^{(p)}\}^{\frac{1}{2}(n_2-p-1)}$$

$$\cdot\quad(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\{W^{(p)}\}^{-r}\ \frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)}\ dr$$

where

$$r = \tfrac{1}{2}n_2 + h - \tfrac{1}{2}(p-1), \quad b_i = \tfrac{1}{2}(i-1), \quad a_i = \tfrac{1}{2}n_1 + k_{p-i+1} + b_i,$$

$$C_p = \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}n_2)} |\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}, \quad (a)_\kappa = \prod_{i=1}^{p} \Gamma(a-\tfrac{1}{2}(i-1))_{k_i},$$

$$(a)_k = a(a+1)\ldots(a+k-1) \quad ,$$

$\sum_\kappa$ is the sum over all partitions κ of the integer k where

$$\kappa = (k_1,k_2,\ldots,k_p), \; k_1 \geq k_2 \geq \ldots \geq k_p > 0, \; \sum_{i=1}^{p} k_i = k, \quad \text{and}$$

$C_\kappa(\underset{\sim}{S})$ is a zonal polynomial; see James [13].

Noting that the integral in (3.6) is in the form of Meijer's G-function we can write the density of $W^{(p)}$ as

$$(3.7) \qquad f(W^{(p)}) = C_p\{W^{(p)}\}^{\frac{1}{2}(n_2-p-1)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa}{k!} \cdot$$

$$\cdot \quad C_\kappa(\underset{\sim}{I}_p-(\lambda\underset{\sim}{\Lambda})^{-1}) \; G^{p,0}_{p,p}\left(W^{(p)}\middle|{}^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots,b_p}\right) \quad .$$

Letting $p = 2$ in (3.7) and using (2.5) we obtain

$$(3.8) \qquad f(W^{(2)}) = C_2\{W^{(2)}\}^{\frac{1}{2}(n_2-3)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa}{k!}$$

$$\cdot \quad C_\kappa(\underset{\sim}{I}_2-(\lambda\underset{\sim}{\Lambda})^{-1}) \; \frac{\{1-W^{(2)}\}^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)}$$

$$\cdot \quad {}_2F_1(a_2-b_2,\; a_1-b_2,\; a_1+a_2-b_1-b_2;\; 1-W^{(2)}) \quad .$$

The probability that $W^{(2)} \leq w$ (≤ 1) can be obtained by integrating (3.8) by parts a_1 times when n_1 is even. Using the relation [8]

$$(3.9) \qquad (d^n/dz^n)[z^{c-1}\ {}_2F_1(a,b;c;z)] = (c-n)_n\ z^{c-n-1}\ {}_2F_1(a,b;c-n;z),$$

and recalling that $\kappa = (k_1,k_2)$, we obtain the cdf of $W^{(2)}$ in terms of a_i's and b_i's as

$$(3.10) \qquad \Pr\{W^{(2)} \leq w\} = |\lambda \underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I}_2-(\lambda\underset{\sim}{\Lambda})^{-1})\ w^{\frac{1}{2}(n_2-1)}}{k!}$$

$$\cdot \Big\{\frac{\Gamma_2(\frac{1}{2}n)(\frac{1}{2}n)_\kappa}{\Gamma_2(\frac{1}{2}n_2)\Gamma(a_1+a_2-b_1-b_2)} \sum_{r=0}^{a} \frac{(a_1+a_2-b_1-b_2-r)_r}{\{\frac{1}{2}(n_2-1)\}_{r+1}}$$

$$\cdot\ w^r(1-w)^{a_1+a_2-b_1-b_2-r-1}$$

$$\cdot\ {}_2F_1(a_2-b_2,a_1-b_2,a_1+a_2-b_1-b_2-r;1-w)+I_w(\tfrac{1}{2}n_2,b)\Big\}$$

where $I_w(a,b)$ denotes the incomplete beta function, a_i, b_i are defined in (3.6), $a = a_1-1$ and $b = a_2-b_2$. When n_1 is odd, after integrating (3.8) by parts a_2 times, the cdf of $W^{(2)}$ is (3.10) with $a = a_2-1$ and $b = a_1-b_2$. Letting $p = 3$ in (3.7) we have

$$(3.11) \qquad f(W^{(3)}) = \frac{\Gamma_3(\frac{1}{2}n)}{\Gamma_3(\frac{1}{2}n_2)}\ |\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}\ \{W^{(3)}\}^{\frac{1}{2}(n_2-4)}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n_1)_\kappa}{k!}\ C_\kappa(\underset{\sim}{I}_3-(\lambda\underset{\sim}{\Lambda})^{-1})G^{3,0}_{3,3}\left(W^{(3)}\middle|\begin{matrix}a_1,a_2,a_3\\ b_1,b_2,b_3\end{matrix}\right)$$

where a_i and b_i are defined in (3.6) .

It is clear $G^{3,0}_{3,3}(W^{(3)}|^{a_1,a_2,a_3}_{b_1,b_2,b_3})$ could be written out in terms of the hypergeometric function using Theorem 1, for computation purposes. Also letting $p = 4$ in (3.7) yields

$$(3.12) \qquad f(W^{(4)}) = \frac{\Gamma_4(\frac{1}{2}n)}{\Gamma_4(\frac{1}{2}n_2)} |\lambda \underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} \{W^{(4)}\}^{\frac{1}{2}(n_2-5)}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa (\frac{1}{2}n_1)_\kappa}{k!} C_\kappa (\underset{\sim}{I}_4 - (\lambda \underset{\sim}{\Lambda})^{-1}) G^{4,0}_{4,4}(W^{(4)}|^{a_1,a_2,a_3,a_4}_{b_1,b_2,b_3,b_4})$$

where a_i's and b_i's are defined in (3.6).

4. The Non-Central Distribution of $W^{(p)}$ in Case 2

Let $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1-\ell_i)$ where $\ell_1, \ell_2, \ldots, \ell_p$ are the ch. roots of the determinantal equation

$$(4.1) \qquad |\underset{\sim}{S}_1 - \ell(\underset{\sim}{S}_1 + \underset{\sim}{S}_2)| = 0$$

where $\underset{\sim}{S}_1$ is a $(p \times p)$ matrix distributed as non-central Wishart with s degrees of freedom, $\underset{\sim}{\Omega}$ is a matrix of non-centrality parameters and $\underset{\sim}{S}_2$ has the Wishart distribution with t degrees of freedom, the covariance matrix in each case being $\underset{\sim}{\Sigma}$. Pillai and Al-Ani [30] obtained the density of $W^{(p)}$ for $p = 2,3,4$. Here we obtain the density of $W^{(p)}$ in general in terms of Meijer's G-functions. As in Section 3, applying Lemma 1 to the expression for $E[W^{(p)}]^h$ obtained by Al-Ani [1] and using (2.4) we find

$$(4.2) \qquad f(W^{(p)}) = C_p\{W^{(p)}\}^{\frac{1}{2}(t-p-1)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!}$$

$$\cdot \; G^{p,0}_{p,p}\left(W^{(p)} \middle| \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_p \end{matrix}\right)$$

where

$$\nu = \tfrac{1}{2}(s+t), \; C_p = \frac{\Gamma_p(\nu)}{\Gamma_p(\frac{1}{2}t)} e^{-\mathrm{tr}\underset{\sim}{\Omega}}, \; b_i = \tfrac{1}{2}(i-1), \; a_i = \tfrac{1}{2}s + k_{p-i+1} + b_i \quad .$$

The probability that $W^{(2)} \leq w \; (\leq 1)$ can be obtained by using (2.5) in (4.2), integrating by parts a_1 times when s is even, then using (3.9) we get the cdf of $W^{(2)}$ as

$$(4.3) \qquad \Pr\{W^{(2)} \leq w\} = e^{-\mathrm{tr}\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{\Omega})}{k!} \left\{ \frac{w^{\frac{1}{2}(t-1)} \Gamma_2(\nu) \; (\nu)_\kappa}{\Gamma_2(\frac{1}{2}t)\Gamma(a_1+a_2-b_1-b_2)} \right.$$

$$\cdot \sum_{r=0}^{a} \frac{(a_1+a_2-b_1-b_2-r)_r}{\{\frac{1}{2}(t-1)\}_{r+1}} \; w^r (1-w)^{a_1+a_2-b_1-b_2-r-1}$$

$$\left. \cdot \; {}_2F_1(a_2-b_2, a_1-b_2; \; a_1+a_2-b_1-b_2-r; 1-w^{(2)}) + I_w(\tfrac{1}{2}t, b) \right\}$$

where $a = a_1-1$, $b = a_2-b_2$ and the a_i's and b_i's are defined in (4.2). When s is odd, we integrate (4.2) by parts a_2 times and find the cdf is (4.3) with $a = a_2-1$, $b = a_1-b_2$. The densities of $W^{(3)}$ and $W^{(4)}$ obtained by Pillai and Al-Ani [30] are special cases of (4.2) as can be verified by letting $p = 3,4$ in (4.2), applying Theorem 1 and making the substitution

$$(a_1,a_3,b_1,b_3) \rightarrow (a_3,a_1,b_3,b_1) \quad .$$

5. The Non-Central Distribution of $W^{(p)}$ in Case 3

Let the columns of $\begin{pmatrix}\underset{\sim}{X}_1\\ \underset{\sim}{X}_2\end{pmatrix}$ be independent normal (p+q)-variates ($p \leq q$, $p+q \leq n$, n is the sample size) with zero means and covariance matrix

$$(5.1) \qquad \underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \bar{\underset{\sim}{\Sigma}}{}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} \cdot$$

Let $\underset{\sim}{R}^2 = \mathrm{diag}(r_1^2, r_2^2, \ldots, r_p^2)$ where r_i^2 are the ch. roots of

$$(5.2) \qquad |\underset{\sim}{X}_1\underset{\sim}{X}{}_2' \, (\underset{\sim}{X}_2\underset{\sim}{X}{}_2')^{-1} \, \underset{\sim}{X}_2\underset{\sim}{X}{}_1' - r^2 \, \underset{\sim}{X}_1\underset{\sim}{X}{}_1'| = 0$$

and $\underset{\sim}{P}^2 = \mathrm{diag}\,(\rho_1^2, \rho_2^2, \ldots, \rho_p^2)$ where ρ_i^2 are the ch. roots of

$$(5.3) \qquad |\underset{\sim}{\Sigma}_{12} \, \underset{\sim}{\Sigma}{}_{22}^{-1} \, \bar{\underset{\sim}{\Sigma}}{}'_{12} - \rho^2 \, \underset{\sim}{\Sigma}_{11}| = 0 \, .$$

Constantine [5] obtained the density of $r_1^2, r_2^2, \ldots, r_p^2$ as

$$(5.4) \qquad f(r_1^2,r_2^2,\ldots,r_p^2) = C|\underset{\sim}{I}_p-\underset{\sim}{P}^2|^{\frac{1}{2}n}|\underset{\sim}{R}^2|^{\frac{1}{2}(q-p-1)}|\underset{\sim}{I}_p-\underset{\sim}{R}^2|^{\frac{1}{2}(n-q-p-1)}$$

$$\cdot \prod_{i<j} (r_i^2-r_j^2) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n)_\kappa C_\kappa(\underset{\sim}{R}^2)C_\kappa(\underset{\sim}{P}^2)}{(\frac{1}{2}q)_\kappa \; C_\kappa(\underset{\sim}{I}_p) \; k!}$$

where

$$C = \pi^{\frac{1}{2}p^2} \, \Gamma_p(\tfrac{1}{2}n)[\Gamma_p(\tfrac{1}{2}q)\Gamma_p(\tfrac{1}{2}(n-q))\Gamma_p(\tfrac{1}{2}p)]^{-1} \, ,$$

To find $E[W^{(p)}]^h$, $W^{(p)} = \prod_{i=1}^{p} (1-r_i^2)$, we multiply (5.4) by $|\underset{\sim}{I}_p-\underset{\sim}{R}^2|^h$, proceed as in Section 3 for Case 1 and we find

$$(5.5)\qquad E[W^{(p)}]^h = \frac{\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}(n-q)+h)}{\Gamma_p(\tfrac{1}{2}(n-q))\Gamma_p(\tfrac{1}{2}n+h)} |\underset{\sim}{I}_p-\underset{\sim}{P}^2|^{\frac{1}{2}n} {}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n;\ \tfrac{1}{2}n+h;\ \underset{\sim}{P}^2)\ .$$

Noting that (5.5) can be obtained from (3.5) by substituting

$$(5.6)\qquad (n_2,n_1,(\lambda\underset{\sim}{\Lambda})^{-1}) \rightarrow (n-q,n,\underset{\sim}{I}_p-\underset{\sim}{P}^2)$$

it can be verified that the density of $W^{(p)}$ in this case is

$$(5.7)\qquad f(W^{(p)}) = C_p\{W^{(p)}\}^{\frac{1}{2}(n-q-p-1)} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\frac{1}{2}n)_\kappa(\frac{1}{2}n)_\kappa C_\kappa(\underset{\sim}{P}^2)}{k!}$$

$$\cdot\ G^{p,0}_{p,p}\left(W^{(p)}\middle|\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_p\end{matrix}\right)$$

where

$$C_p = \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}(n-q))}|\underset{\sim}{I}_p-\underset{\sim}{P}^2|^{\frac{1}{2}n},\quad a_i = \tfrac{1}{2}q+k_{p-i+1}+b_i,\ b_i = \tfrac{1}{2}(i-1)\ .$$

The cdf of $W^{(2)}$ is obtained from (3.10) when q is even by substituting as in (5.6) and using the a_i's as just defined. For q odd the cdf of $W^{(2)}$ follows from that of Case 1 for n_1 odd by making the substitution (5.6) and using the a_i's just defined. The densities of $W^{(p)}$ for $p = 2,3,4$ follow from (3.8), (3.11), (3.12) respectively making substitution (5.6).

CHAPTER III

EXACT DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERION FOR TESTING INDEPENDENCE OF SETS OF VARIATES UNDER THE NULL HYPOTHESIS

1. Introduction and Summary

Let the p-component vector $\underset{\sim}{X}$ be distributed according to $N(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$. We partition $\underset{\sim}{X}$ into q subvectors with $p_1,p_2,\dots,p_q$ components respectively, that is

$$(1.1) \qquad \underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}^{(1)} \\ \underset{\sim}{X}^{(2)} \\ \vdots \\ \underset{\sim}{X}^{(q)} \end{pmatrix} .$$

The vector of means $\underset{\sim}{\mu}$ and the positive definite covariance matrix $\underset{\sim}{\Sigma}$ are partitioned similarly

$$(1.2) \qquad \underset{\sim}{\mu} = \begin{pmatrix} \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{\mu}^{(2)} \\ \vdots \\ \underset{\sim}{\mu}^{(q)} \end{pmatrix} \quad \text{and} \quad \underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} & \cdots & \underset{\sim}{\Sigma}_{1q} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} & \cdots & \underset{\sim}{\Sigma}_{2q} \\ & & & \\ \underset{\sim}{\Sigma}_{q1} & \underset{\sim}{\Sigma}_{q2} & \cdots & \underset{\sim}{\Sigma}_{qq} \end{pmatrix} .$$

The hypothesis H_o to be tested is whether the q sets are mutually independent. Thus we test the hypothesis

$$(1.3) \qquad H_o: N(\underset{\sim}{x}|\underset{\sim}{\mu}, \underset{\sim}{\Sigma}) = \prod_{i=1}^{q} N(\underset{\sim}{x}^{(i)}|\underset{\sim}{\mu}^{(i)}, \underset{\sim}{\Sigma}_{ii})$$

or equivalently $H_o: \underset{\sim}{\Sigma}_{ij} = 0, i \neq j$, against $H_1: \underset{\sim}{\Sigma}_{ij} \neq 0, i \neq j$.

If $\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_N$ is a sample of N observations drawn from $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, where $\underset{\sim}{x}_\alpha$, $\underset{\sim}{\mu}$, and $\underset{\sim}{\Sigma}$ are partitioned as above, then the likelihood ratio criterion is a monotonic increasing function of V defined by [37]

$$(1.4) \qquad V = |\underset{\sim}{A}| \;/\; \prod_{i=1}^{q} |\underset{\sim}{A}_{ii}|$$

where

$$(1.5) \qquad \underset{\sim}{A} = \sum_{\alpha=1}^{N} (\underset{\sim}{x}_\alpha - \bar{\underset{\sim}{x}})(\underset{\sim}{x}_\alpha - \bar{\underset{\sim}{x}})'$$

and is partitioned in the same manner as $\underset{\sim}{\Sigma}$. The corresponding matrix $\underset{\sim}{A}_{ii}$ is defined and partitioned similarly.

In 1935 Wilks [37] obtained the distributions of the likelihood ratio criterion V, in the following special cases; for $q = 3$, i) all values of $p_3, p_1 = p_2 = 1$, ii) $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, iii) $p_1 = 1$, $p_2 = 2$, $p_3 = 2,4$ iv) $p_1 = p_2 = 2$, $p_3 = 2,3$, each expression being a finite series involving incomplete beta functions. Wald and Brookner [36] gave a method for obtaining the exact distributions of V, if not more than one p_i was odd, but exact distributions weren't explicitly obtained. Here we give expressions for the distributions of the likelihood ratio criterion for the cases: $q = 3$; all values of p_1 and i) $p_2 = p_3 = 1$, ii) $p_2 = 2$, $p_3 = 1$, iii) $p_2 = p_3 = 2$, iv) $p_2 = 3$, $p_3 = 1$, v) $p_3 = 3$, $p_2 = 2$, vi) $p_2 = 4$, $p_3 = 1$, vii) $p_2 = 4$, $p_3 = 2$, (See Section 3). $q = 4$, all values of p_1 and i) $p_2 = p_3 = p_4 = 2$, ii) $p_2 = p_3 = 2$, $p_4 = 1$, iii) $p_2 = 2$, $p_3 = p_4 = 1$, iv) $p_2 = p_3 = p_4 = 1$, (See Section 4).

Consul [8] has obtained the cdf's in all cases of Section 3, except case vi), expressing his results as infinite series using Mellin's inversion theorem. Using a transformation suggested by Schatzoff [35] and some results of Gupta [11] many of the cdf's given here are in finite series form. Exact lower 1% and 5% points for Case ii), iii) and v) of Section 3, are given in Tables 5 - 10.

2. Some Preliminary Results

The following results, available in Gupta [11] will be noted. Let X_j be a beta random variable with

$$(2.1) \qquad X_j \sim \beta[f_1-j+1)/2,\ f_2/2] = K_j X_j^{(f_1-j-1)/2}(1-X_j)^{(f_2-2)/2} \quad \begin{matrix} 0 < X_j < 1 \\ f_1 \geq j \end{matrix}$$

where

$$(2.2) \qquad K_j = (1/\beta[(f_1-j+1)/2,\ f_2/2]) \quad .$$

When f_2 is even, $(f_2-2)/2$ is an integer and (2.1) can be expanded using the binomial theorem, giving

$$(2.3) \qquad \beta[(f_1-j+1)/2,\ f_2/2] = K_j \sum_{\ell=0}^{b} (-1)^{\ell}\ X_j^{(f_1-j+2\ell-1)/2}$$

where

$$(2.4) \qquad b = (f_2-2)/2 \ .$$

Making the transformation

$$(2.5) \qquad Y_j = -\log X_j, \qquad dY_j = -\ dX_j/X_j$$

where $\log X \equiv \log_e X$,

we find the density of Y_j is given by

$$(2.6) \qquad Y_j \sim K_j \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} e^{-Y_j(f_1-j+2\ell+1)/2} \qquad Y_j > 0 \ .$$

Considering the relationship of Theorems 8.5.1 and 8.5.2 in Chapter 8 of Anderson [3] we see that for

$$(2.7) \qquad Z_j = X_{2j-1} \cdot X_{2j} \qquad \text{where } X_j \text{ has density (2.1)}$$

$$(2.8) \qquad Z_j \sim C_j \cdot Z_j^{(f_1-2j-1)/2} (1-\sqrt{Z_j})^{f_2-1}$$

where

$$(2.9) \qquad C_j = (2\beta[f_1-2j+1,\ f_2])^{-1} \ .$$

If we make the further transformation as in Schatzoff [35]

$$(2.10) \qquad Y_j' = - \log Z_j$$

then expanding, using the binomial theorem, we get the density of Y_j' as

$$(2.11) \qquad Y_j' \sim C_j \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} \exp[- Y_j'(f_1+\ell-2j+1)/2], \; Y_j' > 0 \quad .$$

Now consider the density and distribution of random variables like $V = V_1 + V_2$ where apart from normalizing constants

$$(2.12) \qquad V_1 \sim v_1^{k} e^{av_1}, \qquad v_1 > 0, \; k \text{ a non-negative integer}$$

and

$$(2.13) \qquad V_2 \sim e^{bv_2}, \qquad v_2 > 0 \quad .$$

The density function of V is easily found by forming the convolution integral

$$(2.14) \qquad V_1^k e^{aV_1} * e^{bV_2} = \int_o^v v_1^k e^{av_1} e^{b(v-v_1)} dv_1$$

$$= e^{bv} \int_o^v v_1^k e^{(a-b)v_1} dv_1$$

where the asterisk denotes the convolution operator. We have two cases

i) $a = b$, then (2.14) is

$$(2.15) \qquad e^{bv} \int_o^v v_1^k dv_1 = e^{bv} \frac{v^{k+1}}{k+1}$$

ii) $a \neq b$ is

$$(2.16) \qquad e^{bv} \int_0^v v_1^k e^{(a-b)v_1} dv_1 =$$

$$e^{av} \left[\sum_{r=1}^{k+1} (-1)^{r+1} \frac{k! \, v^{k-r+1}}{(k-r+1)!(a-b)^r} \right] + e^{bv} \left(\frac{-1}{a-b}\right)^{k+1} k! \quad .$$

Now, more explicitly we will denote V by $V_{p_1,p_2,\dots,p_q;N}$ in the case where we have q sets of variates with p_i variates in the ith set and N observations have been taken on $\underset{\sim}{X}$. Also notice that V is unchanged by permutation of the p_i's. Anderson [3] shows that

$$(2.17) \qquad V_{p_1,p_2,\dots,p_q;N} \sim \prod_{i=2}^{q} \left\{ \prod_{j=1}^{p_i} X_{ij} \right\}$$

where X_{ij} are independent and have density

$$(2.18) \qquad X_{ij} \sim \beta[(n-\bar{p}_i-j+1)/2; \ \bar{p}_i/2]$$

where $\bar{p}_i = p_1+p_2+\dots+p_{i-1}$ and $n = N-1$.

3. Exact Distributions of V When $q = 3$

Distribution of $V_{p_1,1,1;N}$.

For any p_1 value, $p_2 = p_3 = 1$ we have from (2.17)

$$(3.1) \qquad V_{p_1;1,1;N} = X_{21} \cdot X_{31} \quad .$$

Denote X_{2j} by S_j and X_{3j} by T_j. Thus

$$S_j \sim \beta[(f_1-j+1)/2,\ f_2/2] \tag{3.2}$$

where

$$f_1 = n-\bar{p}_2 \quad \text{and} \quad f_2 = \bar{p}_2 \tag{3.3}$$

and

$$T_j \sim \beta[(f_1'-j+1)/2,\ f_2'/2] \tag{3.4}$$

where

$$f_1' = n-\bar{p}_3 = f_1-1 \quad \text{and} \quad f_2' = \bar{p}_3 = f_2+1 \quad . \tag{3.5}$$

Now letting

$$Y_1 = -\log S_1 \quad \text{and} \quad Y_1^* = -\log T_1 \tag{3.6}$$

we have

$$\begin{aligned} - \log V_{p_1,1,1;\ N} &= \log S_1 - \log T_1 \\ &= Y_1 + Y_1^* = W_1 \quad (\text{say}) \quad . \end{aligned} \tag{3.7}$$

It follows from (2.6) that

$$Y_1 \sim (\beta[f_1/2,f_2/2])^{-1} \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} e^{-Y_1(f_1+2\ell)/2} \qquad Y_1 > 0 \tag{3.8}$$

where

(3.9) $$b = (f_2-2)/2$$

and also

(3.10) $$Y_1^* \sim (\beta[(f_1-1)/2,(f_2+1)/2])^{-1} \sum_{m=0}^{b'} (-1)^m \binom{b'}{\ell} e^{-Y_1^*(f_1+2\ell-1)/2}$$

$$Y_1^* > 0$$

where

(3.11) $$b' = (f_2-1)/2 \quad .$$

Now

(3.12) $$W_1 = -\log V_{p_1,1,1;N} \sim \left\{(\beta[f_1/2,f_2/2])^{-1} \sum_{\ell=0}^{b} (-1)^\ell \binom{b}{\ell} e^{-Y_1(f_1+2\ell)/2}\right\} *$$
$$\left\{(\beta[(f_1-1)/2,(f_2+1)/2])^{-1} \sum_{m=0}^{b'} (-1)^m \binom{b'}{\ell} e^{-Y_1^*(f_1+2m-1)/2}\right\}$$

where $*$ denotes the convolution operator. Thus we get the probability density function of $-\log V_{p_1,1,1;N}$ using (2.14) and (2.16), is

(3.13) $$W_1 = -\log V_{p_1,1,1;N} \sim 2\beta_1 \left[\sum_{\ell=0}^{(f_2-2)/2} \sum_{m=0}^{(f_2-1)/2} (-1)^{\ell+m} g_1(\ell,m) \right.$$
$$\left. \cdot \left\{ \frac{e^{-W_1(f_1+2\ell)/2} - e^{-W_1(f_1+2m-1)/2}}{2m-2\ell-1} \right\} \right] \quad .$$

where

$$(3.14) \qquad g_1(\ell,m) = \binom{(f_2-2)/2}{\ell}\binom{(f_2-1)/2}{m}$$

and

$$(3.15) \qquad \beta_1 = (\beta[f_1/2,f_2/2]\beta[(f_1-1)/2,(f_2+1)/2])^{-1} \ .$$

Now substituting

$$(3.16) \qquad W_1 = -\log \ V \qquad dW_1 = -\frac{dV}{V}$$

in (3.13) we obtain

$$(3.17) \qquad V_{p_1,1,1;N} \sim 2\beta_1 \Big[\sum_{\ell=0}^{(f_2-2)/2} \ \sum_{m=0}^{(f_2-1)/2} (-1)^{\ell+m} \frac{g_1(\ell,m)}{2m-2\ell-1} \cdot \left\{V^{(f_1+2\ell-2)/2} - V^{(f_1+2m-3)/2}\right\}\Big] \ .$$

To find the cdf of $V_{p_1,1,1;N}$ we integrate between the limits $(0,v)$ in (3.17), $0 \le v \le 1$ obtaining

$$(3.18) \qquad \Pr\{V_{p_1,1,1;N} \le v\} = 4\beta_1\Big[\sum_{\ell=0}^{(f_2-2)/2} \ \sum_{m=0}^{(f_2-1)/2} (-1)^{\ell+m} \frac{g_1(\ell,m)}{(2m-2\ell-1)} \cdot \left\{\frac{v^{(f_1+2\ell)/2}}{f_1+2\ell} - \frac{v^{(f_1+2m-1)/2}}{f_1+2m-1}\right\}\Big]$$

which is an infinite series for f_2 even or odd.

Distribution of $V_{p_1,2,1;\,N}$

For any p_1 value, $p_2 = 2$, $p_3 = 1$ we have from (2.17)

$$V_{p_1,2,1;\,N} = X_{21} \cdot X_{22} \cdot X_{31} \tag{3.19}$$

where, as in Case i), with $S_j = X_{2j}$ and $T_j = X_{3j}$, S_j and T_j are distributed as (3.2) and (3.4) respectively, but now

$$f_1' = f_1 - 2 \quad \text{and} \quad f_2' = f_2 + 2 \ . \tag{3.20}$$

Now letting

$$Z_1' = S_1 \cdot S_2 \ , \tag{3.21}$$

and using (2.7) and (2.8) it follows

$$Z_1' \sim (2\beta[f_1-1,f_2])^{-1}\, Z_1^{(f_1-3)/2}\, (1-\sqrt{Z_i})^{f_2-1} \ . \tag{3.22}$$

Applying (2.10) and (2.11) we have the density of $Y_1' = -\log Z_1'$ as

$$Y_1' \sim (2\beta[f_1-1,f_2])^{-1} \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-Y_1'(f_1+\ell-1)/2} \ , \ Y_1' > 0 \ . \tag{3.23}$$

Further defining Y_1^* as in (3.6) and using (2.6) we find its density is

$$Y_1^* \sim \frac{1}{\beta[(f_1-2)/2,(f_2+2)/2]} \sum_{m=0}^{f_2-2} (-1)^m \binom{f_2/2}{m} e^{-Y_1^*(f_1+2m-2)/2} \ , Y_1^* > 0. \tag{3.24}$$

Since

$$-\log V_{p_1,2,1;N} = -\log Z_1 - \log T_1 = Y_1' + Y_1^* = W_2 \quad \text{(say)}$$

$$W_2 \sim \left\{(2\beta[f_1-1,f_2])^{-1} \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-Y_1'(f_1+\ell-1)/2}\right\} * \left\{(\beta[(f_1-2)/2,(f_2+2)/2])^{-1} \sum_{m=0}^{f_2/2} (-1)^{m} \binom{f_2/2}{m} e^{-Y_1^{*}(f_1+2m-2)/2}\right\} . \tag{3.25}$$

Now using (2.14) - (2.16) we get the density of W_2 as

$$-\log V_{p_1,2,1;\, N} \sim 2\beta_2\Bigg[\sum_{m=1}^{f_2/2} (-1)^{m+1}\, g_2(2m-1,m)\, W_2\, e^{-W_2(f_1+2m-2)/2} + 2 \sum_{\substack{\ell=0 \\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{f_2/2} (-1)^{\ell+m} \frac{g_2(\ell,m)}{2m-\ell-1} \left\{e^{-W_2(f_1+\ell-1)/2} - e^{-W_2(f_1+2m-2)/2}\right\}\Bigg] \tag{3.26}$$

where

$$\beta_2 = (\beta[f_1-1,f_2]\beta[(f_1-2)/2,(f_2+2)/2])^{-1} \tag{3.27}$$

and

$$g_2(\ell,m) = \binom{f_2-1}{\ell}\binom{f_2/2}{m} \quad . \tag{3.28}$$

Substituting in (3.26)

$$W_2 = -\log V \qquad dW_2 = -\frac{dV}{V} \tag{3.29}$$

we obtain the density of $V_{p_1,2,1;\, N}$ as

$$(3.30)\qquad V_{p_1,2,1;\,N} \sim 2\beta_2 \Big[\sum_{m=1}^{f_2/2} (-1)^{m+1}\, g_2(2m-1,m)(-\log V)\, V^{\frac{1}{2}(f_1+2\ell-4)}$$

$$+\; 2 \sum_{\substack{\ell=0 \\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{f_2/2} (-1)^{\ell+m} \frac{g_2(\ell,m)}{2m-\ell-1} \Big\{ V^{(f_1+\ell-3)/2} - V^{(f_1+2\ell-4)/2} \Big\} \Big] .$$

Integrating (3.30) between the limits $(0,v)$, $0 \leq v \leq 1$, we obtain the c.d.f. of $V_{p_1,2,1;\,N}$ as

$$(3.31)\qquad \Pr\{V_{p_1,2,1;\,N} \leq v\} = \beta_2 \Big[\sum_{m=1}^{f_2/2} (-1)^{m+1} \frac{g_2(2m-1,m)}{a^2}\, v^{a/2} \{2 - a \log v\}$$

$$+\; 2 \sum_{\substack{\ell=0 \\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{f_2/2} (-1)^{\ell+m} \frac{g_2(\ell,m)}{2m-\ell-1} \Big\{ \frac{v^{(f_1+\ell-1)/2}}{f_1+\ell-1} - \frac{v^{a/2}}{a} \Big\} \Big] ,$$

where

$$(3.32)\qquad a = f_1 + 2m - 2 \quad ,$$

and (3.31) is a finite series for f_2 even and an infinite series for f_2 odd.

It can be shown that the expression obtained by Wilks [37] for the c.d.f. of V when $p_1 = 1$, $p_2 = p_3 = 2$ is a special case of (3.31) with $p_1 = 2$. This follows by letting $f_1 = N-3$ and $f_2 = 2$ in (3.31) and letting

$$(3.33) \qquad \beta_W(\tfrac{N-5}{2},2) = \frac{(N-3)(N-4)}{4}\left[\frac{2W^{(N-5)/2}}{N-5} - 2\frac{W^{(N-3)/2}}{N-5} + 4\frac{W^{(N-3)/2}}{(N-3)(N-5)}\right]$$

in Wilks' expression and simplifying.

Distribution of $V_{p_1,2,2;\,N}$

For any p_1 value $p_2 = p_3 = 2$, from (2.17) we have

$$(3.34) \qquad V_{p_1,2,2;\,N} = X_{21} \cdot X_{22} \cdot X_{31} \cdot X_{32}$$

where again S_j and T_j are defined and distributed as in (3.2) and (3.4) and f_1', f_2' are as in (3.20) and f_1 and f_2 as in (3.3). Defining Z_1' as in (3.21) we have the density of $Y_1' = -\log Z_1'$ is (3.23). Now letting

$$(3.35) \qquad Z_1 = T_1 \cdot T_2$$

and

$$(3.36) \qquad U_1 = -\log Z_1$$

from (2.11) we have

$$(3.37) \qquad U_1 \sim (2\beta[f_1'-1,f_2'])^{-1} \sum_{m=0}^{f_2'-1} (-1)^m \binom{f_2'-1}{m} e^{-U_1(f_1+m-1)/2} \qquad U_1 > 0$$

or

$$(3.38) \qquad U_1 \sim (2\beta[f_1-3,f_2+2])^{-1} \sum_{m=0}^{f_2+1} (-1)^m \binom{f_2-1}{m} e^{-U_1(f_1+m-3)/2} \qquad U_1 > 0.$$

Now

$$(3.39)\qquad -\log V_{p_1,2,2;\,N} = -\log Z_1' - \log Z_1 = Y_1' + U_1 = W_3 \quad (\text{say}) ,$$

thus

$$(3.40)\qquad W_3 \sim \left\{ (2\beta[f_1-1,f_2])^{-1} \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-Y_1'(f_1+\ell-1)/2} \right\}$$

$$* \left\{ (2\beta[f_1-3,f_2+2])^{-1} \sum_{m=0}^{f_2+1} (-1)^{m} \binom{f_2+1}{m} e^{-U_1(f_1+m-3)/2} \right\} .$$

Using (2.14) - (2.16) we obtain the density function of W_3 as

$$(3.41)\qquad W_3 \sim \beta_3 \Big[\sum_{\ell=0}^{f_2-1} g_3(\ell,\ell+2)\, W_3\, e^{-\frac{aW_3}{2}}$$

$$+ 2 \sum_{\substack{\ell=0 \\ m\neq\ell+2}}^{f_2-1} \sum_{m=0}^{f_2+1} (-1)^{m+\ell} g_3(\ell,m) \left\{ \frac{e^{-\frac{aW_3}{2}} - e^{-\frac{bW_3}{2}}}{b-a} \right\} \Big]$$

where

$$(3.42)\qquad \beta_3 = (4\beta[f_1-1,f_2]\beta[f_1-3,f_2+2])^{-1} ,$$

$$(3.43)\qquad g_3(\ell,m) = \binom{f_2-1}{\ell}\binom{f_2+1}{m}$$

and

$$(3.44)\qquad a = f_1+\ell-1 \qquad b = f_1+m-3 .$$

Substituting

$$(3.45) \qquad W_3 = -\log V \qquad dW_3 = -\frac{dV}{V}$$

in (3.41), we obtain the density of $V_{p_1,2,2;\,N}$ as

$$(3.46) \qquad V_{p_1,2,2;\,N} \sim \beta_1 \Big[\sum_{\ell=0}^{f_2-1} g_3(\ell,\ell+2)(-\log V)\, V^{(a-2)/2}$$

$$+ 2 \sum_{\ell=0}^{f_2-1} \sum_{\substack{m=0 \\ m\neq\ell+2}}^{f_2+1} (-1)^{m+\ell} \frac{g_3(\ell,m)}{b-a} \{V^{(a-2)/2} - V^{(b-2)/2}\}\Big] .$$

Integrating (3.46) between the limits $(0,v)$, $0 \leq v \leq 1$, the cdf of $V_{p_1,2,2;\,N}$ is

$$(3.47) \qquad \Pr\{V_{p_1,2,2;\,N} \leq v\} = 2\beta_1 \Big[\sum_{\ell=0}^{f_2-1} g_3(\ell,\ell+2) \frac{v^{a/2}}{a^2} \{2-a \log v\}$$

$$+ 2 \sum_{\ell=0}^{f_2-1} \sum_{\substack{m=0 \\ m\neq\ell+2}}^{f_2+1} (-1)^{m+\ell} \frac{g_3(\ell,m)}{b-a} \Big\{\frac{v^{a/2}}{a} - \frac{v^{b/2}}{b}\Big\}\Big]$$

which is a finite series for f_2 even or odd.

It can easily be shown that the expression obtained by Wilks [37] for the cdf of V in the case when $p_1 = 1$, $p_2 = p_3 = 2$ is a special case of (3.47) with $p_1 = 1$. This follows by letting $f_1 = N-2$, $f_2 = 1$ in (3.47) and using (3.33) in Wilks' expression and simplifying. Also the expressions obtained by Wilks [37] in the cases when $p_1 = p_2 = p_3 = 2$ and $p_1 = p_2 = 2$,

$p_3 = 3$ are special cases of (3.47) with $p_1 = 2$, $f_1 = N-3$, $f_2 = 1$ and $p_1 = 3$, $f_1 = N-4$, and $f_2 = 3$ respectively.

In as much as the mechanics of obtaining the remaining results are the same as in the previous cases, only the final results will be given.

Distribution of $V_{p_1,3,1;\,N}$

The cdf of $V_{p_1,3,1;\,N}$ for all values of p_1 is

$$\Pr\{V_{p_1,3,1;\,N} \leq v\} = \beta_4\Big[4 \sum_{m=1}^{(f_2-2)/2} \sum_{t=0}^{(f_2+1)/2} (-1)^{m+t+1} \frac{g_4(2m-1,m,t)}{c-b} \cdot \left\{\frac{v^{b/2}}{b^2}[2-b \log v] - \frac{2v^{b/2}}{b(c-b)} + \frac{2v^{c/2}}{c(c-b)}\right\}$$
$$+ 4 \sum_{m=0}^{(f_2-2)/2} \sum_{t=1}^{(f_2+1)/2} (-1)^{m+t} \frac{g_4(2t-2,m,t)}{(b-c)c^2} v^{c/2}[2-c \log v]$$
$$+ 8 \sum_{\substack{\ell=0\\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{(f_2-2)/2} \sum_{\substack{t=0\\ \ell\neq 2t-2}}^{(f_2+1)/2} \frac{(-1)^{\ell+m+t} g_4(\ell,m,t)}{(b-a)(c-a)} \left\{\frac{v^{a/2}}{a} - \frac{v^{c/2}}{c}\right\}$$
$$- 8 \sum_{\ell=0}^{f_2-1} \sum_{\substack{m=0\\ \ell\neq 2m-1}}^{(f_2-2)/2} \sum_{t=0}^{(f_2+1)/2} \frac{(-1)^{\ell+m+t} g_4(\ell,m,t)}{(b-a)(c-b)} \left\{\frac{v^{b/2}}{b} - \frac{v^{c/2}}{c}\right\}\Big] \tag{3.48}$$

which is an infinite series for f_2 even or odd and where

$$\beta_4 = (2\beta[f_1-1,f_2]\beta[(f_1-2)/2,f_2/2]\beta[(f_1-3)/2,(f_2+3)/2])^{-1}$$

and

$$g_4(\ell,m,t) = \binom{f_2-1}{\ell}\binom{(f_2-2)/2}{m}\binom{(f_2+1)/2}{t}$$

and

$$a = f_1+\ell-1, \quad b = f_1+2m-2, \quad c = f_1+2t-3 \quad .$$

Distribution of $V_{p_1,3,2;\,N}$

The cdf of $V_{p_1,3,2;\,N}$ for all values of p_1 is

$$(3.49) \qquad \Pr\{V_{p_1,3,2;\,N} \le v\} = \beta_5\Big[\sum_{m=1}^{(f_2-2)/2} \frac{(-1)^{m+1} g_5(2m-1,m,2m+2)}{b} v^{b/2} \cdot \left\{(\log v)^2 + \frac{8}{b^2} - \frac{4 \log v}{b}\right\}$$

$$+ 4 \sum_{m=1}^{(f_2-2)/2} \sum_{\substack{t=0 \\ t\neq 2m+2}}^{f_2+2} \frac{(-1)^{m+t+1} g_5(2m-1,m,t)}{c-b} \cdot \left\{\frac{v^{b/2}}{b^2}[2-b \log v] - \frac{2v^{b/2}}{b(c-b)} + \frac{2v^{c/2}}{c(c-b)}\right\}$$

$$+4 \sum_{\substack{\ell=0 \\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{(f_2-2)/2} \frac{(-1)^{m+1} g_5(\ell,m,\ell+3) v^{a/2}}{(b-a)\, a^2}[2-a \log v]$$

$$+8 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{(f_2-2)/2} \sum_{t=0}^{f_2+2} \frac{(-1)^{\ell+m+t} g_5(\ell,m,t)}{(b-a)(c-a)}\left\{\frac{v^{a/2}}{a} - \frac{v^{c/2}}{c}\right\}$$
$$\ell\neq 2m-1, \quad t\neq \ell+3$$

$$-4 \sum_{\substack{\ell=0 \\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{(f_2-2)/2} \frac{(-1)^{m+\ell} g_5(\ell,m,2m+2)\ v^{b/2}}{(b-a)\ b^2} [2-b \log v]$$

$$-8 \sum_{\substack{\ell=0 \\ \ell\neq 2m-1}}^{f_2-1} \sum_{m=0}^{(f_2-2)/2} \sum_{\substack{t=0 \\ t\neq 2m+2}}^{f_2+2} \frac{(-1)^{\ell+m+t} g_5(\ell,m,t)}{(b-a)(c-a)} \left\{\frac{v^{b/2}}{b} - \frac{v^{c/2}}{c}\right\}\Bigg]$$

which is a finite series for f_2 even and an infinite series for f_2 odd and where

$$\beta_5 = (4\beta[f_1-1,f_2]\beta[(f_1-2)/2,f_2/2]\beta[f_1-4,f_2+3])^{-1}$$

and

$$g_5(\ell,m,t) = \binom{f_2-1}{\ell}\binom{f_2-2/2}{m}\binom{f_2+2}{t}$$

and

$$a = f_1+\ell-1, \quad b = f_1+2m-2, \quad c = f_1+t-4 .$$

It can easily be verified that the expression obtained by Wilks [37] for the cdf of V when $p_1 = p_2 = 2$, $p_3 = 3$ is a special case of (3.49) with $p_1 = 2$, $f_1 = N-3$ and $f_2 = 2$.

Distribution of $V_{p_1,4,1;N}$

The cdf of $V_{p_1,4,1;N}$ for all values of p_1 is

$$
\begin{aligned}
(3.50)\qquad \Pr\{V_{p_1,4,1;N} \leq v\} = \beta_6\Big[& \sum_{t=2}^{f_2/2} \frac{(-1)^t g_6(2t-3,2t-1,t)}{c} \Big\{(\log v)^2 + \frac{4v^{c/2}}{c^2}[2-c \log v]\Big\} \\
& + 4 \sum_{\substack{\ell=0 \\ \ell\neq 2t-3}}^{f_2-3} \sum_{t=0}^{(f_2+2)/2} \frac{(-1)^t g_6(\ell,\ell+2,t)}{(c-a)} \Big\{\frac{v^{a/2}}{a}[2-a \log v] - \frac{2v^{a/2}}{a(c-a)} + \frac{2v^{c/2}}{c(c-a)}\Big\} \\
& + 4 \sum_{\substack{m=0 \\ m\neq 2t-1}}^{f_2-1} \sum_{t=2}^{(f_2+2)/2} \frac{(-1)^{m+t+1} g_6(2t-3,m,t) v^{c/2}}{(b-c)\, c^2} [2-c \log v] \\
& + 8 \sum_{\substack{\ell=0 \\ \ell\neq 2t-3}}^{f_2-1} \sum_{\substack{m=0 \\ m\neq \ell+2}}^{f_2-1} \sum_{t=0}^{(f_2+2)/2} \frac{(-1)^{\ell+m+t} g_6(\ell,m,t)}{(c-a)(b-a)} \Big\{\frac{v^{a/2}}{a} - \frac{v^{c/2}}{c}\Big\} \\
& - 4 \sum_{\substack{\ell=0 \\ \ell\neq 2t-3}}^{f_2-1} \sum_{t=1}^{f_2/2} \frac{(-1)^{\ell+t+1} g_6(\ell,2t-1;t) v^{c/2}}{(c-a)\, c^2} [2-c \log v] \\
& - 8 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{f_2-1} \sum_{t=0}^{(f_2+2)/2} \frac{(-1)^{\ell+m+t} g_6(\ell,m,t)}{(b-a)(c-a)} \Big\{\frac{v^{b/2}}{b} - \frac{v^{c/2}}{c}\Big\}\Big] \\
& \qquad m\neq 2t-1,\ m\neq \ell+2
\end{aligned}
$$

which is a finite series for f_2 even and an infinite series for f_2 odd and where

$$\beta_6 = (4\beta[f_1-1,f_2]\beta[f_1-3,f_2]\beta[(f_1-4)/2,(f_2+4)/2])^{-1}$$

and

$$g_6(\ell,m,t) = \binom{f_2-1}{\ell}\binom{f_2-1}{m}\binom{(f_2+2)/2}{t}$$

and

$$a = f_1+\ell-1, \quad b = f_1+m-3, \quad c = f_1+2t-4 \quad .$$

It can be verified that the cdf of V obtained by Wilks [37] in the case where $p_1=1$, $p_2=2$, $p_3=4$ is a special case of (3.50) by letting $p_1=2$, $f_1=N-3$ and $f_2=2$.

Distribution of $V_{p_1,4,2;N}$

The cdf of $V_{p_1,4,2;N}$ for all values of p_1 is given by

$$(3.51) \qquad \Pr\{V_{p_1,4,2;N} \leq v\} = \beta_7\Big[\sum_{\ell=0}^{f_2-3} \frac{(-1)^{\ell} g_7(\ell,\ell+2,\ell+4)v^{a/2}}{a} \cdot \Big\{(\log v)^2 + \frac{8}{a^2} - \frac{4\log v}{a}\Big\}$$

$$+ 4 \sum_{\substack{\ell=0 \\ t\neq\ell+4}}^{f_2-3} \sum_{t=0}^{f_2+3} \frac{(-1)^t g_7(\ell,\ell+2,t)}{c-a}\Big\{\frac{v^{a/2}}{a}[2-a\log v] - \frac{2v^{a/2}}{a(c-a)} + \frac{2v^{c/2}}{c(c-a)}\Big\}$$

$$+ 4 \sum_{\substack{\ell=0}}^{f_2-1} \sum_{\substack{m=0 \\ m\neq\ell+2}}^{f_2-1} \frac{(-1)^m g_7(\ell,m,\ell+4) v^{a/2}}{(b-a)a^2} [2-a \log v]$$

$$+ 8 \sum_{\ell=0}^{f_2-1} \sum_{\substack{m=0 \\ m\neq\ell+2}}^{f_2-1} \sum_{\substack{t=0 \\ t\neq\ell+4}}^{f_2+3} \frac{(-1)^{\ell+m+t} g_7(\ell,m,t)}{(b-a)(c-a)} \left\{\frac{v^{a/2}}{a} - \frac{v^{c/2}}{c}\right\}$$

$$- 4 \sum_{\ell=0}^{f_2-1} \sum_{\substack{m=0 \\ m\neq\ell+2}}^{f_2-1} \frac{(-1)^{\ell} g_7(\ell,m,m+2)\ v^{b/2}}{(b-a)\ b^2} [2-b \log v]$$

$$- 8 \sum_{\ell=0}^{f_2-1} \sum_{\substack{m=0 \\ m\neq\ell+2}}^{f_2-1} \sum_{\substack{t=0 \\ t\neq m+2}}^{f_2+3} \frac{(-1)^{\ell+m+t}\ g_7(\ell,m,t)}{(b-a)(c-b)} \left\{\frac{v^{b/2}}{b} - \frac{v^{c/2}}{c}\right\}\Bigg]$$

which is a finite series for f_2 even or odd and where

$$\beta_7 = (8\beta[f_1-1,f_2]\beta[f_1-3,f_2]\beta[f_1-5,f_2+4])^{-1}$$

and

$$g_7(\ell,m,t) = \binom{f_2-1}{\ell}\binom{f_2-1}{m}\binom{f_2+3}{t}$$

and

$$a = f_1+\ell-1, \quad b = f_1+m-3, \quad c = f_1+t-5 \quad .$$

Further it can be shown that the cdf of V obtained by Wilks [37] for $p_1= 1$, $p_2= 4$, $p_3= 2$ is a special case of (3.51) with $p_1= 1$, $f_1= N-2$, $f_2 = 1$.

4. Exact Distributions of V When $q = 4$

Distribution of $V_{p_1,2,2,2;N}$

The cdf of $V_{p_1,2,2,2;N}$ for all p_1 may be obtained from (3.51) by changing the limits of summation of m and ℓ from f_2-1 and f_2-3 to f_2+1 and f_2-1, where appropriate and replacing β_7 and $g_7(\ell,m,t)$ by β_8 and $g_8(\ell,m,t)$ where

$$\beta_8 = (8\beta[f_1-1,f_2]\beta[f_1-3,f_2+2]\beta[f_1-5,f_2+4])^{-1}$$

and

$$g_8(\ell,m,t) = \binom{f_2-1}{\ell}\binom{f_2+1}{m}\binom{f_2+3}{t} \quad .$$

Distribution of $V_{p_1,2,2,1;N}$

The cdf of $V_{p_1,2,2,1;N}$ for all p_1 may be obtained from (3.50) by changing the upper limits of summation of ℓ,m,t from f_2-3, f_2-1 and $f_2/2$ to f_2-1, f_2+1 and $(f_2+2)/2$ where appropriate and replacing $g_6(\ell,m,t)$ and β_6 by $g_9(\ell,m,t)$ and β_9 where

$$\beta_9 = (4\beta[f_1-1,f_2]\beta[f_1-3,f_2+2]\beta[(f_1-4)/2,(f_2+4)/2])^{-1}$$

and

$$g_9(\ell,m,t) = \binom{f_2-1}{\ell}\binom{f_2+1}{m}\binom{(f_2+2)/2}{t} \quad .$$

Distribution of $V_{p_1,2,1,1;N}$

The cdf of $V_{p_1,2,1,1;N}$ for all p_1 may be obtained from (3.48) by changing the upper limit of summation of m from $(f_2-2)/2$ to $f_2/2$, whenever it appears, and substituting β_{10} and $g_{10}(\ell,m,t)$ for β_4 and $g_4(\ell,m,t)$ where

$$\beta_{10} = (2\beta[f_1-1,f_2]\beta[(f_1-2)/2,(f_2+2)/2]\beta[(f_1-3)/2,(f_1+3)/2])^{-1}$$

and

$$g_{10}(\ell,m,t) = \binom{f_2-1}{\ell}\binom{f_2/2}{m}\binom{(f_2+1)/2}{t} \quad .$$

Distribution of $V_{p_1,1,1,1;N}$

The cdf of $V_{p_1,1,1,1;N}$ for all values of p_1 is

$$\Pr\{V_{p_1,1,1,1;N} \leq v\} = \beta_{11}\Big[4 \sum_{\ell=0}^{(f_2-2)/2} \sum_{m=0}^{(f_2-1)/2} \frac{(-1)^{m+1} g_{11}(\ell,m,\ell+1)_v{}^{a/2}}{a^2(b-a)} \cdot [2-a \log v]$$

$$+\, 8 \sum_{\ell=0}^{(f_2-2)/2} \sum_{\substack{m=0 \\ t\neq\ell+1}}^{(f_2-1)/2} \sum_{t=0}^{f_2/2} \frac{(-1)^{\ell+m+t} g_{11}(\ell,m,t)}{(c-a)(b-a)} \cdot \left\{\frac{v^{a/2}}{a} - \frac{v^{c/2}}{c}\right\}$$

$$- 8 \sum_{\ell=0}^{(f_2-2)/2} \sum_{m=0}^{(f_2-1)/2} \sum_{t=0}^{f_2/2} \frac{(-1)^{\ell+m+t} \quad g_{11}(\ell,m,t)}{(c-b)(b-a)} \left\{\frac{v^{b/2}}{b} - \frac{v^{c/2}}{c}\right\}\Big]$$

which is an infinite series for f_2 even or odd.

Results for many other combinations of p_i's could be obtained using this method but more than two convolutions are involved and the cdf's become quite unwieldy.

5. Computation of Percentage Points

The expressions derived in the preceeding sections were used for tabulation of percentage points of V. Values of $V_{p_1,p_2,p_3;N}$ were first computed on the CDC 6500 to a minimum accuracy of four significant digits based on three arguments (p_1,N,α) where α is the lower probability level. For values of $N > 24$ Anderson's approximation ([3] page 239) was used. These values were then used to obtain correction factors for converting chi-square percentiles with $f = \frac{1}{2}(p^2 - \Sigma\, p_i^2)$ degrees of freedom to exact percentiles of $-\{N - 1.5 - (p^2 - \Sigma\, p_i^3)/3f\} \log V$. Finally tabulation of the correction factors,

$$C = [\text{percentile of } - \{N - 1.5 - (p^3 - \Sigma\, p_i^3)/3f\} \log V \,/ \qquad (\text{percentile of } \chi_f^2) \;,$$

was made. These are given to three decimal places although they were generally obtained to four decimals. The correction factors are presented for $\alpha = 0.01, 0.05$, $M = 1(1)10(2)20, 24, 30, 60, 120, \infty$, $p_1 = 1(1)10$ in Tables 5 and 6 and $p_1 = 2(2)10$ in Tables 7-10.

Table 5. Chi-Square Adjustments to

the $V_{p_1,2,2;N}$ Criterion, Factor C for

Lower 1% Points of V (Upper Percentiles of χ^2)

$M \backslash p_1$	1	2	3	4	5	6	7	8	9	10
1	1.367	1.444	1.503	1.555	1.602	1.646	1.687	1.726	1.762	1.796
2	1.134	1.171	1.203	1.234	1.264	1.292	1.320	1.346	1.371	1.395
3	1.071	1.094	1.115	1.137	1.158	1.179	1.199	1.219	1.239	1.257
4	1.044	1.060	1.075	1.091	1.107	1.124	1.140	1.156	1.171	1.187
5	1.030	1.042	1.053	1.066	1.079	1.092	1.105	1.118	1.131	1.144
6	1.022	1.031	1.040	1.050	1.060	1.071	1.082	1.093	1.104	1.115
7	1.017	1.024	1.031	1.039	1.048	1.057	1.066	1.075	1.085	1.094
8	1.013	1.019	1.025	1.032	1.039	1.047	1.055	1.063	1.071	1.079
9	1.011	1.015	1.021	1.026	1.033	1.039	1.046	1.053	1.060	1.067
10	1.009	1.013	1.017	1.022	1.028	1.033	1.039	1.045	1.052	1.058
12	1.006	1.009	1.013	1.016	1.021	1.025	1.030	1.034	1.039	1.044
14	1.005	1.007	1.010	1.013	1.016	1.019	1.023	1.026	1.031	1.035
16	1.004	1.005	1.008	1.010	1.013	1.016	1.018	1.021	1.025	1.028
18	1.003	1.004	1.006	1.008	1.010	1.013	1.015	1.018	1.021	1.024
20	1.002	1.004	1.005	1.007	1.009	1.011	1.012	1.015	1.017	1.020
24	1.002	1.003	1.004	1.005	1.006	1.007	1.008	1.012	1.014	1.015
30	1.001	1.002	1.002	1.003	1.004	1.005	1.006	1.009	1.010	1.011
40	1.001	1.001	1.002	1.002	1.002	1.003	1.005	1.007	1.007	1.008
60	1.001	1.000	1.001	1.001	1.001	1.002	1.003	1.004	1.004	1.005
120	1.000	1.000	1.000	1.000	1.000	1.000	1.001	1.001	1.002	1.003
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
χ_f^2	20.0902	26.2170	31.9999	37.5662	42.9798	48.2782	53.4858	58.6192	63.6907	68.7095

p_1 = number of variates in the first set; $M = N-p_1-4$; N = number of observations, C = [percentile for $-\{N-1.5-\frac{1}{3}(p^3-\Sigma p_i^3)(p^2-\Sigma p_i^2)^{-1}\log V]/$ (percentile for χ^2 with $\frac{1}{2}(p^2-\Sigma p_i^2)$ degrees of freedom).

Table 6. Chi-Square Adjustments to

the $V_{p_1,2,2;N}$ Criterion, Factor C for

Lower 5% Points of V (Upper Percentiles of χ^2)

M \ p_1	1	2	3	4	5	6	7	8	9	10
1	1.299	1.360	1.409	1.453	1.494	1.532	1.568	1.601	1.633	1.663
2	1.110	1.142	1.171	1.199	1.226	1.252	1.277	1.301	1.324	1.346
3	1.059	1.079	1.098	1.118	1.138	1.157	1.176	1.195	1.213	1.230
4	1.037	1.051	1.065	1.080	1.095	1.110	1.125	1.140	1.154	1.169
5	1.025	1.035	1.046	1.058	1.070	1.082	1.094	1.106	1.119	1.131
6	1.018	1.026	1.035	1.044	1.054	1.064	1.074	1.084	1.095	1.105
7	1.014	1.020	1.027	1.035	1.043	1.051	1.060	1.069	1.077	1.086
8	1.011	1.016	1.022	1.028	1.035	1.042	1.050	1.057	1.065	1.073
9	1.009	1.013	1.018	1.023	1.029	1.035	1.042	1.048	1.055	1.062
10	1.007	1.011	1.015	1.020	1.025	1.030	1.036	1.042	1.047	1.054
12	1.005	1.008	1.011	1.015	1.018	1.023	1.027	1.032	1.037	1.041
14	1.004	1.006	1.008	1.011	1.014	1.018	1.021	1.025	1.029	1.033
16	1.003	1.005	1.007	1.009	1.011	1.014	1.017	1.020	1.024	1.027
18	1.002	1.004	1.005	1.007	1.009	1.012	1.014	1.017	1.020	1.023
20	1.002	1.003	1.004	1.006	1.008	1.010	1.012	1.014	1.017	1.019
24	1.001	1.002	1.003	1.004	1.005	1.006	1.008	1.009	1.012	1.013
30	1.001	1.001	1.002	1.002	1.003	1.004	1.006	1.007	1.008	1.009
40	1.000	1.001	1.001	1.001	1.002	1.002	1.004	1.005	1.006	1.007
60	1.000	1.000	1.000	1.001	1.001	1.001	1.002	1.002	1.003	1.004
120	1.000	1.000	1.000	1.000	1.000	1.000	1.001	1.001	1.001	1.002
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
χ_f^2	15.5073	21.0261	26.2962	31.4104	36.4151	41.3372	46.1943	50.9985	55.7585	60.4809

p_1 = number of variates in the first set; $M = N-p_1-4$; N = number of observations, C = [percentile for $-\{N-1.5-\frac{1}{3}(p^3-\Sigma p_i^3)(p^2-\Sigma p_i^2)^{-1}\log V]$/ percentile for χ^2 with $\frac{1}{2}(p^2-\Sigma p_i^2)$ degrees of freedom).

Table 7. Chi-Square Adjustments to the $V_{p_1,2,1;N}$ Criterion,

Factor C for Lower 1% Points of V (Upper Percentiles of χ^2)

$M \backslash p_1$	2	4	6	8	10
1	1.367				
2	1.134				
3	1.071				
4	1.044				
5	1.030				
6	1.022				
7	1.017				
8	1.013				
9	1.011				
10	1.009				
12	1.006				
14	1.005				
16	1.004				
18	1.003				
20	1.002				
24	1.002	1.002	1.004	1.006	1.009
30	1.001	1.001	1.002	1.004	1.006
40	1.001	1.001	1.001	1.002	1.004
60	1.000	1.000	1.000	1.001	1.002
120	1.000	1.000	1.000	1.000	1.000
∞	1.000	1.000	1.000	1.000	1.000
χ_f^2	20.0902	29.1412	37.5662	45.6417	53.4858

p_1= number of variates in the first set; $M=N-p_1-3$; N=number of observations, C=[percentile for $-\{N-1.5-\frac{1}{3}(p^3-\Sigma p_i^3)(p^2-\Sigma p_i^2)^{-1}\}$ log V]/(percentile for χ^2 with $\frac{1}{2}(p-\Sigma p_i^2)$ degrees of freedom).

Table 8. Chi-Square Adjustments to the $V_{p_1,2,1;N}$ Criterion, Factor C for Lower 5% Points of V (Upper Percentiles of χ^2)

p_1 / M	2	4	6	8	10
1	1.299				
2	1.110				
3	1.059				
4	1.037				
5	1.025				
6	1.018				
7	1.014				
8	1.011				
9	1.009				
10	1.007				
12	1.005				
14	1.004				
16	1.003				
18	1.002				
20	1.002				
24	1.001	1.002	1.004	1.006	1.009
30	1.001	1.001	1.003	1.004	1.006
40	1.000	1.001	1.002	1.003	1,004
60	1.000	1.000	1.001	1.001	1,002
120	1.000	1.000	1.000	1.000	1.001
∞	1.000	1.000	1.000	1.000	1.000
χ_f^2	15.5073	23.6848	31.4140	38.8851	46.1943

p_1= number of variates in the first set; $M=N-p_1-3$, N=number of observations, C=[percentile for $-\{N-1.5-\frac{1}{3}(p^3-\Sigma p_i^3)(p^2-\Sigma p_i^2)^{-1}\}\log V$]/(percentile for χ^2 with $\frac{1}{2}(p^2-\Sigma p_i^2)$ degrees of freedom).

Table 9. Chi-Square Adjustments to the $V_{p_1,3,2;N}$ Criterion, Factor C for Lower 1% Points of V (Upper Percentiles of χ^2)

M \ p_1	2	4	6	8	10
1	1.503				
2	1.203				
3	1.115				
4	1.075				
5	1.053				
6	1.040				
7	1.031				
8	1.025				
9	1.021				
10	1.017				
12	1.013				
14	1.010				
16	1.008				
18	1.006				
20	1.005				
24	1.004	1.004	1.005	1.007	1.010
30	1.002	1.002	1.003	1.005	1.007
40	1.002	1.002	1.002	1.003	1.004
60	1.001	1.001	1.001	1.001	1.002
120	1.000	1.000	1.000	1.000	1.001
∞	1.000	1.000	1.000	1.000	1.000
χ_f^2	31.9999	45.6417	58.6192	71.2014	83.5134

p_1= number of variates in the first set; $M=N-p_1-5$, N=number of observations, $C=[$percentile for $-\{N-1.5-\frac{1}{3}(p^3-\Sigma p_i^3)(p^2-\Sigma p_i^2)^{-1}\}$ log V$]/$(percentile for χ^2 with $\frac{1}{2}(p^2-\Sigma p_i^2)$ degrees of freedom).

Table 10. Chi-Square Adjustments to the $V_{p_1,3,2;N}$ Criterion, Factor C for Lower 5% Points of V (Upper Percentiles of χ^2)

M \ p_1	2	4	6	8	10
1	1.409				
2	1.171				
3	1.098				
4	1.065				
5	1.046				
6	1.035				
7	1.027				
8	1.022				
9	1.018				
10	1.015				
12	1.011				
14	1.008				
16	1.007				
18	1.005				
20	1.004				
24	1.003	1.003	1.005	1.008	1.011
30	1.002	1.002	1.004	1.005	1.007
40	1.001	1.001	1.002	1.003	1.005
60	1.000	1.001	1.001	1.002	1.002
120	1.000	1.000	1.000	1.000	1.001
∞	1.000	1.000	1.000	1.000	1.000
χ^2_f	26.2962	38.8851	50.9985	62.8296	74.4683

p_1= number of variates in the first set; $M=N-p_1-5$, N=number of observations, $C=[\text{percentiles } -\{N-1.5-\frac{1}{3}(p^3-\Sigma p_i^3)(p^2-\Sigma p_i^2)^{-1}\} \log V]/(\text{percentile for } \chi^2 \text{ with } \frac{1}{2}(p^2-\Sigma p_i^2)$ degrees of freedom).

CHAPTER IV

SOME DISTRIBUTION PROBLEMS IN THE MULTIVARIATE COMPLEX GAUSSIAN CASE

1. Introduction and Summary

Let $\underset{\sim}{X}_1 : p \times n$ and $\underset{\sim}{X}_2 : p \times n$ be real random variables having the joint density function

$$(1.1) \qquad (2\pi)^{-pn} |\underset{\sim}{\Sigma}_o|^{-\frac{1}{2}n} \exp\{-\tfrac{1}{2} \operatorname{tr} \underset{\sim}{\Sigma}_o^{-1} (\underset{\sim}{X}-\underset{\sim}{\nu})(\underset{\sim}{X}-\underset{\sim}{\nu})'\}, \quad -\infty \leq \underset{\sim}{X} \leq \infty$$

where

$$\underset{\sim}{X} = \begin{pmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \end{pmatrix}, \quad \underset{\sim}{\Sigma}_o = \begin{pmatrix} \underset{\sim}{\Sigma}_1 & -\underset{\sim}{\Sigma}_2 \\ \underset{\sim}{\Sigma}_2 & \underset{\sim}{\Sigma}_1 \end{pmatrix}, \quad \underset{\sim}{\nu} = \begin{pmatrix} \underset{\sim}{\mu}_1 & -\underset{\sim}{\mu}_2 \\ \underset{\sim}{\mu}_2 & \underset{\sim}{\mu}_1 \end{pmatrix} \begin{pmatrix} \underset{\sim}{M}_1 \\ \underset{\sim}{M}_2 \end{pmatrix},$$

$\underset{\sim}{\Sigma}_1 : p \times p$ is a real symmetric positive definite (p.d.) matrix, $\underset{\sim}{\Sigma}_2 : p \times p$ is a real skew-symmetric matrix, $\underset{\sim}{\mu}_j : p \times q$ and $\underset{\sim}{M}_j : q \times n$ $(j = 1,2)$, are given matrices or their joint density does not contain $\underset{\sim}{\Sigma}_1, \underset{\sim}{\Sigma}_2, \underset{\sim}{\mu}_1, \underset{\sim}{\mu}_2$ as parameters. Then it has been shown by Goodman [10] that the distribution of the complex matrix $\underset{\sim}{Z} = \underset{\sim}{X}_1 + i\underset{\sim}{X}_2$, $(i = (-1)^{\frac{1}{2}})$, is complex Gaussian and its density function is given by

$$(1.2) \qquad N_c(\underset{\sim}{\nu}, \underset{\sim}{\Sigma}_o) = \pi^{-pn} |\underset{\sim}{\Sigma}|^{-n} \exp\{-\operatorname{tr} \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{Z}-\underset{\sim}{\mu}\underset{\sim}{M})(\underset{\sim}{Z}-\overline{\underset{\sim}{\mu}\underset{\sim}{M}})'\}$$

where $\underset{\sim}{\Sigma} = \underset{\sim}{\Sigma}_1 + i\underset{\sim}{\Sigma}_2$ is Hermitian p.d., i.e. $\bar{\underset{\sim}{\Sigma}}' = \underset{\sim}{\Sigma}$, $\underset{\sim}{\mu} = \underset{\sim}{\mu}_1 + i\underset{\sim}{\mu}_2$ and $\underset{\sim}{M} = \underset{\sim}{M}_1 + i\underset{\sim}{M}_2$. Goodman [10], Wooding [38], James [13], Al-Ani [1], and Khatri [16], [17], [19], [20] have studied distributions derived from a sample of a complex p-variate normal distribution.

Some concepts which are important and necessary notation are given below.

$$\tilde{\Gamma}_m(a) = \pi^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \Gamma(a-i+1)$$

$$[a]_\kappa = \prod_{i=1}^{m} (a-i+1)_{k_i} = \tilde{\Gamma}_m(a,\kappa)/\tilde{\Gamma}_m(a)$$

where $\kappa = (k_1, k_2, \ldots, k_p)$ is a partition of the integer k and

$$\tilde{\Gamma}_m(a,\kappa) = \pi^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \Gamma(a+k_i-i+1) \quad .$$

The hypergeometric functions are defined as

$${}_p\tilde{F}_q(a_1,\ldots,a_p;\ b_1,\ldots,b_q;\underset{\sim}{A},\underset{\sim}{B}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\prod_{i=1}^{p} [a_i]_\kappa\ \tilde{C}_\kappa(\underset{\sim}{A})\ \tilde{C}_\kappa(\underset{\sim}{B})}{\prod_{i=1}^{q} [b_i]_\kappa\ \tilde{C}_\kappa(\underset{\sim}{I}_m) k!}$$

or when $\underset{\sim}{B} = \underset{\sim}{I}_m$ we denote it by

$${}_p\tilde{F}_q(a_1,\ldots,a_p;\ b_1,\ldots,b_q;\ \underset{\sim}{A})$$

and $\tilde{C}_\kappa(\underset{\sim}{A})$ is a zonal polynomial of a Hermitian matrix $\underset{\sim}{A}$ and is given as a symmetric function of the characteristic roots of $\underset{\sim}{A}$. (See Section 6)

The non-central distributions of the characteristic roots concerning the classical problems of the covariance model, MANOVA model, and canonical correlation coefficients have been found by James [13] and Khatri [16], [19]. Here for the three cases mentioned, we give the general moment and the density which is expressed in terms of Meijer's G-function [25],[26], for $W^{(p)} = \prod_{i=1}^{p} (1-w_i)$, where the w_i, $i = 1,2,\ldots,p$ are the characteristic roots in the above cases. The moments and densities are analogous to those given in the real case in Chapter II. Further the density functions of U and Pillai's V criteria in the complex central case are obtained for $p = 2$ and from the non-central complex multivariate F distribution various independence relationships are shown and independent beta variables are obtained. The last section is devoted to complex zonal polynomials. A method for computing them in terms of elementary symmetric function (esf's) is given and they are tabulated through degree 8 in Tables 11-14.

2. Density Functions of $W^{(p)}$ in the Non-Central Case

Testing the Equality of Two Covariance Matrices

Let $\underset{\sim}{X}:(p \times n_1) \sim N_c(\underset{\sim}{0},\underset{\sim}{\Sigma}_1)$ and $\underset{\sim}{Y}:(p \times n_2) \sim N_c(\underset{\sim}{0},\underset{\sim}{\Sigma}_2)$ be independent and $n_1 \geq p$. Then Khatri [19] has shown the density function of the characteristic roots, $0 < f_1 < \ldots < f_p$ of $(\underset{\sim}{X}\,\underset{\sim}{\bar{X}}')(\underset{\sim}{Y}\,\underset{\sim}{\bar{Y}}')^{-1}$ can be written as

$$(2.1) \qquad C(p)|\underset{\sim}{\Lambda}|^{-n_1}\ {}_1F_0(n,;\underset{\sim}{I}_p-\underset{\sim}{\Lambda}^{-1},\underset{\sim}{F}(\underset{\sim}{I}_p+\underset{\sim}{F})^{-1})\ \frac{|\underset{\sim}{F}|^{n_1-p}}{|\underset{\sim}{I}_p+\underset{\sim}{F}|^{n}} \prod_{i>j} (f_i-f_j)^2$$

where

(2.2) $$C(p) = \frac{\pi^{p(p-1)}\tilde{\Gamma}_p(n)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}, \quad n = n_1+n_2, \quad \underset{\sim}{F} = \mathrm{diag}(f_1,\ldots,f_p)$$

and $\underset{\sim}{\Lambda}$ is a diagonal matrix whose diagonal elements are the characteristic roots of $(\underset{\sim}{\Sigma_1}\underset{\sim}{\Sigma_2^{-1}})$. Transforming

(2.3) $$w_i = f_i/(1 + f_i)$$

we find the density of $0 < w_1 < \ldots < w_p$ is

(2.4) $$C(p)|\underset{\sim}{\Lambda}|^{-n_1} \, {}_1\tilde{F}_0(n;\underset{\sim}{I_p}-\underset{\sim}{\Lambda}^{-1},\underset{\sim}{W})|\underset{\sim}{W}|^{n_1-p}|\underset{\sim}{I_p}-\underset{\sim}{W}|^{n_2-p} \prod_{i>j} (w_i-w_j)^2$$

where

$$\underset{\sim}{W} = \mathrm{diag}\,(w_1,w_2,\ldots,w_p) \quad .$$

To find $E[W^{(p)}]^h$ where $W^{(p)} = \prod_{i=1}^{p} (1-w_i)$ we multiply (2.4) by $|\underset{\sim}{I_p}-\underset{\sim}{W}|^h$ and transform $\underset{\sim}{T} \rightarrow \underset{\sim}{U}\,\underset{\sim}{W}\,\underset{\sim}{\bar{U}'}$ where $\underset{\sim}{U}$ is unitary, i.e. $\underset{\sim}{U}\,\underset{\sim}{\bar{U}'} = \underset{\sim}{I}$, and $\underset{\sim}{T}$ is Hermitian p.d. . Using the Jacobian of transformation given by Khatri [16]

(2.5) $$J(\underset{\sim}{T};\underset{\sim}{U},\underset{\sim}{W}) = \prod_{i>j} (w_i-w_j)^2 \, h_2(\underset{\sim}{U})$$

and integrating out $\underset{\sim}{U}$ and $\underset{\sim}{W}$ using

(2.6) $$\int_{\underset{\sim}{U}\,\underset{\sim}{\bar{U}'} = \underset{\sim}{I}} h_2(\underset{\sim}{U}) = \frac{\pi^{p(p-1)}}{\tilde{\Gamma}_p(p)}$$

and

$$(2.7)\qquad \int_{\underset{\sim}{\bar{S}}=\underset{\sim}{S}>0} |\underset{\sim}{S}|^{q-p}|\underset{\sim}{I}_p-\underset{\sim}{S}|^{n+h-q-p}\tilde{C}_\kappa(\underset{\sim}{S})d\underset{\sim}{S} = \frac{\tilde{\Gamma}_p(q,\kappa)\tilde{\Gamma}_p(n+h-q)\tilde{C}_\kappa(\underset{\sim}{I}_p)}{\tilde{\Gamma}_p(n+h,\kappa)}$$

we get after simplifying

$$(2.8)\qquad E[W^{(p)}]^h = |\underset{\sim}{\Lambda}|^{-n_1}\frac{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(n_2+h)}{\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(n+h)}\ {}_2F_1(n,n_1;n+h;\underset{\sim}{I}_p-\underset{\sim}{\Lambda}^{-1}) .$$

Before finding the density of $W^{(p)}$, below are stated some needed results on Mellins transforms [7], [8], [9], and Meijer's G-function [25], [26].

If s is any complex variate and $f(x)$ is a function of a real variable x, such that

$$(2.9)\qquad F(s) = \int_0^\infty x^{s-1}\ f(x)\ dx$$

exists, then under certain regularity conditions

$$(2.10)\qquad f(x) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} x^{-s}\ F(s)\ ds \quad .$$

$F(s)$ is called the Mellin transform of $f(x)$ and $f(x)$ is the inverse Mellin transform of $F(s)$. Meijer [25], [26] defined the G-function by

$$(2.11)\qquad G^{m,n}_{p,q}\left(x\middle|{a_1,\ldots,a_p \atop b_1,\ldots,b_q}\right) = (2\pi i)^{-1}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, x^s ds$$

where C is a curve separating the singularities of $\prod_{j=1}^{m} \Gamma(b_j - s)$ from those of $\prod_{j=1}^{n} \Gamma(1-a_j+s)$, $q \geq 1$, $0 \leq n \leq p \leq q$, $0 \leq m \leq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$. Using (2.9) and (2.10) we see from (2.8) that the density of $f(W^{(p)})$ has the form

$$f(W^{(p)}) = C_p \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa [n_1]_\kappa}{k!} \tilde{C}_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Lambda}^{-1}) \{W^{(p)}\}^{n_2 - p} \tag{2.12}$$

$$\cdot (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \{W^{(p)}\}^{-r} \frac{\prod_{i=1}^{p} \Gamma(r+b_i)}{\prod_{i=1}^{p} \Gamma(r+a_i)} dr$$

where

$$C_p = \frac{\tilde{\Gamma}_p(n)}{\tilde{\Gamma}_p(n_2)} |\underset{\sim}{\Lambda}|^{-n_1}, \quad b_i = i-1, \quad a_i = n_1 + k_{p-i+1} + b_i \quad . \tag{2.13}$$

Noting that the integral in (2.12) is in the form of Meijers G-function we can write the density of $W^{(p)}$ as

$$f(W^{(p)}) = C_p \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa [n_1]_\kappa}{k!} \tilde{C}_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Lambda}^{-1}) \{W^{(p)}\}^{n_2 - p} \tag{2.14}$$

$$\cdot G^{p,0}_{p,p}\left(W^{(p)} \Big| {a_1, \ldots, a_p \atop b_1, \ldots, b_p}\right) \quad .$$

Using the fact that

$$(2.15) \qquad G_{2,2}^{2,0}\left(x\middle|\begin{matrix}a_1,a_2\\ b_1,b_2\end{matrix}\right) = \frac{x^{b_1}(1-x)^{a_1+a_2-b_1-b_2-1}}{\Gamma(a_1+a_2-b_1-b_2)}$$

$$\cdot\ {}_2F_1(a_2-b_2,a_1-b_2;\ a_1+a_2-b_1-b_2;1-x) \quad 0 < x < 1$$

we find the density of $W^{(2)}$ to be

$$(2.16) \qquad f(W^{(2)}) = C_2 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n]_\kappa [n_1]_\kappa}{k!} \tilde{C}_\kappa(\underset{\sim}{I}_2-\Delta^{-1})\{W^{(2)}\}^{n_2-2}$$

$$\cdot\ \frac{\{1-W^{(2)}\}^{2n_1+k-1}}{\Gamma(2n_1+k)}\ {}_2F_1(n_1+k_1,n_1+k_2-1;2n_1+k;1-W^{(2)})$$

where $\kappa = (k_1,k_2)$. Using the results of Consul [9] for $p = 3$ and Al-Ani [1] for $p = 4$ we could also write out the densities of $W^{(3)}$ and $W^{(4)}$.

MANOVA Model

Suppose $\underset{\sim}{X}:p \times m \sim N_c(\underset{\sim}{\mu},\underset{\sim}{\Sigma})$ and $\underset{\sim}{Y}:p \times n \sim N_c(\underset{\sim}{0},\underset{\sim}{\Sigma})$ are independent with $m \geq p$. Then the joint density of the characteristic roots $0 < f_1 < \ldots < f_p$ of $(\underset{\sim}{X}\,\underset{\sim}{\bar{X}}')(\underset{\sim}{Y}\,\underset{\sim}{\bar{Y}}')^{-1}$ is given by Khatri [19] as

$$(2.15) \qquad C'(p)e^{-\mathrm{tr}\underset{\sim}{\Omega}}\ {}_1\tilde{F}_1(m+n;m;\underset{\sim}{\Omega},(\underset{\sim}{I}_p+\underset{\sim}{F}^{-1})^{-1})\frac{|\underset{\sim}{F}|^{m-p}}{|\underset{\sim}{I}_p+\underset{\sim}{F}|^{m+n}} \prod_{i>j}(f_i-f_j)^2$$

where

$$C'(p) = \frac{\tilde{\Gamma}_p(m+n)\pi^{p(p-1)}}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(p)} \quad , \quad \underset{\sim}{F} = \text{diag}\,(f_1,\ldots,f_p)$$

and $\underset{\sim}{\Omega} = \text{diag}(\omega_1,\ldots,\omega_p)$ where ω_i are the characteristic roots of $\bar{\underset{\sim}{\mu}}'\, \underset{\sim}{\Sigma}^{-1}\, \underset{\sim}{\mu}$. Now proceeding as in the previous case we obtain $E[W^{(p)}]^h$, $W^{(p)} = \prod_{i=1}^{p} (1-w_i)$ where

$$w_i = f_i/(1+f_i)$$

$$(2.16) \qquad E[W^{(p)}]^h = e^{-tr\underset{\sim}{\Omega}} \frac{\tilde{\Gamma}_p(m+n)\tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(m+n+h)}\ {}_1\tilde{F}_1(m+n;m+n+h;\underset{\sim}{\Omega}) \quad .$$

Using Mellin's transform and Meijer's G-function as in the previous case we get the density of $W^{(p)}$ as

$$(2.17) \qquad f(W^{(p)}) = e^{-tr\underset{\sim}{\Omega}} \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[m+n]_\kappa \tilde{C}_\kappa(\underset{\sim}{\Omega})}{k!} \{W^{(p)}\}^{n-p}$$

$$\cdot\ G^{p,0}_{p,p}\left(W^{(p)} \middle| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_p \end{matrix}\right)$$

where

$$a_i = m+k_{p-i+1}+b_i , \quad b_i = i-1 \quad .$$

As in the covariance model case, we could also obtain the density explicitly for $p = 2,3,4$.

Canonical Correlation

Let

$$(2.18)\qquad \begin{bmatrix} \underset{\sim}{X}:\ p \times n \\ \underset{\sim}{Y}:\ q \times n \end{bmatrix} \sim N_c \left[\underset{\sim}{0},\ \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}'_{12} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} \right]$$

$n \geq p+q$ and $q \geq p$. Then the joint density of the characteristic roots $0 < r_1^2 < \ldots < r_p^2$ of $(\underset{\sim}{X}\,\underset{\sim}{\bar{Y}}')(\underset{\sim}{Y}\,\underset{\sim}{\bar{Y}}')^{-1}(\underset{\sim}{Y}\,\underset{\sim}{\bar{X}}')(\underset{\sim}{X}\,\underset{\sim}{\bar{X}}')^{-1}$ is given by Khatri [19] as

$$(2.19)\qquad C''(p)|\underset{\sim}{I}_p-\underset{\sim}{P}^2|^n\,{}_2\tilde{F}_1(n,n;q;\underset{\sim}{P}^2,\underset{\sim}{R}^2)|\underset{\sim}{R}^2|^{q-p}|\underset{\sim}{I}_p-\underset{\sim}{R}^2|^{n-q-p}\prod_{i>j}(r_i^2-r_j^2)^2$$

where

$$(2.20)\qquad C''(p) = \frac{\tilde{\Gamma}_p(n)\pi^{p(p-1)}}{\tilde{\tilde{\Gamma}}_p(n-q)\tilde{\Gamma}_p(q)\tilde{\Gamma}_p(p)}\ ,\quad \underset{\sim}{R}^2 = \text{diag}\,(r_1^2,\ldots,\ r_p^2)$$

and $\underset{\sim}{P}^2 = \text{diag}\,(\rho_1,\ldots,\rho_p)$ where ρ_i are the characteristic roots of $\underset{\sim}{\Sigma}_{12}\,\underset{\sim}{\Sigma}_{22}^{-1}\,\underset{\sim}{\Sigma}_{21}\,\underset{\sim}{\Sigma}_{11}^{-1}$. Proceeding as in the previous cases we find $E[W^{(p)}]^h$, $W^{(p)} = \prod_{i=1}^{p}(1-r_i^2)$, by substituting in (2.8) as follows

$$(2.21)\qquad (n_1,n_2,\underset{\sim}{\Lambda}) \rightarrow (n,n-q,\ (\underset{\sim}{I}_p-\underset{\sim}{P}^2)^{-1})\quad .$$

Further the density of $W^{(p)}$ is obtained from (2.14) by making the above substitution and letting

$$a_i = q+k_{p-i+1}+b_i, \quad b_i = i-1 \quad .$$

As in the other cases the densities could be written out explicitly for $p = 2,3,4$.

3. The Density Function of Pillai's V-Statistic in the Central Case For Two Roots

If $\underset{\sim}{P}^2 = \underset{\sim}{0}$ in (2.19) we have the density function of the characteristic roots $r_1^2, r_2^2, \ldots, r_p^2$ in the central case. Letting $p = 2$ we have

$$(3.1) \qquad f_1(r_1^2, r_2^2) = C''(2)|\underset{\sim}{R}^2|^{q-2}|\underset{\sim}{I}_p-\underset{\sim}{R}^2|^{n-q-2}(r_1^2-r_2^2)^2 \quad .$$

Let $V = r_1^2 + r_2^2$ and $G = r_1^2 r_2^2$, $0 < V < 1$. To find the density function of V we make the above transformation and find

$$(3.2) \qquad f_2(V,G) = C''(2)G^{q-2}(1-V+G)^{n-q-2}(V^2-4G)^{\frac{1}{2}} \quad .$$

Integrating G between the limits 0 to $V^2/4$, [31] and writing $(1-V+G)^{n-q-2}$ as a finite series we have

$$(3.3) \qquad f(V) = C''(2) \sum_{r=0}^{n-q-2} \binom{n-q-2}{r}(1-V)^{n-q-r-2}$$

$$\cdot \left\{ \int_0^{V^2/4} G^{q+r-2}(V^2-4G)^{\frac{1}{2}} dG \right\} \quad .$$

Integrating the expression in the brackets by parts we find the density function of V to be

$$(3.4) \qquad f_3(V) = C''(2) \sum_{r=0}^{n-q-2} \binom{n-q-2}{r} (1-V)^{n-q-r-2}$$

$$\cdot \frac{(q+r-2)!V^{2(q+r)-1}}{2^{q+r-1}3\cdot 5 \ldots (2(q+r)-1)} \;, \quad 0 < V < 1 \;.$$

To obtain the density function of V in the range $1 \leq V \leq 2$ we change $r_i^2 \to 1-r_i^2$ in (3.1) and transform as before to get

$$(3.5) \qquad f_4(V,G) = C''(2)(1-V+G)^{q-2}G^{n-q-2}(V^2-4G)^{\frac{1}{2}} \;.$$

Writing $(1-V+G)^{q-2}$ as a series and integrating G between the limits 0 to $V^2/4$ we have

$$(3.6) \qquad f_5(V) = C''(2) \sum_{r=0}^{q-2} \binom{q-2}{r} (1-V)^{q-r-2}$$

$$\cdot \int_0^{V^2/4} G^{n+r-q-2} \, (V^2-4G)^{\frac{1}{2}} \, dG \;.$$

Evaluating the integral by parts yields

$$(3.7) \qquad f_5(V) = C''(2) \sum_{r=0}^{q-2} \binom{q-2}{r} (1-V)^{q-r-2} \frac{(n+r-q-2)!V^{2(n+r-q)-1}}{2^{n+r-q-1} \; 3\cdot 5 \ldots 2(n+r-q)-1} \;.$$

Transforming $V' = 2-V$, $1 \leq V \leq 2$ we find

$$(3.8) \qquad f_6(V) = C''(2) \sum_{r=0}^{q-2} \binom{q-2}{r} (V-1)^{q-r-2} \frac{(n+r-q-2)!(2-V)^{2(n+r-q)-1}}{2^{n+r-q-1}3\cdot 5 \ldots (2(n+r-q)-1)} \;,$$

$$1 \leq V \leq 2 \;.$$

By making the following changes in the parameters in (3.1)

$$(q,n-q,r_i^2) \rightarrow (m,n,w_i)$$

or

$$(q,n-q,r_i^2) \rightarrow (n_1,n_2,w_i)$$

we obtain the central density of the characteristic roots in the MANOVA or equality of two matrices cases, respectively. Thus the results of this section and the next aren't restricted to the canonical correlation case, but extend to the two cases mentioned above as well.

4. The Density Function of the U-Statistic in the Central Case For Two Roots

To obtain the density function of U we make the transformation in (3.1)

$$r_i^2 = \lambda_i(1+\lambda_i)^{-1}$$

and find

$$(4.1) \qquad g_1(\lambda_1,\lambda_2) = C''(2)|\underset{\sim}{Q}|^{q-2}|\underset{\sim}{I}_p+\underset{\sim}{Q}|^{-n}(\lambda_1-\lambda_2)^2$$

where $\underset{\sim}{Q} = \text{diag}(\lambda_1,\lambda_2)$. Letting $U = \lambda_1+\lambda_2$ and $G = \lambda_1\lambda_2$ we see the joint density of U and G can be put in the form

$$(4.2) \qquad g_2(U,G) = C''(2)\, G^{q-2}(1+\frac{U}{2})^{-2n}\,(U^2-4G)^{\frac{1}{2}}$$

$$\cdot \left[1 - \frac{U^2-4G}{4(1+\frac{U}{2})^2}\right]^{-n} \quad .$$

Writing the part in brackets as a series and integrating G between the limits 0 to $U^2/4$ yields

$$(4.3) \qquad g_3(U) = C''(2)(1+\frac{U}{2})^{-2n} \sum_{r=0}^{\infty} \frac{(-1)^r \binom{-n}{r}}{4^r (1+\frac{U}{2})^{2r}} \cdot \left\{ \int_0^{U^2/4} (U^2-4G)^{r+\frac{1}{2}} G^{q-2} \, dG \right\} .$$

Integrating the expression in the brackets by parts, we find the density of U for $p = 2$ is

$$(4.4) \qquad g_3(U) = C''(2) \sum_{r=0}^{\infty} (-1)^r \frac{\binom{-n}{r}(q-2)! \, U^{2r+2(q-3)+5}}{4^{r+q-1}(1+\frac{U}{2})^{2r+2n}} \cdot \frac{1}{(r+\frac{3}{2})(r+\frac{5}{2}) \dots (r+\frac{2(q-3)+5}{2})} .$$

5. Complex Multivariate Beta Distribution and Independent Beta Variables

If $\underset{\sim}{X}:p \times m$ and $\underset{\sim}{Y}:p \times n$ are independent complex matrix variates $m \geq p$, whose columns are independent complex p-variate with covariance matrix $\underset{\sim}{\Sigma}$, and if $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$ and $E(\underset{\sim}{Y}) = \underset{\sim}{0}$, then the distribution of

$$(5.1) \qquad \underset{\sim}{F} = \underset{\sim}{\bar{X}}'(\underset{\sim}{Y}\,\underset{\sim}{\bar{Y}}')^{-1} \underset{\sim}{X}$$

depends on parameters

$$(5.2) \qquad \underset{\sim}{\Omega} = \underset{\sim}{\bar{\mu}}' \underset{\sim}{\Sigma}^{-1} \underset{\sim}{\mu}$$

and is [13]

$$(5.3)\qquad f(\underset{\sim}{F}) = k_1 e^{-\mathrm{tr}\underset{\sim}{\Omega}}\ {}_1\tilde{F}_1(m+n;m;\underset{\sim}{\Omega}(\underset{\sim}{I}_p+\underset{\sim}{F}^{-1})^{-1})|\underset{\sim}{F}|^{m-p}|\underset{\sim}{I}_p+\underset{\sim}{F}|^{-(m+n)}(d\underset{\sim}{F})$$

where

$$(5.4)\qquad k_1 = \frac{\tilde{\Gamma}_p(m+n)}{\tilde{\Gamma}_p(m)\tilde{\Gamma}_p(n)} \quad .$$

Since the density of $\underset{\sim}{F}$ for $p \geq m$ can be obtained from (5.3) by making the changes

$$(5.5)\qquad (p,m,n) \rightarrow (m,p,m+n-p)$$

it suffices to work only with (5.3). Making the transformation

$$(5.6)\qquad \underset{\sim}{L} = (\underset{\sim}{I}_p + \underset{\sim}{F}^{-1})^{-1}$$

in (5.3) and noting $J(\underset{\sim}{L};\underset{\sim}{F}) = |\underset{\sim}{I}_p-\underset{\sim}{L}|^{-2p}$ [16] we have,

$$(5.7)\qquad f(\underset{\sim}{L}) = k_1 e^{-\lambda^2}\ {}_1\tilde{F}_1(m+n;m;\lambda^2\underset{\sim}{L})|\underset{\sim}{L}|^{m-p}|\underset{\sim}{I}_p-\underset{\sim}{L}|^{n-p}(d\underset{\sim}{L}) \quad .$$

Proceeding in a manner similar to Khatri and Pillai [21] let

$$(5.8)\qquad \underset{\sim}{L} = \begin{array}{cc} \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \underset{\sim}{\ell} & \underset{\sim}{L}_{11} \end{pmatrix} & \begin{array}{c} 1 \\ p-1 \end{array} \\ \begin{array}{cc} 1 & \quad p-1 \end{array} & \end{array} \ , \quad \underset{\sim}{L}_{22} = \underset{\sim}{L}_{11} - \underset{\sim}{\ell}\,\underset{\sim}{\ell}'/\ell_{11}$$

and note that $|\underset{\sim}{L}| = \ell_{11}|\underset{\sim}{L}_{22}|$ and

$$(5.9) \qquad |\underset{\sim}{I}_p - \underset{\sim}{L}| = (1-\ell_{11})|\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22} - \underset{\sim}{\ell}\,\underset{\sim}{\ell}'/[\ell_{11}(1-\ell_{11})]| \quad .$$

Now it can be shown that ℓ_{11} and $\{\underset{\sim}{L}_{22}, \underset{\sim}{v} = \underset{\sim}{\ell}/[\ell_{11}(1-\ell_{11})^{\frac{1}{2}}]\}$ are independently distributed and their respective distributions are

$$(5.10) \qquad f_1(\ell_{11}) = [\beta(m,n)]^{-1} e^{-\lambda^2} \, {}_1\tilde{F}_1(m+n;m;\lambda^2 \ell_{11}) \ell_{11}^{m-1}(1-\ell_{11})^{n-1}$$

and

$$(5.11) \qquad f_2(\underset{\sim}{L}_{22}, \underset{\sim}{v}) = k_2 |\underset{\sim}{L}_{22}|^{m-p} |\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22} - \bar{\underset{\sim}{v}}' \underset{\sim}{v}|^{n-p} \quad ,$$

where

$$(5.12) \qquad k_2 = k_1 \, \beta(m,n) \quad .$$

For further independence, we can use the transformation

$$\underset{\sim}{u} = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-\frac{1}{2}} \underset{\sim}{v} \quad .$$

With Jacobian of transformation $|\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22}|^{-1}$ it can be shown that $\underset{\sim}{u}$ and $\underset{\sim}{L}_{22}$ are independently distributed and their respective distributions are

$$(5.13) \qquad f_3(\underset{\sim}{u}) = \pi^{-(p-1)}[\Gamma(n)/\Gamma(n-p+1)](1-\bar{\underset{\sim}{u}}' \underset{\sim}{u})^{n-p}$$

and

$$(5.14) \qquad f_4(\underset{\sim}{L}_{22}) = k_3 |\underset{\sim}{L}_{22}|^{m-(p-1)-1} |\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22}|^{n+1-(p-1)-1}$$

where

$$k_3 = \pi^{(p-1)}[\Gamma(n-p+1)/\Gamma(n)]k_2 \quad .$$

Notice that $\underset{\sim}{L}_{22}$: (p-1) x (p-1) is the central complex multivariate beta distribution with m and n+1 degrees of freedom. Making the transformation

$$(5.15) \qquad x_i = u_i/(1-\bar{u}_1u_1- \ldots - \bar{u}_{p-1}u_{p-1}), \; i = 1,2,\ldots p-1, \; u_0 = 0$$

in (5.13) with Jacobian of transformation $\prod_{i=1}^{p-1} (1-\bar{x}_ix_i)^{p-i-1}$, we obtain the density of $\underset{\sim}{X} = (x_1,x_2,\ldots,x_{p-1})'$ as

$$(5.16) \qquad f(\underset{\sim}{X}) = \pi^{-(p-1)} \prod_{i=1}^{p-1} \frac{\Gamma(n-i+1)}{\Gamma(n-i)} (1-\bar{x}_ix_i)^{n-i-1} \quad .$$

After making the transformation of $x_j = a_j + ib_j$ to polar coordinates (r_j, θ_j), we find with $\underset{\sim}{r} = (r_1,\ldots,r_{p-1})'$

$$(5.17) \qquad f(\underset{\sim}{r}) = \prod_{i=1}^{p-1} \frac{\Gamma(n-i+1)}{\Gamma(n-i)} (1-r_i^2) \; 2r_i \; dr_i \quad .$$

Finally the transformation $w_i = r_i^2$ yields independent real beta variates and their respective densities are given by

$$(5.18) \qquad f_i(w_i) = [\beta(1,n-i)]^{-1} (1-w_i)^{n-i-1} \quad .$$

6. Complex Zonal Polynomials

The zonal polynomials of a Hermitian matrix $\underset{\sim}{A}$ [13], are given by

$$(6.1) \qquad \tilde{C}_\kappa(\underset{\sim}{A}) = \chi_{[\kappa]}(1)\, \chi_{\{\kappa\}}(\underset{\sim}{A}) \quad ,$$

where $\kappa = (k_1, k_2, \ldots, k_m)$ is a partition of the integer k and $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group and is given by

$$(6.2) \qquad \chi_{[\kappa]}(1) = k! \prod_{i<j}^{m} (k_i - k_j - i + j) \Big/ \prod_{i=1}^{m} (k_i + m - i)! \quad ,$$

$\chi_{\{\kappa\}}(\underset{\sim}{A})$ is the character of the representation $\{\kappa\}$ of the linear group and is given as a symmetric function of the characteristic roots $e_1, e_2, \ldots, e_m$ of $\underset{\sim}{A}$ by

$$(6.3) \qquad \chi_{\{\kappa\}}(\underset{\sim}{A}) = |(e_i^{k_j+m-j})| / |(e_i^{m-j})|$$

where the determinants are Vandermonde type. Further the following equality is satisfied

$$(6.4) \qquad \sum_{\kappa} \tilde{C}_\kappa(\underset{\sim}{A}) = (a_1)^k$$

where a_i is the ith esf of the e_i's. Using the following lemma obtained by Pillai [27] we can get the zonal polynomials as a linear combination of the esf's. Tables 11-14 give $\chi_{\{\kappa\}}(\underset{\sim}{A})$ and $\chi_{[k]}(1)$ through degree 8.

Lemma: Let $D(g_s, g_{s-1}, \ldots, g_1)$, $(g_j \geq 0,\ j = 1,2,\ldots,s)$, denote the determinant

$$(6.5)\qquad D(g_s, g_{s-1}, \ldots, g_1) = \begin{vmatrix} e_s^{g_s} & e_s^{g_{s-1}} & \cdots & e_s^{g_1} \\ \vdots & & & \\ e_1^{g_s} & e_1^{g_{s-1}} & & e_1^{g_1} \end{vmatrix} .$$

If $a_r (r \leq s)$ denotes the rth esf in s e's, then

$$(6.6)\qquad \text{i)}\quad a_r D(g_s, g_{s-1}, \ldots, g_1) = \Sigma' \, D(g'_s, g'_{s-1}, \ldots, g'_1)$$

where $g'_j = g_j + \delta$, $j = 1,2,\ldots,s$, $\delta = 0,1$ and Σ' denotes the sum over the $\binom{s}{r}$ combinations of s g's taken r at a time for which r indices $g'_j = g_j + 1$ such that $\delta = 1$ while for other indices $g'_j = g_j$ such that $\delta = 0$.

ii) $(a_r)^k (a_h)^\ell \, D(g_s, g_{s-1}, \ldots, g_1)$, $k, \ell \geq 0$, can be expressed as a sum of $\binom{s}{r}^k \binom{s}{h}^\ell$ determinants obtained by performing on $D(g_s, g_{s-1}, \ldots, g_1)$ in any order (i) k times and (i) ℓ times with $r=h$. However if at least two of the indices in any determinant are equal, the corresponding term in the summation vanishes.

An example will suffice to show how $\chi_{\{\kappa\}}(\underset{\sim}{A})$ for any degree can be obtained from those of lower degree. Here we obtain $\chi_{\{\kappa\}}(\underset{\sim}{A})$ for $k=3$. Let

$$(6.7)\qquad D = |(e_i^{m-j})|$$

and

(6.8) $$D(k_1+m-1,k_2+m-2,\ldots,1) = |(e_i^{k_j+m-j})| \quad .$$

When $k = 2$ we have

(6.9) $$(a_1^2-a_2)D = D(m+1,m-2,m-3,\ldots,1) \quad \text{for} \quad \kappa = (2)$$

and

(6.10) $$a_2D = D(m,m-1,m-3,\ldots,1) \quad \text{for} \quad \kappa = (1^2) \quad .$$

Multiplying (6.9) and (6.10) by a_1, using Pillai's lemma, gives

(6.11) $$(a_1^3-a_1a_2)D = D(m+2,m-2,\ldots,1) + D(m+1,m-1,m-3,\ldots,1)$$

and

(6.12) $$a_1a_2D = D(m+1,m-1,m-3,\ldots,1) + D(m,m-1,m-2,m-4,\ldots,1).$$

But since

(6.13) $$a_3D = D(m,m-1,m-2,m-4,\ldots,1)$$

we have substituting in (6.12)

(6.14) $$(a_1a_2 - a_3)D = D(m+1,m-1,m-3,\ldots,1) \quad .$$

When $\kappa = (1^3)$ and $\kappa = (21)$ in (6.8), we obtain (6.13) and (6.14) respectively. Thus

$$\chi_{\{1^3\}}(\underset{\sim}{A}) = a_3 \quad \text{and} \quad \chi_{\{21\}}(\underset{\sim}{A}) = a_1a_2-a_3 \quad .$$

Substituting (6.14) in (6.11) we find

$$(a_1^3-2a_1a_2+a_3)D = D(m+2,m-2,\ldots,1) \tag{6.15}$$

and thus

$$\chi_{\{3\}}(\underset{\sim}{A}) = a_1^3-2a_1a_2+a_3 \quad .$$

Table 11. Complex Zonal Polynomials of 1st - 5th Degree

<table>
<tr><th></th><th colspan="7">In terms of elementary symmetric functions of the Latent Roots of $\underset{\sim}{A}$</th><th>$\chi_{[\kappa]}(1)$</th></tr>
<tr><td></td><td colspan="7">1st Degree</td><td></td></tr>
<tr><td>$\chi_{\{1\}}$</td><td colspan="7">a_1</td><td>1</td></tr>
<tr><td></td><td colspan="7">2nd Degree</td><td></td></tr>
<tr><td>$\chi_{\{2\}}$</td><td colspan="7">$a_1^2 - a_2$</td><td>1</td></tr>
<tr><td>$\chi_{\{1^2\}}$</td><td colspan="7">a_2</td><td>1</td></tr>
<tr><td></td><td colspan="7">3rd Degree</td><td></td></tr>
<tr><td>$\chi_{\{3\}}$</td><td colspan="7">$a_1^3 - 2a_1a_2 + a_3$</td><td>1</td></tr>
<tr><td>$\chi_{\{21\}}$</td><td colspan="7">$a_1a_2 - a_3$</td><td>2</td></tr>
<tr><td>$\chi_{\{1^3\}}$</td><td colspan="7">a_3</td><td>1</td></tr>
<tr><td></td><td colspan="7">4th Degree</td><td></td></tr>
<tr><td></td><td>a_1^4</td><td>$a_1^2a_2$</td><td>a_2^2</td><td>a_1a_3</td><td>a_4</td><td></td><td></td><td></td></tr>
<tr><td>$\chi_{\{4\}}$</td><td>1</td><td>-3</td><td>1</td><td>2</td><td>-1</td><td></td><td></td><td>1</td></tr>
<tr><td>$\chi_{\{31\}}$</td><td></td><td>1</td><td>-1</td><td>-1</td><td>1</td><td></td><td></td><td>3</td></tr>
<tr><td>$\chi_{\{2^2\}}$</td><td></td><td></td><td>1</td><td>-1</td><td>0</td><td></td><td></td><td>2</td></tr>
<tr><td>$\chi_{\{21^2\}}$</td><td></td><td></td><td></td><td>1</td><td>-1</td><td></td><td></td><td>3</td></tr>
<tr><td>$\chi_{\{1^4\}}$</td><td></td><td></td><td></td><td></td><td>1</td><td></td><td></td><td>1</td></tr>
<tr><td></td><td colspan="7">5th Degree</td><td></td></tr>
<tr><td></td><td>a_1^5</td><td>$a_1^3a_2$</td><td>$a_1a_2^2$</td><td>$a_1^2a_3$</td><td>a_2a_3</td><td>a_1a_4</td><td>a_5</td><td></td></tr>
<tr><td>$\chi_{\{5\}}$</td><td>1</td><td>-4</td><td>3</td><td>3</td><td>-2</td><td>-2</td><td>1</td><td>1</td></tr>
<tr><td>$\chi_{\{41\}}$</td><td></td><td>1</td><td>-2</td><td>-1</td><td>2</td><td>1</td><td>-1</td><td>4</td></tr>
<tr><td>$\chi_{\{32\}}$</td><td></td><td></td><td>1</td><td>-1</td><td>-1</td><td>1</td><td>0</td><td>5</td></tr>
<tr><td>$\chi_{\{31^2\}}$</td><td></td><td></td><td></td><td>1</td><td>-1</td><td>-1</td><td>1</td><td>6</td></tr>
<tr><td>$\chi_{\{2^21\}}$</td><td></td><td></td><td></td><td></td><td>1</td><td>-1</td><td>0</td><td>5</td></tr>
<tr><td>$\chi_{\{21^3\}}$</td><td></td><td></td><td></td><td></td><td></td><td>1</td><td>-1</td><td>4</td></tr>
<tr><td>$\chi_{\{1^5\}}$</td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td>1</td></tr>
</table>

Table 12. Complex Zonal Polynomials for 6th Degree

	a_1^6	$a_1^4a_2$	$a_1^2a_2^2$	$a_1^3a_3$	a_2^3	$a_1a_2a_3$	$a_1^2a_4$	a_3^2	a_2a_4	a_1a_5	a_6	$\chi_{[\kappa]}(1)$
$\chi_{\{6\}}$	1	-5	6	4	-1	-6	-3	1	2	2	-1	1
$\chi_{\{51\}}$		1	-3	-1	1	4	1	-1	-2	-1	1	5
$\chi_{\{42\}}$			1	-1	-1	0	1	0	1	-1	0	9
$\chi_{\{41^2\}}$				1	0	-2	-1	1	1	1	-1	10
$\chi_{\{3^2\}}$					1	-2	1	1	-1	0	0	5
$\chi_{\{321\}}$						1	-1	-1	0	1	0	16
$\chi_{\{31^3\}}$							1	0	-1	-1	1	10
$\chi_{\{2^3\}}$								1	-1	0	0	5
$\chi_{\{2^21^2\}}$									1	-1	0	9
$\chi_{\{21^4\}}$										1	-1	5
$\chi_{\{1^6\}}$											1	1

Table 13. Complex Zonal Polynomials for 7th Degree

	a_1^7	$a_1^5a_2$	$a_1^3a_2^2$	$a_1^4a_3$	$a_1a_2^3$	$a_1^2a_2a_3$	$a_1^3a_4$	$a_2^2a_3$	$a_1a_3^2$	$a_1a_2a_4$	$a_1^2a_5$	a_3a_4	a_2a_5	a_1a_6	a_7	$X_{[\kappa]}(1)$
$X_{\{7\}}$	1	-6	10	5	-4	-12	-4	3	3	6	3	-2	-2	-2	1	1
$X_{\{61\}}$		1	-4	-1	3	6	1	-3	-2	-4	-1	2	2	1	-1	6
$X_{\{52\}}$			1	-1	-2	1	1	2	-1	0	-1	0	-1	1	0	14
$X_{\{51^2\}}$				1	0	-3	-1	1	2	2	1	-2	-1	-1	1	15
$X_{\{43\}}$					1	-2	1	-1	2	0	-1	-1	1	0	0	14
$X_{\{421\}}$						1	-1	-1	-1	1	1	1	0	-1	0	35
$X_{\{41^3\}}$							1	0	0	-2	-1	1	1	1	-1	20
$X_{\{3^21\}}$								1	-1	-1	1	1	-1	0	0	21
$X_{\{32^2\}}$									1	-1	0	-1	1	0	0	21
$X_{\{321^2\}}$										1	-1	-1	0	1	0	35
$X_{\{31^4\}}$											1	0	-1	-1	1	15
$X_{\{2^31\}}$												1	-1	0	0	14
$X_{\{2^21^3\}}$													1	-1	0	14
$X_{\{21^5\}}$														1	-1	6
$X_{\{1^7\}}$															1	1

Table 14. Complex Zonal Polynomials for 8th Degree

	a_1^8	$a_1^6a_2$	$a_1^4a_2^2$	$a_1^5a_3$	$a_1^2a_2^3$	$a_1^3a_2a_3$	$a_1^4a_4$	a_2^4	$a_1a_2^2a_3$	$a_1^2a_3^2$	$a_1^2a_2a_4$	$a_1^3a_5$
$\chi_{\{8\}}$	1	-7	15	6	-10	-20	-5	1	12	6	12	4
$\chi_{\{71\}}$		1	-5	-1	6	8	1	-1	-9	-3	-6	-1
$\chi_{\{62\}}$			1	-1	-3	2	1	1	3	-2	-1	-1
$\chi_{\{61^2\}}$				1	0	-4	-1	0	3	3	3	1
$\chi_{\{53\}}$					1	-2	1	-1	1	2	-1	-1
$\chi_{\{521\}}$						1	-1	0	-2	-1	2	1
$\chi_{\{51^3\}}$							1	0	0	0	-3	-1
$\chi_{\{4^2\}}$								1	-3	1	2	-1
$\chi_{\{431\}}$									1	-1	-1	1
$\chi_{\{42^2\}}$										1	-1	0
$\chi_{\{421^2\}}$											1	-1
$\chi_{\{41^4\}}$												1
$\chi_{\{3^22\}}$												
$\chi_{\{3^21^2\}}$												
$\chi_{\{32^21\}}$												
$\chi_{\{321^3\}}$												
$\chi_{\{31^5\}}$												
$\chi_{\{2^4\}}$												
$\chi_{\{2^31^2\}}$												
$\chi_{\{2^21^4\}}$												
$\chi_{\{21^6\}}$												
$\chi_{\{1^8\}}$												

Table 14. (Cont'd.)

	$a_2a_3^2$	$a_2^2a_4$	$a_1a_3a_4$	$a_1a_2a_5$	$a_1^2a_6$	a_4^2	a_3a_5	a_2a_6	a_1a_7	a_8	$\chi_{[\kappa]}(1)$
$\chi_{\{8\}}$	-3	-3	-6	-6	-3	1	2	2	2	-1	1
$\chi_{\{71\}}$	3	3	4	4	1	-1	-2	-2	-1	1	7
$\chi_{\{62\}}$	-1	-2	2	0	1	0	0	1	-1	0	20
$\chi_{\{61^2\}}$	-2	-1	-4	-2	-1	1	2	1	1	-1	21
$\chi_{\{53\}}$	-1	2	-2	0	1	0	1	-1	0	0	28
$\chi_{\{521\}}$	2	0	0	-1	-1	0	-1	0	1	0	64
$\chi_{\{51^3\}}$	0	1	2	2	1	-1	-1	-1	-1	1	35
$\chi_{\{4^2\}}$	2	-2	-2	2	0	1	-1	0	0	0	14
$\chi_{\{431\}}$	-1	0	3	-1	-1	-1	0	1	0	0	70
$\chi_{\{42^2\}}$	-1	1	-1	1	0	0	1	-1	0	0	56
$\chi_{\{421^2\}}$	0	-1	-1	1	1	1	0	0	-1	0	90
$\chi_{\{41^4\}}$	0	0	0	-2	-1	0	1	1	1	-1	35
$\chi_{\{3^22\}}$	1	-1	-1	1	0	1	-1	0	0	0	42
$\chi_{\{3^21^2\}}$		1	-1	-1	1	0	1	-1	0	0	56
$\chi_{\{32^21\}}$			1	-1	0	-1	0	1	0	0	70
$\chi_{\{321^3\}}$				1	-1	0	-1	0	1	0	64
$\chi_{\{31^5\}}$					1	0	0	-1	-1	1	21
$\chi_{\{2^4\}}$						1	-1	0	0	0	14
$\chi_{\{2^31^2\}}$							1	-1	0	0	28
$\chi_{\{2^21^4\}}$								1	-1	0	20
$\chi_{\{21^6\}}$									1	-1	7
$\chi_{\{1^8\}}$										1	1

CHAPTER V

AN APPROXIMATION TO THE DISTRIBUTION OF THE LARGEST ROOT OF A MATRIX AND PERCENTAGE POINTS

1. Introduction and Summary

Khatri [16] has pointed out that one can handle all the classical problems of point estimation and testing hypotheses concerning the parameters of complex multivariate normal populations much as one handles those for multivariate normal populations in real variates. Further, he suggested the maximum latent root statistic for testing the reality of a covariance matrix [17] . The joint distribution of the latent roots $w_1, w_2, \ldots, w_q$ under certain null hypotheses can be written as [15], [16]

$$C_1\left\{\prod_{j=1}^{q} w_j^m(1-w_j)^n\right\}\left\{\prod_{i>j}(w_i - w_j)^2\right\} \tag{1.1}$$

where

$$C_1 = \prod_{j=1}^{q} \Gamma(m+n+q+j)/\{\Gamma(n+j)\Gamma(m+j)\Gamma(j)\} \tag{1.2}$$

and

$$0 \leq w_1 \leq w_2 \leq \ldots \leq w_q \leq 1.$$

Khatri [15] has derived the distribution of w_q (or w_1) in a determinant form as follows

$$(1.3) \qquad \Pr\{w_q \leq x;m,n\} = C_1 \begin{vmatrix} \beta_o & \beta_1 & \cdots & \beta_{q-1} \\ \beta_1 & \beta_2 & \cdots & \beta_q \\ & & & \\ \cdot & \cdot & \cdots & \cdot \\ \beta_{q-1} & \beta_q & & \beta_{2q-2} \end{vmatrix} = |(\beta_{i+j-2})|$$

where C_1 is defined in (1.2),

$$(1.4) \qquad \beta_{i+j-2} = \int_o^x w^{m+i+j-2} (1-w)^n \, dw$$

for $i,j=1,2,\ldots,q$, and (β_{i+j-2}) is a qxq matrix. $\Pr\{w_1 \leq x;m,n\}$ can be obtained from (1.3) using

$$(1.5) \qquad \Pr\{w_1 \leq x;m,n\} = 1-\Pr\{w_q \leq 1-x;n,m\}.$$

In this paper an approximation to the cdf of w_q at the upper end is obtained and upper 1% and 5% points are given for q=2,3,4,5,6 in Tables 16-25. A general form of the approximation is given with the results for q=2 and 3 written out explicitly.

2. Case For Two Roots

It is easily seen from (1.3) that letting $q = 2$ and expanding the determinant

$$(2.1) \qquad \Pr\{w_2 \le x;m,n\} = C_1\left\{\int_0^x w^m(1-w)^n\,dw \int_0^x w^{m+2}(1-w)^n\,dw - \left(\int_0^x w^{m+1}(1-w)^n dw\right)^2\right\}.$$

This can be written as a finite sum for integral values of m and n integrating by parts and we obtain

$$(2.2) \qquad \Pr\{w_2 \le x;m,n\} = C_1\ \{\beta_o\beta_2 - \beta_1^2\}$$

where β_ℓ is defined in (1.4) or a form which is more convenient for our purposes is

$$(2.3) \qquad \beta_\ell = \beta(m+\ell+1,n+1) - \sum_{j=0}^{m+\ell} \frac{(m+\ell)!}{j!}\ \frac{x^j(1-x)^{m+n+\ell-j+1}}{\prod_{k=0}^{m+\ell-j}(n+k+1)}$$

where $\beta(a,b)$ is the usual beta function. For small values of m, the approximation is obtained by neglecting all terms involving $(1-x)^{2n}$ and higher powers. Expanding (2.2) and neglecting terms involving $(1-x)^{2n}$ and higher powers we have

$$(2.4) \qquad \Pr\{w_2 \le x;m,n\} \doteq C_1\left[\beta(m+1,n+1)\beta_2 - \beta(m+3,n+1)\sum_{j=0}^{m}\frac{m!}{j!}\ \frac{x^j(1-x)^{m+n-j+1}}{\prod_{k=0}^{m-j}(n+k+1)} - \beta(m+2,n+1)\beta_1 + \beta(m+2,n+1)\sum_{j=0}^{m+1}\frac{(m+1)!}{j!}\ \frac{x^j(1-x)^{m+n-j+2}}{\prod_{k=0}^{m-j+1}(n+k+1)}\right].$$

Adding and subtracting $\beta(m+3,n+1)\beta(m+1,n+1)$ and $\beta(m+2,n+1)^2$ in (2.4) we find

$$(2.5) \qquad \Pr\{w_2 \leq x;m,n\} \doteq C_1[\beta(m+1,n+1)\beta_2 + \beta(m+3,n+3)\beta_o - 2\beta(m+2,n+1)\beta_1] - 1.$$

Noting from (2.2) with $x=1$ that

$$(2.6) \qquad C_1 = [\beta(m+3,n+1)\beta(m+1,n+1) - \beta(m+2,n+1)^2]^{-1},$$

and simplifying in (2.5) we find

$$(2.7) \quad \Pr\{w_2 \leq x;m,n\} \doteq \frac{m+n+2}{\beta(m+2,n+2)}\left[\beta_2 + \frac{(m+1)_2}{(m+n+2)_2}\beta_o - \frac{2m}{m+n+2}\beta_1\right] - 1$$

where

$$(2.8) \qquad (a)_k = a(a+1) \ldots (a+k-1).$$

This approximation is very simple for computational use and no products of incomplete beta functions are involved. Upper 1% and 5% points using (2.7) are given in Tables 16,17. The error involved in using this approximation has been computed and the difference between the exact and approximate percentage points occurs in the seventh place. (See Table 15).

3. Case For Three Roots

When there are 3 roots, we have

$$\Pr\{w_3 \leq x;m,n\} = C_1\{\beta_o\beta_2\beta_4 + 2\beta_1\beta_2\beta_3 - \beta_2^3 - \beta_o\beta_3^2 - \beta_1^2\beta_4\} \tag{3.1}$$

where β_ℓ is defined in (2.3) and C_1 in (1.2). As in the two-roots case we write the β_k's as finite sums, expand, neglect terms involving $(1-x)^{2n}$ and higher powers and find

$$\begin{aligned}\Pr\{w_3 \leq x;m,n\} \doteq C_1\{&[\beta(3)\beta(5)-\beta(4)^2]\beta_o + 2[\beta(3)\beta(4) - \beta(2)\beta(5)]\beta_1 \\ &+[2\beta(2)\beta(4)+\beta(1)\beta(5)-3\beta(3)^2]\beta_2 + 2[\beta(2)\beta(3)-\beta(1)\beta(4)]\ \beta_3 \\ &+ [\beta(1)\beta(3)-\beta(2)^2]\ \beta_4\} - 2\end{aligned} \tag{3.2}$$

where $\beta(j) = \beta(m+j,n+1)$. Simplifying in (3.2) we find

$$\begin{aligned}\Pr\{w_3 \leq x;m,n\} \doteq [\beta(m+1,n+3)]^{-1}\Big\{&\frac{(m+2)_2}{2}\beta_o - 2(m+3)(m+n+4)\beta_1 \\ &+ \frac{3\{m(m+n+6)+2n+7\}(m+n+4)}{m+1}\beta_2 - \frac{2(m+n+3)_3}{m+1}\beta_3 + \frac{(m+n+4)(m+n+3)_3}{2(m+1)_2}\Big\} - 2.\end{aligned} \tag{3.3}$$

As in the two-roots case, no products of incomplete beta functions are involved. Upper 1% and 5% points using (3.3) are given in Tables 18 and 19. Comparison of some exact and approximate values is made in Table 15.

4. Case For q Roots

Denote the product $\beta_{i_1}\beta_{i_2}\cdots\beta_{i_q}$ occuring in $|(\beta_\ell)|$ by $\beta_{i_1i_2\cdots i_q}$ where i_j are any of the integers $0,1,\ldots,2q-2$. It can be seen from cases with $q=2$ and 3 that $\beta_{i_1i_2\cdots i_q}$ is approximated by $\beta'_{i_1i_2\cdots i_q}$

$$(4.1) \qquad \beta'_{i_1i_2\cdots i_q} = \sum_{j=1}^{q} \prod_{\substack{k=1\\k\neq j}}^{q} \beta(m+i_k+1,n+1)\beta_{i_j}$$

where β_{i_j} is defined in (1.4). Thus the distribution of w_q can be approximated by

$$(4.2) \qquad \Pr\{w_q \leq x;m,n\} \doteq C_1\,|(\beta_\ell)|' - (q-1)$$

where $|(\beta_\ell)|'$ is obtained by replacing $\beta_{i_1i_2\cdots i_q}$ in $|(\beta_\ell)|$ by $\beta'_{i_1i_2\cdots i_q}$. By collecting the coefficients of incomplete beta functions we get the following form which is simpler for computer computations

$$(4.3) \qquad \Pr\{w_q \leq x;m,n\} = C_1 \sum_{k=0}^{2q-2} D'_k\,\beta_k - (q-1),$$

where D'_k is the sum of the cofactors of $\beta(k+1)$ in the qxq matrix

$$\begin{vmatrix} \beta(1) & \beta(2) & \cdots & \beta(q) \\ \beta(2) & \beta(3) & \cdots & \beta(q+1) \\ \cdot & \cdot & \cdots & \cdot \\ \beta(q) & \beta(q+1) & \cdots & \beta(2q-1) \end{vmatrix}$$

where $\beta(j)$ is defined as $\beta(m+j,n+1)$, the usual beta function. Letting q=2 and 3 in (4.3) we get (2.5) and (3.2) respectively.

5. Computation of Percentage Points

Based on the results of the preceeding sections upper 1% and 5% points were computed for q=2,3,4,5,6 on the CDC 6500 computer. Results are given to five significant figures for the arguments m=0(1)5,7,10,15 and n=5(5) 30(10)40(20)120(40)200,300,500,1000. As can be seen from the comparisons below the percentage points from the exact and approximate cdf's agree through five figures and generally six.

Table 15. Comparison of Percentage Points From the Exact and Approximate CDF's

			1%		5%	
q	m	n	exact	approximate	exact	approximate
2	0	30	.246078	.246078	.194089	.194089
2	10	160	.142231	.142231	.124867	.124867
3	0	30	.332512	.332512	.280221	.280222
3	10	160	.166031	.166031	.148441	.148441
4	0	30	.353382	.353384	.404003	.404003
4	15	200	.183258	.183258	.167724	.167725
5	0	5	.906746	.906746	.867886	.867887
6	5	100	.278743	.278744	.254438	.254439

In addition to the above, the approximate expression is attractive for two reasons; first, computation time is less for the approximation because we don't evaluate a determinant at each step in the iteration scheme, as we do for the exact case; second, round off error is less troublesome in the approximate expression.

Table 16. Upper 1% Points of the Largest Root for q=2

n \ m	0	1	2	3	4	5	7	10	15
5	0.73163	0.78184	0.81537	0.83962	0.85809	0.87266	0.89424	0.91559	0.93676
10	.53116	.59150	.63618	.67120	.69962	.72326	.76050	.80033	.84333
15	.41313	.47060	.51546	.55216	.58304	.60955	.65296	.70184	.75786
20	.33720	.38956	.43175	.46720	.49774	.52449	.56946	.62199	.68488
25	.28459	.33194	.37091	.40426	.43346	.45942	.50389	.55722	.62327
30	0.24608	0.28901	0.32489	0.35600	0.38357	0.40834	0.45138	0.50409	0.57110
40	.19356	.22946	.26007	.28710	.31144	.33364	.37300	.42265	.48826
60	.13558	.16236	.18570	.20672	.22599	.24387	.27631	.31867	.37738
80	.10430	.12558	.14434	.16141	.17722	.19203	.21924	.25548	.30712
100	.084737	.10237	.11803	.13237	.14574	.15832	.18165	.21311	.25878
120	0.071353	0.086398	0.099823	0.11218	0.12374	0.13467	0.15505	0.18277	0.22353
160	.054221	.065846	.076284	.085945	.095034	.10367	.11900	.14223	.17563
200	.043723	.053191	.061725	.069652	.077135	.084271	.097732	.11640	.14461
300	.029460	.035927	.041786	.047253	.052438	.057405	.066832	.080034	.10029
500	.017829	.021785	.025384	.028756	.031966	.035051	.040938	.049252	.062166
1000	.0089721	.010980	.012811	.014533	.016176	.017760	.020794	.025107	.031871

Table 17. Upper 5% Points of the Largest Root for q=2

m n	0	1	2	3	4	5	7	10	15
5	0.63265	0.69818	0.74271	0.77533	0.80039	0.82031	0.85006	0.87975	0.90950
10	.43902	.50675	.55776	.59825	.63143	.65925	.70356	.75138	.80371
15	.33433	.39500	.44309	.48290	.51673	.54598	.59435	.64947	.71347
20	.26957	.32303	.36671	.40383	.43608	.46455	.51286	.56993	.63918
25	.22572	.27305	.31252	.34665	.37678	.40376	.45038	.50695	.57791
30	0.19409	0.23638	0.27217	0.30350	0.33149	0.35681	0.40118	0.45612	0.52687
40	.15156	.18626	.21619	.24286	.26705	.28925	.32893	.37949	.44714
60	.10534	.13073	.15308	.17337	.19210	.20957	.24150	.28358	.34255
80	.080711	.10068	.11846	.13475	.14994	.16423	.19067	.22619	.27734
100	.065413	.081862	.096599	.11019	.12294	.13500	.15749	.18808	.23290
120	0.054990	0.068967	0.081547	0.093203	0.10417	0.11459	0.13413	0.16093	0.20070
160	.041699	.052443	.062169	.071230	.079800	.087984	.10344	.12487	.15720
200	.033582	.042306	.050231	.057639	.064668	.071400	.084170	.10200	.12918
300	.022589	.028522	.033937	.039022	.043868	.048529	.057421	.069958	.089343
500	.013651	.017268	.020583	.023707	.026695	.029578	.035105	.042961	.055254
1000	.0068625	.0086933	.010376	.011966	.013491	.014966	.017805	.021864	.028277

Table 18. Upper 1% Points of the Largest Root for q=3

m n	0	1	2	3	4	5	7	10	15
5	0.82375	0.85143	0.87131	0.88637	0.89821	0.90777	0.92231	0.93711	0.95222
10	.64650	.68669	.71788	.74301	.76379	.78132	.80934	.83983	.87331
15	.52543	.56765	.60185	.63041	.65476	.67586	.71074	.75044	.79641
20	.44092	.48177	.51575	.54480	.57008	.59238	.63013	.67453	.72803
25	.37927	.41773	.45033	.47865	.50366	.52602	.56453	.61094	.66864
30	0.33251	0.36841	0.39925	0.42638	0.45059	0.47246	0.51062	0.55750	0.61724
40	.26649	.29770	.32502	.34944	.37158	.39185	.42791	.47348	.53372
60	.19053	.21482	.23651	.25628	.27451	.29148	.32234	.36267	.41848
80	.14819	.16793	.18576	.20217	.21745	.23180	.25824	.29346	.34355
100	.12123	.13782	.15290	.16688	.17997	.19234	.21531	.24630	.29118
120	0.10256	0.11685	0.12991	0.14206	0.15349	0.16433	0.18458	0.21214	0.25258
160	.078406	.089581	.099855	.10947	.11857	.12724	.14356	.16603	.19957
200	.063457	.072627	.081090	.089040	.096582	.10380	.11744	.13636	.16491
300	.042971	.049297	.055165	.060702	.065983	.071057	.080708	.094234	.11495
500	.026110	.030012	.033647	.037089	.040386	.043565	.049643	.058234	.071564
1000	.013180	.015173	.017034	.018803	.020501	.022143	.025297	.029783	.036813

Table 19. Upper 5% Points of the Largest Root for q=3

m n	0	1	2	3	4	5	7	10	15
5	0.75420	0.79166	0.81882	0.83952	0.85589	0.86916	0.88944	0.91022	0.93158
10	.57006	.61690	.65359	.68337	.70816	.72917	.76298	.80005	.84112
15	.45433	.50053	.53828	.57004	.59728	.62101	.66049	.70582	.75881
20	.37674	.41989	.45608	.48723	.51450	.53868	.57988	.62874	.68820
25	.32148	.36119	.39512	.42479	.45114	.47481	.51582	.56568	.62826
30	0.28022	0.31672	0.34830	0.37625	0.40134	0.42410	0.46405	0.51355	0.57723
40	.22286	.25394	.28133	.30595	.32839	.34903	.38593	.43292	.49563
60	.15800	.18168	.20295	.22243	.24048	.25734	.28815	.32870	.38530
80	.12235	.14138	.15866	.17463	.18957	.20365	.22970	.26463	.31471
100	.099816	.11569	.13021	.14371	.15641	.16845	.19090	.22138	.26585
120	0.084286	0.097902	0.11040	0.12208	0.13311	0.14361	0.16329	0.19024	0.23006
160	.064281	.074868	.084645	.093829	.10255	.11089	.12664	.14844	.18120
200	.051949	.060606	.068628	.076190	.083393	.090302	.10341	.12169	.14944
300	.035108	.041052	.046587	.051828	.056842	.061672	.070891	.083873	.10388
500	.021298	.024951	.028365	.031610	.034725	.037737	.043516	.051719	.064517
1000	.010738	.012597	.014341	.016002	.017602	.019153	.022140	.026407	.033128

Table 20. Upper 1% Points of the Largest Root for q=4

m n	0	1	2	3	4	5	7	10	15
5	0.87509	0.89200	0.90477	0.91478	0.92285	0.92951	0.93985	0.95067	0.96202
10	.72325	.75144	.77404	.79266	.80830	.82167	.84335	.86735	.89420
15	.60746	.63947	.66606	.68862	.70810	.72512	.75355	.78631	.82470
20	.52116	.55371	.58136	.60532	.62638	.64508	.67699	.71484	.76085
25	.45542	.48714	.51454	.53863	.56008	.57936	.61278	.65332	.70404
30	0.40400	0.43437	0.46093	0.48453	0.50576	0.52502	0.55881	0.60055	0.65398
40	.32914	.35648	.38080	.40274	.42276	.44116	.47403	.51571	.57097
60	.23973	.26184	.28189	.30031	.31739	.33335	.36244	.40055	.45336
80	.18837	.20673	.22355	.23915	.25374	.26751	.29292	.32684	.37510
100	.15510	.17073	.18514	.19860	.21128	.22328	.24564	.27585	.31961
120	0.13180	0.14539	0.15798	0.16978	0.18094	0.19156	0.21144	0.23854	0.27830
160	.10134	.11209	.12211	.13156	.14054	.14914	.16534	.18768	.22103
200	.082305	.091193	.099506	.10737	.11488	.12208	.13571	.15467	.18326
300	.056004	.062195	.068016	.073551	.078854	.083965	.093709	.10739	.12835
500	.034164	.038014	.041648	.045118	.048455	.051682	.057869	.066630	.08023
1000	.017298	.019276	.021149	.022942	.024672	.026350	.029579	.034183	.041402

Table 21. Upper 5% Points of the Largest Root for q=4

m n	0	1	2	3	4	5	7	10	15
5	0.82407	0.84740	0.86511	0.87905	0.89033	0.89965	0.91420	0.92947	0.94557
10	.66010	.69366	.72072	.74312	.76203	.77824	.80465	.83405	.86715
15	.54470	.58047	.61034	.63583	.65792	.67730	.70983	.74754	.79206
20	.46207	.49715	.52713	.55323	.57625	.59679	.63198	.67400	.72548
25	.40064	.43405	.46306	.48868	.51158	.53225	.56823	.61217	.66756
30	0.35338	0.38485	0.41250	0.43718	0.45947	0.47976	0.51552	0.55997	0.61732
40	.28566	.31338	.33815	.36059	.38114	.40009	.43408	.47745	.53537
60	.20626	.22818	.24813	.26652	.28363	.29966	.32900	.36765	.42156
80	.16132	.17929	.19582	.21121	.22565	.23929	.26457	.29848	.34706
100	.13243	.14763	.16169	.17485	.18728	.19909	.22114	.25109	.29473
120	0.11231	0.12545	0.13767	0.14916	0.16004	0.17043	0.18992	0.21663	0.25604
160	.086129	.096463	.10612	.11526	.12396	.13230	.14807	.16992	.20271
200	.069843	.078351	.086332	.093904	.10114	.10810	.12132	.13976	.16772
300	.047422	.053318	.058876	.064172	.069258	.074168	.083555	.096781	.11714
500	.028878	.032528	.035983	.039288	.042473	.045559	.051488	.059913	.073044
1000	.014602	.016471	.018245	.019948	.021593	.023191	.026275	.030684	.037625

Table 23. Upper 5% Points of the Largest Root for q=5

m n	0	1	2	3	4	5	7	10	15
5	0.86789	0.88338	0.89554	0.90537	0.91348	0.92029	0.93111	0.94276	0.95554
10	.72458	.74939	.76985	.78708	.80180	.81455	.83558	.85934	.88648
15	.61439	.64258	.66652	.68720	.70529	.72128	.74834	.78004	.81787
20	.53107	.55989	.58488	.60684	.62637	.64388	.67408	.71042	.75528
25	.46679	.49507	.51993	.54208	.56199	.58005	.61165	.65046	.69969
30	0.41600	0.44323	0.46744	0.48921	0.50897	0.52703	0.55900	0.59893	0.65067
40	.34122	.36596	.38829	.40865	.42738	.44471	.47589	.51580	.56926
60	.25055	.27081	.28941	.30666	.32277	.33789	.36565	.40227	.45343
80	.19780	.21474	.23044	.24514	.25899	.27209	.29643	.32913	.37599
100	.16335	.17785	.19137	.20410	.21616	.22763	.24911	.27831	.32088
120	0.13911	0.15175	0.16360	0.17480	0.18544	0.19562	0.21476	0.24101	0.27975
160	.10726	.11730	.12677	.13577	.14438	.15264	.16829	.18999	.22256
200	.087264	.094493	.10347	.11098	.11818	.12512	.13832	.15676	.18473
300	.059521	.065347	.070882	.076184	.081291	.086232	.095694	.10904	.12959
500	.036382	.040017	.043484	.046817	.050039	.053168	.059191	.067761	.081120
1000	.018450	.020325	.022114	.023839	.025514	.027144	.030294	.034807	.041912

Table 24. Upper 1% Points of the Largest Root for q=6

m n	0	1	2	3	4	5	7	10	15
5	0.92768	0.93537	0.94156	0.94665	0.95091	0.95456	0.95924	0.97249	0.98011
10	.81665	.83213	.84508	.85611	.86561	.87389	.88751	.90160	.93327
15	.71783	.73757	.75451	.76926	.78223	.79376	.81335	.83734	.87568
20	.63648	.65818	.67714	.69391	.70887	.72234	.74560	.77358	.80940
25	.57007	.59245	.61226	.62999	.64598	.66052	.68603	.71760	.76176
30	0.51545	0.53781	0.55780	0.57585	0.59227	0.60731	0.63398	0.66750	0.71631
40	.43163	.45300	.47238	.49012	.50646	.52159	.54884	.58380	.62935
60	.32472	.34325	.36034	.37622	.39107	.40502	.43062	.46433	.51201
80	.25991	.27589	.29078	.30474	.31790	.33037	.35352	.38460	.42872
100	.21656	.23051	.24358	.25591	.26761	.27874	.29958	.32785	.36905
120	0.18556	0.19790	0.20951	0.22051	0.23098	0.24099	0.25982	0.28558	0.32356
160	.14423	.15421	.16365	.17265	.18126	.18953	.20521	.22692	.25928
200	.11793	.12630	.13424	.14183	.14912	.15615	.16952	.18818	.21628
300	.080997	.086935	.092602	.098044	.10329	.10837	.11811	.13183	.15284
500	.049794	.053543	.057137	.060602	.063956	.067216	.073494	.082419	.096317
1000	.025362	.027312	.029186	.030998	.032758	.034473	.037790	.042539	.049989

Table 25. Upper 5% Points of the Largest Root for q=6

m n	0	1	2	3	4	5	7	10	15
5	0.89716	0.90796	0.91667	0.92385	0.92988	0.93502	0.94291	0.95382	0.96990
10	.77232	.79117	.80699	.82049	.83217	.84237	.85933	.87831	.92571
15	.66925	.69183	.71127	.72824	.74321	.75654	.77926	.80640	.84101
20	.58762	.61157	.63256	.65117	.66782	.68283	.70888	.74044	.78006
25	.52259	.54671	.56812	.58733	.60471	.62053	.64836	.68281	.72777
30	0.46998	0.49367	0.51492	0.53415	0.55169	0.56778	0.59639	0.63236	0.68045
40	.39058	.41271	.43285	.45131	.46836	.48418	.51275	.54948	.59853
60	.29127	.30999	.32731	.34344	.35855	.37277	.39893	.43353	.48214
80	.23199	.24793	.26282	.27680	.29001	.30255	.32588	.35729	.40229
100	.19268	.20649	.21945	.23170	.24334	.25444	.27525	.30359	.34498
120	0.16474	0.17688	0.18833	0.19919	0.20955	0.21947	0.23816	0.26383	0.30178
160	.12768	.13744	.14668	.15550	.16396	.17209	.18753	.20898	.24115
200	.10422	.11236	.12010	.12751	.13463	.14151	.15462	.17296	.20076
300	.071406	.077150	.082639	.087917	.093015	.097956	.10743	.12083	.14145
500	.043810	.047420	.050884	.054227	.057468	.060620	.066699	.07536	.088882
1000	.022280	.024150	.025950	.027692	.029385	.031037	.034236	.038825	.046054

CHAPTER VI

SUMMARY AND CONCLUSION

The study of some central and non-central distribution problems in real and complex multivariate analysis has been carried out in this work. The main objective has been to investigate the distributions of characteristic roots, and functions of characteristic roots of certain matrices, with emphasis on various likelihood ratio criteria, in the central and non-central cases. Although in the first chapter a general form was found for the first three moments of $U_i^{(p)}$ and the first two moments of $V_i^{(p)}$, both in the non-central (linear) MANOVA case, it would be useful to obtain expressions for the general moments of both statistics. Further it is hoped that a general expression can be obtained for $a_{\kappa,\tau}$ coefficients given in Chapter I, although such expressions were given in some cases there.

In Chapter II we make use of Mellin's transform and Meijers G-function to find the non-central distributions of Wilks' Λ-criterion in the cases of MANOVA, canonical correlation and the equality of two covariance matrices. The densities were found for any p and the cdf of Λ was explicitly found for $p = 2$ but the cdf remains to be found for higher p values. Also power computations could be carried out. The exact distributions of the likelihood ratio criterion $V_{p_1,p_2,\ldots,p_q;N}$ for testing the independence of q-sets of variates under the null hypothesis are found in Chapter III for various combinations of p_i values and general N using the convolution operation. Due to the fact that the expressions become unwieldy, results were obtained when no more than two convolutions were necessary, but the work could be extended to more than two convolutions. χ^2 correction factors, from which lower 1% and 5% points can be obtained, were tabled in special cases but could be obtained for other cases using the results of Chapter III.

In Chapter IV the density functions of Wilks' Λ-criterion in the complex case for the three situations mentioned in Chapter II, are obtained using Mellin's transform and expressed in terms of Meijer's G-function. Results are obtained concerning the complex multivariate beta distribution and independent beta variates. While the density functions of the U-statistic and of Pillai's V-criterion were found for $p = 2$ in the complex central case, results haven't been obtained for larger p values. Using Pillai's lemma on the product of an esf and a Vandermonde determinant, zonal polynomials of a Hermitian matrix were found through degree 8 and results for higher degrees could be found in the same manner. In Chapter V, an approximation to the distribution of the maximum characteristic root in the complex case is obtained. Although a computational form, using determinants, was found, it is felt that further research might yield a general term for the coefficient of each incomplete beta function involved. Tables were given for upper 1% and 5% points for values of $q = 2(1)6$. These tables could be easily extended using the results of Chapter V.

REFERENCES

[1] Al-Ani, S. (1968). Some distribution problems concerning characteristic roots and vectors in multivariate analysis. Mimeograph Series No.162, Department of Statistics, Purdue University.

[2] Anderson, T.W. (1946). The non-central Wishart distribution and certain problems of multivariate statistics. Ann. Math. Stat. 17, 409-431.

[3] Anderson, T.W. (1958). An Introduction to Multivariate Statistical Analysis, Wiley, New York.

[4] Anderson, T.W. and Girshick, M.A. (1944). Some extensions of the Wishart distribution. Ann. Math. Stat. 15, 345-357.

[5] Constantine, A.G. (1963). Some Non-Central Distribution Problems in Multivariate Analysis. Ann. Math. Stat. 34, 1270-1285.

[6] Constantine, A.G. (1966). The distribution of Hotelling's generalized T_o^2. Ann. Math. Stat. 37, 215-225.

[7] Consul, P.C. (1966). On Some Inverse Mellin Integral Transforms. Academic Royale Des Science de Belgique.52, 547-561.

[8] Consul, P.C. (1967). On the Exact Distributions of Likelihood Ratio Criteria for Testing Independence of Sets of Variates Under the Null Hypothesis. Ann. Math. Stat. 38, 1160-1169.

[9] Consul, P.C. (1967). On the Exact Distribution of the W Criterion for Testing the Sphericity in a p-variate Normal Distribution. Ann. Math. Stat. 38, 1170-1174.

[10] Goodman, N.R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution. (An introduction) Ann. Math. Stat. 34, 152-176.

[11] Gupta, A.K. (1968). Some central and non-central distribution problems in multivariate analysis. Mimeograph Series No. 139, Department of Statistics, Purdue University.

[12] James, A.T. (1961). The distribution of non-central means with known covariance. Ann. Math. Stat. 32, 874-882.

[13] James, A.T. (1964). Distribution of matrix and latent roots derived from normal samples. Ann. Math. Stat. 35, 475-501.

[14] Khatri, C.G. (1964). Distribution of the generalized multiple correlation matrix in the dual case. Ann. Math. Stat. 35, 1801-1806.

[15] Khatri, C.G. (1964). Distribution of the largest or the smallest characteristic root under the null hypothesis concerning complex multivariate normal populations. Ann. Math. Stat. 35, 1807-1810.

[16] Khatri, C.G. (1965). Classical statistical analysis based on certain multivariate complex Gaussian distribution. Ann. Math. Stat. 36,98-114.

[17] Khatri, C.G. (1965). A test for reality of a covariance matrix in certain complex Gaussian distributions. Ann. Math. Stat. 36, 115-119.

[18] Khatri, C.G. (1967). Some distribution problems connected with the characteristic roots of $S_1S_2^{-1}$. Ann. Math. Stat. 38, 944-948.

[19] Khatri, C.G. (1968). Noncentral distributions of the ith largest characteristic roots of three matrices concerning complex multivariate normal populations. Mimeograph Series No. 149, Department of Statistics, Purdue University.

[20] Khatri, C.G. (1968). On the moments of two matrices in three situations for complex multivariate normal populations. Mimeograph Series No. 161, Department of Statistics, Purdue University.

[21] Khatri, C.G. and Pillai, K.C.S. (1965). Some results on the non-central multivariate beta distribution and moments of traces of two matrices. Ann. Math. Stat. 36, 1511-1520.

[22] Khatri, C.G. and Pillai, K.C.S. (1965). Further results on the non-central multivariate beta distribution and moments of traces of two matrices. Mimeograph Series No. 38, Department of Statistics, Purdue University.

[23] Khatri, C.G. and Pillai, K.C.S. (1968). On the moments of elementary symmetric functions of two matrices and an approximation to a distribution. Ann. Math. Stat. 39, 1274-1281.

[24] Kshirsager, A.M. (1961). The non-central multivariate beta distribution. Ann. Math. Stat. 32, 104-111.

[25] Meijer, C.S. (1946). Nederl. Akad. Wetensch. Proc., 49.

[26] Meijer, C.S. (1946). On the G-function I. Indagations Mathematicae, 8, 124-134.

[27] Pillai, K.C.S. (1964). On the moments of elementary symmetric functions of the roots of two matrices. Ann. Math. Stat. 35, 1704-1712.

[28] Pillai, K.C.S. (1965). On elementary symmetric functions of the roots of two matrices in multivariate analysis. Biometrika 52, 499-506.

[29] Pillai, K.C.S. (1968). Moment generating function of Pillai's $V^{(s)}$ criterion. Ann. Math. Stat. 39, 877-880.

[30] Pillai, K.C.S. and Al-Ani, S. (1967). On the Distribution of Some Functions of the Roots of a Covariance Matrix and Noncentral Wilks' Λ. Mimeograph Series No. 125, Department of Statistics, Purdue University.

[31] Pillai, K.C.S. and Jayachandran, K. (1967). Power comparisons of tests of two multivariate hypotheses based on four criteria. Biometrika 54, 195-210.

[32] Pillai, K.C.S. and Gupta, A.K. (1968). On the non-central distribution of the second elementary symmetric function of the roots of a matrix. Ann. Math. Stat. 39, 833-839.

[33] Roy, S.N. (1957). Some Aspects of Multivariate Analysis. Wiley, New York.

[34] Roy, S.N. and Gnanadesikan, R. (1959). Some contributions to ANOVA in one or two dimensions: II. Ann. Math. Stat. 30, 318-340.

[35] Schatzoff, M. (1966). Exact distribution of Wilks' likelihood ratio criterion. Biometrika 53, 347-358.

[36] Wald, A. and Brookner, R.J. (1941). On the distribtuion of Wilks' statistic for testing independence of several groups of variables. Ann. Math. Stat. 12, 137-152.

[37] Wilks, S.S. (1935). On the independence of k sets of normally distributed statistical variables. Econometrica 3, 309-326.

[38] Wooding, R.A. (1956). The multivariate distribution of complex normal variables. Biometrika 43, 212-215.