Distributions of Vectors Corresponding to the Largest Roots of Three Matrices[*]

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1. Introduction and summary. The distributions of the characteristic vectors corresponding to the largest roots (CVCLR) of matrices are studied in this paper (1) in the single sample case and (2) in the two sample case, when the samples are drawn p-variate normal populations. For $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, the distributions of CVCLR considered in (1) are for the special cases when (i) $\underset{\sim}{\mu} = \underset{\sim}{0}$, (ii) $\underset{\sim}{\Sigma} = \underset{\sim}{I}$ and (iii) the rank of $\underset{\sim}{\mu}$ is one. The case $\underset{\sim}{\mu} = \underset{\sim}{0}$ has been studied earlier by Sugiyama [11]. In the two sample case, for $N(\underset{\sim}{\mu}, \underset{\sim}{\Sigma}_1)$ and $N(\underset{\sim}{0}, \underset{\sim}{\Sigma}_2)$, the cases discussed are the same as above except that (ii) should be replaced by $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2 = \underset{\sim}{\Sigma}$ and $\underset{\sim}{\mu}$ is of rank one, and in (iii) $\underset{\sim}{\mu}$ could be random as well. It may be noted that in the two sample case when $\underset{\sim}{\mu}$ is fixed and $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, this case corresponds to the MANOVA model, and when $\underset{\sim}{\mu}$ is random and $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ corresponds to canonical correlation situation. Case (ii) with $\underset{\sim}{\mu} = \underset{\sim}{0}$ in the two sample case deals with the testing of the equality of two covariance matrices. Further, the last section is devoted to the testing of hypothetical vectors of $\underset{\sim}{\Sigma}_1$ in the fields of $\underset{\sim}{\Sigma}_2$ suggesting relevant criteria and deriving their distributions. In the single sample case, tests of hypothetical principal components were developed earlier [7], [8].

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2. Notations and some useful results. Let $\underset{\sim}{\Gamma}: n \times n$ be an orthogonal matrix such that first $p(\leq n)$ columns have random elements and the other $(n-p)$ columns depend on these random elements. We shall write $d\underset{\sim}{\Gamma}^{(n,p)}$, a normalized measure over this space, i.e.

$$\int_{O(n)} d\underset{\sim}{\Gamma}^{(n,p)} = 1 \ .$$

In terms of Roy's notations [10] let $J(\underset{\sim}{\Gamma}) = 2^n / \left|\frac{\partial(L\ L')}{\partial(\underset{\sim}{L}_D)}\right|_{\underset{\sim}{L}_I}$ = a function of random elements. Then we shall write

$$d\underset{\sim}{\Gamma}^{(n,p)} = \Gamma_p(\tfrac{1}{2}n)\ J(\underset{\sim}{\Gamma})/\Pi^{\frac{1}{2}pn} \ .$$

The following lemma has been established by Khatri [5].

Lemma 1. Let $\underset{\sim}{U}: p \times n = \underset{\sim}{L}\begin{pmatrix} \alpha & 0 \\ 0 & \underset{\sim}{V} \end{pmatrix} \underset{\sim}{M}'$ be a transformation such that first column vectors of $\underset{\sim}{L}: p \times p$ and $\underset{\sim}{M}: n \times n$ contain random elements, $\alpha \neq 0$, α^2 is the maximum characteristic (max. ch.) root of $\underset{\sim}{U}\,\underset{\sim}{U}'$ and $\underset{\sim}{V}: (p-1) \times (n-1)$ is a random matrix. Then, the jacobian of the transformation is given by

$$J(\underset{\sim}{U}; \underset{\sim}{L}, \alpha, \underset{\sim}{V}, \underset{\sim}{M}) = |\alpha|^{n-p} |\alpha^2 \underset{\sim}{I}_{p-1} - \underset{\sim}{V}\ \underset{\sim}{V}'| \Pi^{\frac{1}{2}(p+n)} \{\Gamma(\tfrac{n}{2})\Gamma(\tfrac{p}{2})\}^{-1} d\underset{\sim}{\Gamma}^{(p,1)} d\underset{\sim}{M}^{(n,1)} \ .$$

Lemma 2. Let $\underset{\sim}{Y}: m \times q$ be a random matrix and $\underset{\sim}{A}: m \times m$ be a symmetric matrix. Then

$$\int_{\mathcal{D}} |\underset{\sim}{I}_m - \underset{\sim}{Y}\ \underset{\sim}{Y}'|^{\alpha}\ C_{\kappa}(\underset{\sim}{A}\ \underset{\sim}{Y}\ \underset{\sim}{Y}')d\underset{\sim}{Y}$$

$$= \frac{C_{\kappa}(\underset{\sim}{A})\ \Pi^{\frac{1}{2}mq}\ (\tfrac{1}{2}q)_{\kappa}\ \Gamma_q(\alpha + \frac{q+1}{2})}{\Gamma_q(\alpha+\frac{m+q+1}{2})(\alpha+\frac{m+q+1}{2})_{\kappa}} \qquad \text{if } \ m \geq q, \ \ k_{q+1} = 0 \ ;$$

$$= \ 0 \qquad \text{if } \ m \geq q \text{ and } k_{q+1} \neq 0 \ ;$$

$$= \frac{C_{\kappa}(\underset{\sim}{A})\ \Pi^{\frac{1}{2}mq}(\tfrac{1}{2}q)_{\kappa}\ \Gamma_m(\alpha+\frac{q+1}{2})}{\Gamma_m(\alpha+\frac{m+q+1}{2})(\alpha+\frac{m+q+1}{2})_{\kappa}} \qquad \text{if } \ m \leq q \ ;$$

where $\mathcal{D} = \mathcal{D}\{\underset{\sim}{Y}$ such that $\underset{\sim}{I}_m - \underset{\sim}{Y}\ \underset{\sim}{Y}'$ is positive definite$\}$, $\kappa = \{k_1,\ldots,k_m\}$, $\Gamma_m(x) = \Pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(x - \frac{i-1}{2})$, $(x)_{\kappa} = \Gamma_m(x,\kappa)/\Gamma_m(x)$ and $\Gamma_m(x,\kappa) = \Pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(x+k_i - \frac{i-1}{2})$.

Proof: Let us write

$$g = \int_{\mathcal{D}} |\underset{\sim}{I}_m - \underset{\sim}{Y}\ \underset{\sim}{Y}'|^{\alpha}\ C_{\kappa}(\underset{\sim}{A}\ \underset{\sim}{Y}\ \underset{\sim}{Y}')d\underset{\sim}{Y} \int_{O(m)} d\underset{\sim}{H}$$

i.e.

(2.1) $$g = \int_{\mathcal{D}} |\underset{\sim}{I}_m - \underset{\sim}{Y}\ \underset{\sim}{Y}'|^{\alpha} \int_{O(m)} C_{\kappa}(\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{Y}\ \underset{\sim}{Y}'\underset{\sim}{H}')d\underset{\sim}{H}\ d\underset{\sim}{Y}\ , \text{ using } \ \underset{\sim}{Y} \rightarrow \underset{\sim}{H}\ \underset{\sim}{Y} \ .$$

James [3] has proved the following important result:

(2.2) $$\int_{O(m)} C_{\kappa}(\underset{\sim}{A}\ \underset{\sim}{H}\ \underset{\sim}{B}\ \underset{\sim}{H}')d\underset{\sim}{H} = C_{\kappa}(\underset{\sim}{A})\ C_{\kappa}(\underset{\sim}{B})\ /\ k!\ C_{\kappa}(\underset{\sim}{I}_m) \ ,$$

where A and B are symmetric matrices and $C_\kappa(Z)$ is a zonal polynomial corresponding to the partition $\kappa = \{k_1, k_2, \ldots, k_m\}$, $k_1 \geq k_2 \geq \ldots \geq k_m \geq 0$ and $k = \sum_{i=1}^{m} k_i$. Moreover, we may note that

$$(2.3) \qquad \begin{aligned} C_\kappa(Y\,Y') &= C_\kappa(Y'Y) && \text{if} \quad m \geq q, \; k_{q+1} = 0 \; ; \\ &= \;\; 0 && \text{otherwise for} \;\; m \geq q \; . \end{aligned}$$

Let $m \geq q$. Then using (2.2) and (2.3) in (2.1), we get

$$(2.4) \qquad \begin{aligned} g &= C_\kappa(A)\,\{C_\kappa(I_m)\}^{-1} \int_{\mathcal{D}} |I_m - Y\,Y'|^\alpha\, C_\kappa(Y'Y)\, dY \quad \text{if} \quad k_{q+1} = 0; \\ &= 0, \quad \text{otherwise} \; . \end{aligned}$$

Now, integrating Y such that $Y'\,Y = S$, we get

$$\int_{\mathcal{D}} |I_m - Y\,Y'|^\alpha\, C_\kappa(Y'Y)dY = \Pi^{\frac{1}{2}mq} \int_{S>0} |I_q - S|^\alpha |S|^{\frac{1}{2}(m-q-1)} C_\kappa(S)dS/\Gamma_q(\tfrac{1}{2}m)$$

and using Constantine's result [1], we get

$$(2.5) \qquad \int_{\mathcal{D}} |I_m - Y\,Y'|^\alpha\, C_\kappa(Y'Y)dY = \frac{\Pi^{\frac{1}{2}mq}\, \Gamma_q(\frac{1}{2}m,\kappa)\Gamma_q(\alpha+\frac{1}{2}q+\frac{1}{2})\; C_\kappa(I_q)}{\Gamma_q(\frac{1}{2}m)\;\Gamma_q(\alpha+\frac{m+q+1}{2}\;,\;\kappa)} \; .$$

Using (2.5) in (2.4) and noting $C_\kappa(I_q)/C_\kappa(I_m) = (\frac{1}{2}q)_\kappa/(\frac{1}{2}m)_\kappa$, we get the first part of lemma 2 for $m \geq q$. Similarly, the second part of lemma 2 can be proved. Thus, lemma 2 is established.

Lemma 3. Let $Y: m \times q$ be a random matrix and $A: m \times m$ be a positive definite matrix. Then

$$(2.6) \qquad \int_{\mathscr{D}} |I_m - Y\,Y'|^{\alpha}\, C_{\kappa}(A\,Y\,Y')\, \exp(-\mathrm{tr}\, A\,Y\,Y')\, dY$$

$$= \sum_{j=k}^{\infty} \sum_{J} (-1)^{j+k} k!\, a_{J,\kappa} (j!)^{-1} \int_{\mathscr{D}} |I_m - Y\,Y'|^{\alpha}\, C_J(A\,Y\,Y')\, dY \;,$$

where $C_J(I_m - Z)/C_J(I_m) = \sum_{t=0}^{j} \sum_{\tau} (-1)^t a_{J,\tau}\, C_{\tau}(Z)/C_{\tau}(I_m)$, [2], [9], and

$\mathscr{D} = \mathscr{D}\{Y$ such that $I - Y\,Y'$ is positive definite$\}$.

Proof. Let the left hand side of (2.6) be denoted as $g_{\kappa}(A)$. Then

$$G = \sum_{k=0}^{\infty} \sum_{\kappa} g_{\kappa}(A)\, C_{\kappa}(Z)/k!\, C_{\kappa}(I_m)$$

$$= \int_{O(m)} \int_{\mathscr{D}} |I_m - Y\,Y'|^{\alpha} \exp[\mathrm{tr}\, A^{\frac{1}{2}}(H'Z\,H - I)A^{\frac{1}{2}}Y\,Y']\, dY\, dH$$

$$= \sum_{j=0}^{\infty} \sum_{J} \frac{1}{j!} \int_{\mathscr{D}} |I_m - Y\,Y'|^{\alpha} \int_{O(m)} C_J[H'(Z-I)H\, A^{\frac{1}{2}}Y\, Y'A^{\frac{1}{2}}]\, dY\, dH$$

$$= \sum_{j=0}^{\infty} \sum_{J} \frac{C_J(Z-I_m)}{j!\, C_J(I_m)} \int_{\mathscr{D}} |I_m - Y\,Y'|^{\alpha}\, C_J(A\,Y\,Y')\, dY \quad .$$

This proves lemma 3.

3. Distribution of CVCLR in the single sample case. Here, we shall consider the density function of $\underset{\sim}{X}:p \times n$ given by

$$(3.1)\qquad (2\Pi)^{-\frac{1}{2}pn}\,|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \exp[-\tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{\mu}\,\underset{\sim}{\mu}' + \mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{X}\,\underset{\sim}{\mu}' - \tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\underset{\sim}{X}\,\underset{\sim}{X}']$$

and we are interested in obtaining the density function of the first column of $\underset{\sim}{L}$ or $\underset{\sim}{M}$ given by the following transformation

$$\underset{\sim}{X} = \underset{\sim}{L}\begin{pmatrix} \alpha & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{Y} \end{pmatrix}\underset{\sim}{M}', \quad \underset{\sim}{L}:p \times p,\ \underset{\sim}{M}:n \times n \text{ and } \underset{\sim}{Y}:(p-1) \times (n-1)\ .$$

Using lemma 1, the joint density function of $\underset{\sim}{L}$, $\underset{\sim}{M}$, $\underset{\sim}{Y}$ and α is given by

$$(3.2)\qquad c|\alpha|^{n-p}|\alpha^2\underset{\sim}{I}_{p-1} - \underset{\sim}{Y}\,\underset{\sim}{Y}'|\ |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \exp[-\tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{\mu}\,\underset{\sim}{\mu}' + \mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{L}\begin{pmatrix} \alpha & 0 \\ 0 & \underset{\sim}{Y} \end{pmatrix}\underset{\sim}{M}'\underset{\sim}{\mu}'$$
$$- \tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\,\underset{\sim}{L}\begin{pmatrix} \alpha^2 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{Y}\,\underset{\sim}{Y}' \end{pmatrix}\underset{\sim}{L}']\ \ d\underset{\sim}{L}^{(p,1)}d\underset{\sim}{M}^{(n,1)},$$

where $c^{-1} = (2\Pi)^{\frac{1}{2}pn}\,\Gamma(\frac{n}{2})\,\Gamma(\frac{p}{2})/\Pi^{\frac{1}{2}(p+n)}$. In order to integrate α, $\underset{\sim}{Y}$, $\underset{\sim}{M}$ or $\underset{\sim}{L}$ we consider the following particular cases, because the general problem is extremely difficult.

Case 1. Let $\underset{\sim}{\mu} = \underset{\sim}{0}$. In this case, $\underset{\sim}{M}$ and $(\lambda = \alpha^2, \underset{\sim}{Y}, \underset{\sim}{L})$ are independently distributed and their respective densities are given by

$$(3.3)\qquad d\underset{\sim}{M}^{(n,1)}$$

and

$$(3.4) \qquad c\, \lambda^{\frac{1}{2}(n-p-1)} |\lambda \underset{\sim}{I} - \underset{\sim}{YY'}|\ |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \exp[-\tfrac{1}{2}\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L} \begin{pmatrix} \lambda & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{YY'} \end{pmatrix} \underset{\sim}{L'}]\ d\underset{\sim}{L}^{(p,1)} \quad .$$

Integrating over λ and $\underset{\sim}{Y}$, we get the same density of $\underset{\sim}{L}$ as given by Sugiyama [11] when $n \geq p$, and using lemma 2 it can be written as

$$(3.5) \qquad c_1\ |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \sum_{j=0}^{\infty} (-1)^j (\underset{\sim}{\ell}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\ell})^{-j-\frac{1}{2}pn} \binom{\frac{1}{2}pn+j-1}{j} \sum_{J} (\tfrac{n-1}{2})_J \{(\tfrac{n+p+1}{2})_J\}^{-1}$$

$$C_J(\underset{\sim}{L}_1' \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L}_1)\ d\underset{\sim}{L}^{(p,1)} \quad ,$$

when $n \geq p$ and $J = \{j_1, \ldots, j_{p-1}\}$; and while $n \leq p$ and $J = \{j_1, \ldots, j_{n-1}\}$,

$$(3.6) \qquad c_2 |\Sigma|^{-\frac{1}{2}n} \sum_{j=0}^{\infty} (-1)^j (\ell'\Sigma^{-1}\ell)^{-\frac{1}{2}pn-j} \binom{\frac{1}{2}pn+j-1}{j} \sum_{J} (\tfrac{n-1}{2})_J \{(\tfrac{n+p+1}{2})_J\}^{-1}$$

$$C_J(\underset{\sim}{L}_1' \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L}_1)\ d\underset{\sim}{L}^{(p,1)} \quad ,$$

where

$$c_1 = \Pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}pn)\Gamma_{p-1}(\tfrac{p+2}{2})/\{\Gamma_{p-1}(\tfrac{n+p+1}{2})\Gamma(\tfrac{p}{2})\Gamma(\tfrac{n}{2})\}, \qquad \underset{\sim}{L} = (\underset{\sim}{\ell}, \underset{\sim}{L}_1) \qquad \text{and}$$

$$c_2 = \Pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}pn)\Gamma_{n-1}(\tfrac{n+2}{2})/\{\Gamma_{n-1}(\tfrac{n+p+1}{2})\Gamma(\tfrac{p}{2})\Gamma(\tfrac{n}{2})\}$$

We may note that (3.5) can be rewritten as

$$(3.5') \qquad c_1 |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \sum_{j=0}^{\infty} \binom{\frac{1}{2}pn+j-1}{j} (\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1})^{-\frac{1}{2}pn-j} \sum_{J} (\tfrac{p+2}{2})_J \{(\tfrac{n+p+1}{2})_J\}^{-1}$$

$$C_J(\underset{\sim}{L}_1' \underset{\sim}{\Sigma}^{-1} \underset{\sim}{L}_1)\ d\underset{\sim}{L}^{(p,1)} \quad .$$

<u>Case 2</u>. Let $\underset{\sim}{\Sigma} = \underset{\sim}{I}$. Then (3.2) can be rewritten as

$$(3.2') \quad c|\alpha|^{n-p}|\alpha^2\underset{\sim}{I}_{p-1} - \underset{\sim}{Y}\underset{\sim}{Y}'| \exp[-\tfrac{1}{2}\mathrm{tr}\ \underset{\sim}{\mu}\ \underset{\sim}{\mu}' - \tfrac{1}{2}\alpha^2 - \tfrac{1}{2}\mathrm{tr}\ \underset{\sim}{Y}\underset{\sim}{Y}'$$

$$+ \mathrm{tr}\begin{pmatrix} \alpha & 0 \\ \underset{\sim}{0} & \underset{\sim}{Y} \end{pmatrix} \underset{\sim}{M}'\underset{\sim}{\mu}'\underset{\sim}{L}]d\underset{\sim}{L}^{(p,1)}d\underset{\sim}{M}^{(n,1)} \quad .$$

Here, we shall obtain the joint density function of $\underset{\sim}{M}$ and $\underset{\sim}{L}$. Let $\underset{\sim}{M} = (\underset{\sim}{m}, \underset{\sim}{M}_1)$ and $\underset{\sim}{L} = (\underset{\sim}{\ell}, \underset{\sim}{L}_1)$. Then $\mathrm{tr}\ \begin{pmatrix} \alpha & 0 \\ \underset{\sim}{0} & \underset{\sim}{Y} \end{pmatrix}\ \underset{\sim}{M}'\underset{\sim}{\mu}'\underset{\sim}{L} = \underset{\sim}{m}'\underset{\sim}{\mu}\ \underset{\sim}{\ell} + \mathrm{tr}\ \underset{\sim}{Y}\ \underset{\sim}{M}_1'\underset{\sim}{\mu}'\underset{\sim}{L}_1$. We consider the following integration:

$$(3.7) \quad \int_{\underset{\sim}{Y}} |\alpha^2\underset{\sim}{I}_{p-1} - \underset{\sim}{Y}\ \underset{\sim}{Y}'| \exp[-\tfrac{1}{2}\mathrm{tr}\ \underset{\sim}{Y}\ \underset{\sim}{Y}' + \mathrm{tr}\ \underset{\sim}{Y}\ \underset{\sim}{M}_1'\ \underset{\sim}{\mu}'\underset{\sim}{L}_1]\ d\underset{\sim}{Y}$$

$$= \sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa(\tfrac{1}{2}\underset{\sim}{M}_1'\underset{\sim}{\mu}'\underset{\sim}{L}_1\underset{\sim}{L}_1'\underset{\sim}{\mu}\ \underset{\sim}{M}_1)}{k!(\frac{n-1}{2})_\kappa\ C_\kappa(\underset{\sim}{I}_{p-1})} \int_{\underset{\sim}{Y}} |\alpha^2\underset{\sim}{I}_{p-1} - \underset{\sim}{Y}\ \underset{\sim}{Y}'|C_\kappa(\tfrac{1}{2}\underset{\sim}{Y}\ \underset{\sim}{Y}')e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{Y}\underset{\sim}{Y}'}\ d\underset{\sim}{Y} \quad ,$$

and, using lemma 3,

$$(3.8) \quad \int_{\underset{\sim}{Y}} |\alpha^2\underset{\sim}{I}_{p-1} - \underset{\sim}{Y}\ \underset{\sim}{Y}'|C_\kappa(\tfrac{1}{2}\underset{\sim}{Y}\underset{\sim}{Y}')\ \exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{Y}\underset{\sim}{Y}')d\underset{\sim}{Y}$$

$$= \text{coefficient of } \frac{C_\kappa(\underset{\sim}{\theta})}{C_\kappa(\underset{\sim}{I}_{p-1})} \text{ in } \left\{\sum_{j=0}^{\infty} (-\frac{\alpha^2}{2})^j\ \frac{1}{j!}\ |\alpha|^{pn+p-n-1}\right.$$

$$\left.\sum_{J} \frac{(\frac{n-1}{2})_J\ \Gamma_{p-1}(\frac{p+2}{2})}{\Gamma_{p-1}(\frac{n+p+1}{2}, J)}\ C_J(\underset{\sim}{I}_{p-1} - \underset{\sim}{\theta})\Pi^{\frac{1}{2}(p-1)(n-1)}\right\} \text{ if } n \geq p;$$

$$= \text{coefficient of } \frac{C_\kappa(\underset{\sim}{\theta})}{k!C_\kappa(\underset{\sim}{I}_{n-1})} \text{ in } \left\{\sum_{j=0}^{\infty} (-\frac{\alpha^2}{2})^j\ \Pi^{\frac{1}{2}(p-1)(n-1)}\right.$$

$$\left.|\alpha|^{pn+p-n-1}(j!)^{-1}\sum_{J}\frac{(\frac{p-1}{2})_J\Gamma_{n-1}(\frac{n+2}{2})}{\Gamma_{n-1}(\frac{n+p+1}{2}, J)}\ C_J(\underset{\sim}{I}_{n-1} - \underset{\sim}{\theta})\right\} \text{ if } \ n \leq p \quad .$$

Using (3.8) in (3.7) and then (3.7) in (3.2') and integrating over α, we get the joint density function of $\underset{\sim}{L}$ and $\underset{\sim}{M}$ as

$$(3.9)\qquad c_3 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=k}^{\infty} \sum_{J} \frac{2^i \Gamma(\frac{1}{2}pn+i+j)}{(2i)!(\frac{n-1}{2})_\kappa} \frac{(-1)^j (\frac{n-1}{2})_J (\underset{\sim}{m}'\underset{\sim}{\mu}'\underset{\sim}{\ell})^{2i}}{j!\ (\frac{n+p+1}{2})_J}\ a_{\kappa,J}$$

$$C_\kappa(\tfrac{1}{2}\underset{\sim}{M}_1'\underset{\sim}{\mu}'\underset{\sim}{L}_1\underset{\sim}{L}_1'\underset{\sim}{\mu}\ \underset{\sim}{M}_1)\ d\underset{\sim}{L}^{(p,1)} d\underset{\sim}{M}^{(n,1)}$$

for $n \geq p$, while for $n \leq p$

$$(3.10)\qquad c_4 \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \sum_{\kappa} \sum_{J} \frac{2^i \Gamma(\frac{1}{2}pn+i+j)(-1)^j (\frac{p-1}{2})_J}{(2i)!\ (\frac{p-1}{2})_\kappa\ j!(\frac{n+p+1}{2})_J}\ (\underset{\sim}{m}'\underset{\sim}{\mu}'\ell)^{2i}\ a_{\kappa,J}$$

$$C_\kappa(\tfrac{1}{2}\underset{\sim}{\mu}'\underset{\sim}{L}_1\underset{\sim}{L}_1'\underset{\sim}{\mu}\ \underset{\sim}{M}_1\underset{\sim}{M}_1')\ \ d\underset{\sim}{L}^{(p,1)}\ d\underset{\sim}{M}^{(n,1)}$$

where $c_3 = c_1/\Gamma(\frac{1}{2}pn)$ and $c_4 = c_2/\Gamma(\frac{1}{2}pn)$ and $a_{\kappa,J}$ is the coefficient of $C_\kappa(\underset{\sim}{\theta})$ in $C_J(\underset{\sim}{I}-\underset{\sim}{\theta})$. (Note that in (3.9), $\kappa = \{k_1,\ldots,k_{p-1}\}$ and $J = \{j_1,\ldots,j_{p-1}\}$ while in (3.10), $\kappa = \{k_1,\ldots,k_{n-1}\}$ and $J = \{j_1,\ldots,j_{n-1}\}$.)

Case 3. Let the rank of $\underset{\sim}{\mu}:p \times n$ be one and we try to obtain the density function of $\underset{\sim}{L}$. The density function of $\underset{\sim}{M}$ requires the integration

$$\int_{O(p)} \exp[\mathrm{tr}\ \underset{\sim}{C}\,\underset{\sim}{H}\,\underset{\sim}{A}\,\underset{\sim}{H}' + \mathrm{tr}\ \underset{\sim}{H}\,\underset{\sim}{B}]\ d\underset{\sim}{H}$$

where $\underset{\sim}{C}$ and $\underset{\sim}{A}$ are symmetric matrices of order $p \times p$ and $\underset{\sim}{B}:p \times p$ is a square matrix of rank equal to that of $\underset{\sim}{\mu}$. The result for the above integral is unknown and hence we omit this problem at present.

From (3.2), integrating $\underset{\sim}{M}$ and $\underset{\sim}{Y}$, the joint density function of α and $\underset{\sim}{L}$ can be written as

(3.11) $$c|\alpha|^{n-p}|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\exp[-\tfrac{1}{2}\mathrm{tr}\,\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\,\underset{\sim}{\mu}'] \sum_{k=0}^{\infty}\{k!(\tfrac{n}{2})_k\}^{-1}d\underset{\sim}{L}^{(p,1)}f_k(\alpha,\underset{\sim}{L}),$$

where

(3.12) $$f_k(\alpha,\underset{\sim}{L}) = \int_{\underset{\sim}{Y}} |\alpha^2\underset{\sim}{I}_{p-1}-\underset{\sim}{Y}\underset{\sim}{Y}'|\;\{\mathrm{tr}[\tfrac{1}{4}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{L}\begin{pmatrix}\alpha^2 & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{Y}\,\underset{\sim}{Y}'\end{pmatrix}\underset{\sim}{L}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\,\underset{\sim}{\mu}']\}^k$$

$$\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{L}\begin{pmatrix}\alpha^2 & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{Y}\,\underset{\sim}{Y}'\end{pmatrix}\underset{\sim}{L}']d\underset{\sim}{Y}$$

or

(3.13) $$f(\theta,\alpha,\underset{\sim}{L}) = \sum_{k=0}^{\infty}\frac{\theta^k}{k!}\;f_k(\alpha,\underset{\sim}{L}) = \int_{\underset{\sim}{Y}} |\alpha^2\underset{\sim}{I}_{p-1}-\underset{\sim}{Y}\,\underset{\sim}{Y}'|\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{L}\begin{pmatrix}\alpha^2 & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{Y}\,\underset{\sim}{Y}'\end{pmatrix}$$

$$\underset{\sim}{L}'(\underset{\sim}{I}-2\theta\;\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\;\underset{\sim}{\mu}')]d\underset{\sim}{Y}$$

$$= e^{-\frac{1}{2}\alpha^2\underset{\sim}{\ell}'(\underset{\sim}{\Sigma}^{-1}-\frac{1}{2}\theta\;\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\;\underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1})\underset{\sim}{\ell}}\;\sum_{k=0}^{\infty}\sum_{\kappa}$$

$$\frac{(-1)^k\;C_\kappa[\frac{1}{2}\underset{\sim}{L}_1'(\underset{\sim}{\Sigma}^{-1}-\frac{1}{2}\theta\;\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\;\underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1})\underset{\sim}{L}_1]}{k!\;C_\kappa(\underset{\sim}{I}_{p-1})}\;\;g_\kappa(\alpha)\quad,$$

where

(3.14) $$g_\kappa(\alpha) = \int_{\underset{\sim}{Y}} |\alpha^2\underset{\sim}{I}_{p-1}-\underset{\sim}{Y}\;\underset{\sim}{Y}'|\;C_\kappa(\underset{\sim}{Y}\;\underset{\sim}{Y}')d\underset{\sim}{Y}$$

$$= |\alpha|^{np-n+p-1+2k}\pi^{\frac{1}{2}(p-1)(n-1)}(\tfrac{n-1}{2})_\kappa\;\Gamma_{p-1}(\tfrac{p+2}{2})\;C_\kappa(\underset{\sim}{I}_{p-1})\;/$$

$$\Gamma_{p-1}(\tfrac{n+p+1}{2},\kappa)\quad\text{if}\quad n \geq p\;;$$

$$= |\alpha|^{np-n+p-1+2k}\pi^{\frac{1}{2}(p-1)(n-1)}(\tfrac{p-1}{2})_\kappa\;\Gamma_{n-1}(\tfrac{n+2}{2})C_\kappa(\underset{\sim}{I}_{n-1})\;/$$

$$\Gamma_{n-1}(\tfrac{n+p+1}{2},\kappa)\quad\text{if}\quad n \leq p\;.$$

Substituting (3.14) in (3.13), we get $f(\theta,\alpha,\underset{\sim}{L})$ and the coefficient of $\theta^k/k!$ from $f(\theta,\alpha,\underset{\sim}{L})$ gives $f_k(\alpha,\underset{\sim}{L})$. Then using this value in (3.11), we get the joint density function of $\underset{\sim}{L}$ and α. Integrating α, we get the density function of $\underset{\sim}{L}$ as

$$(3.15) \qquad |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \exp[-\tfrac{1}{2}\mathrm{tr}\ \underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\ \underset{\sim}{\mu}'] \sum_{k=0}^{\infty} \{k!(\tfrac{n}{2})_k\}^{-1} d\underset{\sim}{L}^{(p,1)}\ f_k(\underset{\sim}{L})$$

where

$$(3.16) \qquad f(\theta,\underset{\sim}{L}) = \sum_{k=0}^{\infty} \theta^k f_k(\underset{\sim}{L})/k!$$

$$= c_1 \sum_{k=0}^{\infty} (-1)^k \binom{\frac{1}{2}pn+k-1}{k} \{\underset{\sim}{\ell}'(\underset{\sim}{\Sigma}^{-1}-\tfrac{1}{2}\theta\ \underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\ \underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1})\underset{\sim}{\ell}\}^{-\frac{1}{2}pn-k}$$

$$\sum_{\kappa} (\tfrac{n-1}{2})_\kappa\ \{(\tfrac{n+p+1}{2})_\kappa\}^{-1}\ C_\kappa[\underset{\sim}{L}_1'(\underset{\sim}{\Sigma}^{-1}-\tfrac{1}{2}\theta\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\ \underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1})\underset{\sim}{L}_1] \text{ if } n \geq p;$$

$$= c_2 \sum_{k=0}^{\infty} (-1)^k \binom{\frac{1}{2}pn+k-1}{k} \{\underset{\sim}{\ell}'(\underset{\sim}{\Sigma}^{-1}-\tfrac{1}{2}\theta\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\ \underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1})\underset{\sim}{\ell}\}^{-\frac{1}{2}pn-k}$$

$$\sum_{\kappa} (\tfrac{n-1}{2})_\kappa\ \{(\tfrac{n+p+1}{2})_\kappa\}^{-1} C_\kappa[\underset{\sim}{L}_1'(\underset{\sim}{\Sigma}^{-1}-\tfrac{1}{2}\theta\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\ \underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1})\underset{\sim}{L}_1] \text{ if } n \leq p;$$

where c_1 and c_2 are defined in (3.5) and (3.6). Note that for $n \leq p$, $\kappa = \{k_1,\ldots,k_{n-1}\}$, while for $n \geq p$, $\kappa = \{k_1,\ldots,k_{p-1}\}$. For $n \geq p$, (3.16) can be written as

$$(3.16') \quad f(\theta,\underset{\sim}{L}) = c_1 \sum_{k=0}^{\infty} \binom{\frac{1}{2}pn+k-1}{k} \{tr\underset{\sim}{\Sigma}^{-1}-\tfrac{1}{2}\theta tr\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\ \underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\}^{-\frac{1}{2}pn-k}$$

$$\sum_{\kappa} \left(\frac{p+2}{2}\right)_{\kappa} \left\{\left(\frac{n+p+1}{2}\right)_{\kappa}\right\}^{-1} C_{\kappa}[\underset{\sim}{L}_1'(\underset{\sim}{\Sigma}^{-1}-\tfrac{1}{2}\theta\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\ \underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1})\underset{\sim}{L}_1] \ .$$

Note that explicit expression will be obtained by evaluating the coefficient of $\theta^k/k!$ from $f(\theta,\underset{\sim}{L})$. When $\underset{\sim}{\mu} = \underset{\sim}{0}$, we get the results mentioned in (3.5) and (3.6).

4. Distributions of CVCLR's in the two sample case. Let us consider the joint density function of $\underset{\sim}{S}:p \times p$ and $\underset{\sim}{X}:p \times n$ as given by

$$(4.1) \quad \{2^{\frac{1}{2}p(m+n)}\Pi^{\frac{1}{2}pn}\Gamma_p(\tfrac{1}{2}m)|\underset{\sim}{\Sigma}_2|^{\frac{1}{2}m}|\underset{\sim}{\Sigma}_1|^{\frac{1}{2}n}\}^{-1}|\underset{\sim}{S}|^{\frac{1}{2}(m-p-1)}$$

$$\exp[-\tfrac{1}{2}tr\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{S}-\tfrac{1}{2}tr\underset{\sim}{\Sigma}_1^{-1}(\underset{\sim}{X}-\underset{\sim}{\mu})(\underset{\sim}{X}-\underset{\sim}{\mu})'] \ ,$$

where $\underset{\sim}{\mu}:p \times n$ is a matrix having fixed or random elements. The joint distribution of the random elements of $\underset{\sim}{\mu}$ will have measure $d\mu$, say. First we shall consider the case when $\underset{\sim}{\mu}$ is fixed.

Let us use the transformations

$$(4.2) \qquad \underset{\sim}{S} = \underset{\sim}{H}\ \underset{\sim}{D}_{\lambda}\ \underset{\sim}{H}', \quad \underset{\sim}{D}_{\lambda}^{-\frac{1}{2}}\ \underset{\sim}{H}'\underset{\sim}{X} = \underset{\sim}{\Gamma}\ \begin{pmatrix} \alpha & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{Y} \end{pmatrix}\ \underset{\sim}{\Delta}'$$

where $\underset{\sim}{H}:p \times p$ is a random orthogonal matrix, $\underset{\sim}{D}_{\lambda} = diag(\lambda_1,\ldots,\lambda_p)$, the first column vectors of orthogonal matrices $\underset{\sim}{\Gamma}:p \times p$ and $\underset{\sim}{\Delta}:n \times n$ have random elements, $\alpha \neq 0$, α^2 = maximum ch. root of $\underset{\sim}{X}\ \underset{\sim}{X}'\underset{\sim}{S}^{-1}$ and $\underset{\sim}{Y}:(p-1) \times (n-1)$ is a random matrix.

Note that the first column vector of $\underset{\sim}{\Delta}$ is the ch. vector of the matrix $\underset{\sim}{X}'\underset{\sim}{S}^{-1}\underset{\sim}{X}$ corresponding to α^2, while the first column vector of $(\underset{\sim}{H}\ \underset{\sim}{D}_{\lambda}^{\frac{1}{2}}\ \underset{\sim}{\Gamma})$ is the ch. ~~vector~~ ~~of~~ the ~~matrix~~ $(\underset{\sim}{X}\ \underset{\sim}{X}'\underset{\sim}{S}^{-1})$ corresponding to α^2 .

With the help of lemma 1, the jacobian of the transformation given in (4.2) can be obtained as

$$(4.3) \qquad J(\underset{\sim}{S},\underset{\sim}{X};\underset{\sim}{H},\underset{\sim}{D}_{\lambda},\underset{\sim}{Y},\alpha,\underset{\sim}{\Gamma},\underset{\sim}{\Delta}) = \Pi^{\frac{1}{2}(p^2+p+n)}\{\Gamma_p(\tfrac{1}{2}p)\Gamma(\tfrac{n}{2})\Gamma(\tfrac{1}{2}p)\}^{-1}|\alpha|^{n-p}|\underset{\sim}{D}_{\lambda}|^{\frac{1}{2}n}$$

$$|\alpha^2\underset{\sim}{I}_{p-1}-\underset{\sim}{Y}\,\underset{\sim}{Y}'|\{\prod_{i<j=1}^{p}(\lambda_i-\lambda_j)\}d\underset{\sim}{H}^{(p,p)}d\underset{\sim}{\Gamma}^{(p,1)}d\underset{\sim}{\Delta}^{(n,1)} \ .$$

Using (4.3) and (4.2) in (4.1), we get the joint density function of $\underset{\sim}{Y},\underset{\sim}{H},\underset{\sim}{D}_{\lambda}$, α, $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Delta}$ when $\underset{\sim}{\mu}$ is fixed. Now, for the joint density of the required joint ch. vectors, we use the following transformations:

First we consider the tramsformations when $n \leq p$. Let $\underset{\sim}{Y} = \underset{\sim}{\Gamma}_1\binom{\underset{\sim}{D}_{\alpha}}{\underset{\sim}{0}}\underset{\sim}{\Delta}_1'$ where $\underset{\sim}{D}_{\alpha} = \text{diag}\,(\alpha_2,\ldots,\alpha_n)$, $\alpha_i \neq 0$, $\underset{\sim}{\Delta}_1:(n-1) \times (n-1)$ is a random orthogonal matrix and the first $(n-1)$ column vectors of an orthogonal matrix $\underset{\sim}{\Gamma}_1:(p-1) \times (p-1)$ have random elements. Let us introduce a random orthogonal matrix $\underset{\sim}{\Gamma}_2:(p-n) \times (p-n)$ with measure $d\underset{\sim}{\Gamma}_2^{(p-n,p-n)}$ and be independent of $\underset{\sim}{Y},\underset{\sim}{H},\underset{\sim}{D}_{\lambda},\alpha,\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Delta}$. Then, use the transformations

$$(4.4) \qquad \underset{\sim}{Y} = \underset{\sim}{\Gamma}_1\binom{\underset{\sim}{D}_{\alpha}}{\underset{\sim}{0}}\underset{\sim}{\Delta}_1', \quad \underset{\sim}{A} = \underset{\sim}{H}\,\underset{\sim}{D}_{\lambda}^{\frac{1}{2}}\,\underset{\sim}{\Gamma}\begin{pmatrix}1 & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{\Gamma}_1\end{pmatrix}\begin{pmatrix}\underset{\sim}{I}_n & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{\Gamma}_2\end{pmatrix} \ .$$

(Note that the first column vector of $\underset{\sim}{A}$ is the required vector as defined above. If $\underset{\sim}{A} = (\underset{\sim}{A}_1,\underset{\sim}{A}_2)$, $\underset{\sim}{A}_1:p \times n$ and $\underset{\sim}{A}_2:p \times (p-n)$, then $\underset{\sim}{A}_2$ depends on random measure $d\underset{\sim}{\Gamma}_2$ and hence we shall integrate $\underset{\sim}{A}_2$). The jacobian of the transformations is

$$(4.5)\qquad J(\underset{\sim}{Y},\underset{\sim}{\Gamma},\underset{\sim}{\Gamma}_2,\underset{\sim}{D}_\lambda \underset{\sim}{H};\underset{\sim}{A},\underset{\sim}{D}_\alpha,\underset{\sim}{\Delta}_1) = J(\underset{\sim}{Y};\underset{\sim}{\Gamma}_1,\underset{\sim}{D}_\alpha,\underset{\sim}{\Delta}_1)J(\underset{\sim}{\Gamma}_1,\underset{\sim}{\Gamma},\underset{\sim}{\Gamma}_2,\underset{\sim}{D}_\lambda,\underset{\sim}{H};\underset{\sim}{A})$$

$$= \frac{\Pi^{\frac{1}{2}(n-1)(p+n-2)}\left(\prod_{i=2}^{n}|\alpha_i|^{p-n}\prod_{i<j=2}^{n}(\alpha_i^2-\alpha_j^2)\left(\prod_{i=1}^{p}\lambda_i\right)^{\frac{1}{2}}\{\Gamma_p(\tfrac{1}{2}p)\}^2 d\underset{\sim}{\Delta}_1^{(n-1,n-1)}\right.}{\Pi^{\frac{1}{2}p^2}\Gamma_{n-1}(\frac{p-1}{2})\Gamma_{n-1}(\frac{n-1}{2})\prod_{i<j=1}^{p}(\lambda_i-\lambda_j)d\underset{\sim}{\Gamma}^{(p,1)}d\underset{\sim}{\Gamma}_2^{(p-n,p-n)}\ d\underset{\sim}{H}^{(p,p)}}\ .$$

Using this in the joint density function of $\underset{\sim}{\Gamma}_2,\underset{\sim}{\Gamma},\underset{\sim}{D}_\lambda,\underset{\sim}{H},\underset{\sim}{\Delta}$ and $\underset{\sim}{Y}$, we get the joint density function of $\underset{\sim}{A},\underset{\sim}{\Delta}_1,\underset{\sim}{\Delta}$ and $\underset{\sim}{D}_\alpha = \text{diag}(\alpha_1,\alpha_2,\ldots,\alpha_n)$ (where $\alpha = \alpha_1$) as

$$(4.6)\qquad c_5|\underset{\sim}{A}'\underset{\sim}{A}|^{\frac{1}{2}(m+n-p)}\left(\prod_{i=1}^{n}|\alpha_i|\right)^{p-n}\left(\prod_{i<j=1}^{n}(\alpha_i^2-\alpha_j^2)\right)\exp[-\tfrac{1}{2}\text{tr}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{A}\underset{\sim}{A}'-\tfrac{1}{2}\text{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{A}\begin{pmatrix}\underset{\sim}{D}_\alpha^2 & \underset{\sim}{0}\\ \underset{\sim}{0} & \underset{\sim}{0}\end{pmatrix}\underset{\sim}{A}']$$

$$\exp[\text{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{A}\begin{pmatrix}\underset{\sim}{D}_\alpha\\ \underset{\sim}{0}\end{pmatrix}\begin{pmatrix}1 & \underset{\sim}{0}'\\ \underset{\sim}{0} & \underset{\sim}{\Delta}_1\end{pmatrix}\underset{\sim}{\Delta}'\underset{\sim}{\mu}'-\tfrac{1}{2}\text{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{\mu}\ \underset{\sim}{\mu}']d\underset{\sim}{\Delta}_1^{(n-1,n-1)}d\underset{\sim}{\Delta}^{(n,1)}\ ,$$

where

$$c_5^{-1} = \{2^{\frac{1}{2}p(m+n)}\Gamma_p(\tfrac{1}{2}m)\Gamma_n(\tfrac{1}{2}n)\Gamma_n(\tfrac{1}{2}p)\Pi^{\frac{1}{2}p^2}\}\{\Pi^{\frac{1}{2}n^2}\Gamma_p(\tfrac{1}{2}p)\}^{-1}\{|\underset{\sim}{\Sigma}_2|^{\frac{1}{2}m}|\underset{\sim}{\Sigma}_1|^{\frac{1}{2}n}\}\quad .$$

Now, $\underset{\sim}{A} = (\underset{\sim}{A}_1,\underset{\sim}{A}_2)$, $\underset{\sim}{A}_1$:p x n. We integrate $\underset{\sim}{A}_2$ from (4.6). This requires the value of the following integral,

$$(4.7)\qquad \int_{\underset{\sim}{A}_2}|\underset{\sim}{A}_2'(\underset{\sim}{I}-\underset{\sim}{A}_1(\underset{\sim}{A}_1'\underset{\sim}{A}_1)^{-1}\underset{\sim}{A}_1')A_2|^{\frac{1}{2}(m+n-p)}\exp[-\tfrac{1}{2}\text{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{A}_2\underset{\sim}{A}_2']\ d\underset{\sim}{A}_2\quad .$$

Let us write $\underset{\sim}{T}_1 = \underset{\sim}{A}_1(\underset{\sim}{A}_1'\underset{\sim}{A}_1)^{-\frac{1}{2}}$ and $\underset{\sim}{T}_2\underset{\sim}{T}_2' = \underset{\sim}{I}_n - \underset{\sim}{T}_1\underset{\sim}{T}_1'$ such that $\underset{\sim}{T} = (\underset{\sim}{T}_1,\underset{\sim}{T}_2)$ is an orthogonal matrix. Let $\underset{\sim}{Q}_1 = \underset{\sim}{T}_1'\ \underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{T}_1$, $\underset{\sim}{Q}_2 = \underset{\sim}{T}_1'\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{T}_2$ and $\underset{\sim}{Q}_3 = \underset{\sim}{T}_2'\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{T}_2$. Then using the transformation $\underset{\sim}{T}'\underset{\sim}{A}_2 = \begin{pmatrix}\underset{\sim}{A}_{21}\\ \underset{\sim}{A}_{22}\end{pmatrix}$, (4.7) becomes

$$(4.8)\qquad \int_{\underset{\sim}{A}_{22}}\int_{\underset{\sim}{A}_{21}}|\underset{\sim}{A}_{22}'\underset{\sim}{A}_{22}|^{\frac{1}{2}(m+n-p)}\exp[-\tfrac{1}{2}\text{tr}\underset{\sim}{Q}_1\underset{\sim}{A}_{21}\underset{\sim}{A}_{21}'-\text{tr}\underset{\sim}{Q}_2\underset{\sim}{A}_{22}\underset{\sim}{A}_{21}'-\tfrac{1}{2}\text{tr}\underset{\sim}{Q}_3\underset{\sim}{A}_{22}\underset{\sim}{A}_{21}']d\underset{\sim}{A}_{21}d\underset{\sim}{A}_{22}$$

$$= (2\Pi)^{\frac{1}{2}n(p-n)}|\underset{\sim}{Q}_1|^{-\frac{1}{2}(p-n)}\Pi^{\frac{1}{2}(p-n)^2}\{\Gamma_{p-n}(\tfrac{p-n}{2})\}^{-1}\int_{\underset{\sim}{S}>0}|\underset{\sim}{S}|^{\frac{1}{2}(m+n-p-1)}$$

$$\exp[-\tfrac{1}{2}\text{tr}(\underset{\sim}{Q}_3-\underset{\sim}{Q}_2'\underset{\sim}{Q}^{-1}\underset{\sim}{Q}_2)\underset{\sim}{S}]d\underset{\sim}{S}$$

$$= 2^{\frac{1}{2}n(p-n)+\frac{1}{2}(p-n)(m)}\Pi^{\frac{1}{2}n(p-n)+\frac{1}{2}(p-n)^2}|\underset{\sim}{Q}_1|^{-\frac{1}{2}(p-n)}|\underset{\sim}{Q}_3-\underset{\sim}{Q}_2'\underset{\sim}{Q}_1^{-1}\underset{\sim}{Q}_2|^{-\frac{1}{2}m}$$

$$\Gamma_{p-n}(\tfrac{m}{2})\{\Gamma_{p-n}(\tfrac{p-n}{2})\}^{-1}$$

$$= 2^{\frac{1}{2}(p-n)(m+n)}\Pi^{\frac{1}{2}p(p-n)}|\underset{\sim}{\Sigma}_2|^{\frac{1}{2}m}|\underset{\sim}{A}_1'\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{A}_1|^{\frac{1}{2}(m+n-p)}|\underset{\sim}{A}_1'\underset{\sim}{A}_1|^{-\frac{1}{2}(m+n-p)}$$

$$\Gamma_{p-n}(\tfrac{m}{2})\{\Gamma_{p-n}(\tfrac{p-n}{2})\}^{-1}\quad .$$

Hence, we get the joint density function of $\underset{\sim}{A}_1, \underset{\sim}{D}_\alpha, \underset{\sim}{\Delta}_1$ and $\underset{\sim}{\Delta}$ as

$$(4.9) \quad c_6 |\underset{\sim}{A}_1' \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{A}_1|^{\frac{1}{2}(m+n-p)} (\prod_{i=1}^{n} |\alpha_i|)^{p-n} (\prod_{i<j=1}^{n} (\alpha_i^2 - \alpha_j^2)) \exp[-\tfrac{1}{2} tr \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{A}_1 \underset{\sim}{A}_1' - \tfrac{1}{2} tr \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{A}_1 \underset{\sim}{D}_\alpha^2 \underset{\sim}{A}_1']$$

$$\exp[tr \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{A}_1 \underset{\sim}{D}_\alpha \begin{pmatrix} 1 & 0 \\ \underset{\sim}{0} & \underset{\sim}{\Delta}_1' \end{pmatrix} \underset{\sim}{\Delta}' \underset{\sim}{\mu}' - \tfrac{1}{2} tr \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{\mu}\, \underset{\sim}{\mu}'] \; d\underset{\sim}{\Delta}_1^{(n-1,n-1)} d\underset{\sim}{\Delta}^{(n,1)} \text{ for } n \leq p$$

and $$c_6^{-1} = \{2^{\frac{1}{2}n(n+m)} \Gamma_n(\tfrac{m+n-p}{2}) \Gamma_n(\tfrac{n}{2}) \Pi^{\frac{1}{2}pn - \frac{1}{2}n^2}\} |\underset{\sim}{\Sigma}_1|^{\frac{1}{2}n} \quad .$$

Similarly, for $n \geq p$, it is easy to obtain the joint density function of $\underset{\sim}{A}: p \times p$, $\underset{\sim}{\Delta}_1: (n-1) \times (n-1)$ having random elements in first $(p-1)$ columns, $\underset{\sim}{\Delta}$ and $\underset{\sim}{D}_\alpha = diag(\alpha_1, \ldots, \alpha_p)$ as

$$(4.10) \quad c_7 |\underset{\sim}{A}' \underset{\sim}{A}|^{\frac{1}{2}(m+n-p)} (\prod_{i=1}^{p} |\alpha_i|)^{n-p} (\prod_{i<j=1}^{p} (\alpha_i^2 - \alpha_j^2)) \exp[-\tfrac{1}{2} tr \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{A} \underset{\sim}{A}' - \tfrac{1}{2} tr \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{A} \underset{\sim}{D}_\alpha^2 \underset{\sim}{A}']$$

$$\exp(tr \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{A} (\underset{\sim}{D}_\alpha \underset{\sim}{0}) \begin{pmatrix} 1 & 0 \\ \underset{\sim}{0} & \underset{\sim}{\Delta}_1' \end{pmatrix} \underset{\sim}{\Delta}' \underset{\sim}{\mu}' - \tfrac{1}{2} tr \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{\mu}\, \underset{\sim}{\mu}') d\underset{\sim}{\Delta}_1^{(n-1,p-1)} d\underset{\sim}{\Delta}^{(n,1)} \text{ for } n \geq p$$

and $$c_7^{-1} = \{2^{\frac{1}{2}p(m+n)} \Gamma_p(\tfrac{1}{2}m) \Gamma_p(\tfrac{1}{2}n)\} \{|\underset{\sim}{\Sigma}_2|^{\frac{1}{2}m} |\underset{\sim}{\Sigma}_1|^{\frac{1}{2}n}\} \quad .$$

Now, the problem is to obtain the density function of the first column vector of $\underset{\sim}{A}$(or $\underset{\sim}{A}_1$) and/or $\underset{\sim}{\Delta}$. The general problem is extremely difficult and hence we consider below some particular cases.

__Case 1__. Let $\underset{\sim}{\mu} = \underset{\sim}{0}$. Let $\underset{\sim}{\Delta}, \underset{\sim}{\Delta}_1$ and $(\underset{\sim}{A}$ or $\underset{\sim}{A}_1, \underset{\sim}{D}_\alpha^2)$ are independently distributed and their density functions are respectively given by

$$d\underset{\sim}{\Delta}^{(n,1)}, \quad d\underset{\sim}{\Delta}_1^{(n-1,p-1)} \text{ if } n \geq p \text{ or } d\underset{\sim}{\Delta}_1^{(n-1,n-1)} \text{ if } n \leq p$$

and if $\alpha_i^2 = \lambda_i$, $i = 1, 2, \ldots, p$ or n,

$$(4.9') \quad c_6 |\underset{\sim}{A}_1' \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{A}_1|^{\frac{1}{2}(m+n-p)} (\prod_{i=1}^{n} \lambda_i)^{\frac{1}{2}(p-n-1)} (\prod_{i<j=1}^{n} (\lambda_i - \lambda_j)) \exp[-\tfrac{1}{2} tr \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{A}_1 \underset{\sim}{A}_1' - \tfrac{1}{2} tr \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{A}_1 \underset{\sim}{D}_\lambda \underset{\sim}{A}_1']$$

for $n \leq p$, $A_1: p \times n$ and $D_\lambda = diag(\lambda_1, \ldots, \lambda_n)$ and

$$(4.10') \quad c_7 |\underset{\sim}{A}' \underset{\sim}{A}|^{\frac{1}{2}(m+n-p)} (\prod_{i=1}^{p} \lambda_i)^{\frac{1}{2}(n-p-1)} (\prod_{i<j=1}^{p} (\lambda_i - \lambda_j)) \exp[-\tfrac{1}{2} tr \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{A} \underset{\sim}{A}' - \tfrac{1}{2} tr \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{A} \underset{\sim}{D}_\lambda \underset{\sim}{A}']$$

for $n \geq p$, $\underset{\sim}{A}: p \times p$ and $\underset{\sim}{D}_\lambda = diag(\lambda_1, \ldots, \lambda_p)$.

From (4.9'), we shall try to obtain the density function of the first column vector of $\underset{\sim}{A}_1$. Let $\underset{\sim}{A}_1 = (\underset{\sim}{a}, \underset{\sim}{A}_3)$, $\underset{\sim}{\Sigma}_2 = (\underset{\sim}{\Sigma}_2^{\frac{1}{2}})^2$, $\underset{\sim}{Z} = \underset{\sim}{\Sigma}_2^{\frac{1}{2}} \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{\Sigma}_2^{\frac{1}{2}}$ and $\underset{\sim}{b} = \underset{\sim}{\Sigma}_2^{-\frac{1}{2}}$ a.

Let us transform $\underset{\sim}{B} = \underset{\sim}{\Sigma}_2^{-\frac{1}{2}}\underset{\sim}{A}_3$. Then the joint density function of $\underset{\sim}{B}:p \times (n-1)$, $\underset{\sim}{a}$ and $\underset{\sim}{D}_\lambda$ is given by

$$(4.11)\quad c_6|\underset{\sim}{\Sigma}_2|^{\frac{1}{2}(n-1)}(\underset{\sim}{b}'\underset{\sim}{b})^{\frac{1}{2}(m+n-p)}|\underset{\sim}{B}'(\underset{\sim}{I}_p-\underset{\sim}{b}\underset{\sim}{b}'/(\underset{\sim}{b}'\underset{\sim}{b}))\underset{\sim}{B}|^{\frac{1}{2}(m+n-p)}(\prod_{i=1}^{n}\lambda_i)^{\frac{1}{2}(p-n-1)}$$

$$(\prod_{i<j=1}^{n}(\lambda_i-\lambda_j))\exp[-\tfrac{1}{2}\underset{\sim}{b}'\underset{\sim}{b}-\tfrac{1}{2}\lambda_1\underset{\sim}{b}'\underset{\sim}{Z}\underset{\sim}{b}-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}'\underset{\sim}{B}-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}'\underset{\sim}{Z}\underset{\sim}{B}\underset{\sim}{D}_1],$$

where $\quad \underset{\sim}{D}_1 = \mathrm{diag}(\lambda_2,\lambda_3,\ldots,\lambda_n)$.

For integration over $\underset{\sim}{B}:p \times (n-1)$, we require the following integral:

$$(4.12)\quad h(\underset{\sim}{b},\underset{\sim}{D}_1) = \int_B|\underset{\sim}{B}'(\underset{\sim}{I}_p-\underset{\sim}{b}\underset{\sim}{b}'/(\underset{\sim}{b}'\underset{\sim}{b}))\underset{\sim}{B}|^{\frac{1}{2}(m+n-p)}\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}'\underset{\sim}{B}-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}'\underset{\sim}{Z}\underset{\sim}{B}\underset{\sim}{D}_1]d\underset{\sim}{B} \quad .$$

Introduce a random orthogonal matrix $\underset{\sim}{H}:(n-1) \times (n-1)$ with measure $d\underset{\sim}{H}^{(n-1,n-1)}$, use the transformation $\underset{\sim}{B} \to \underset{\sim}{B}\underset{\sim}{H}$ and then integrate over $\underset{\sim}{H}$. This reduces (4.12) to

$$(4.13)\quad h(\underset{\sim}{b},\underset{\sim}{D}_1) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(-\frac{1}{2}\underset{\sim}{D}_1)}{k!C_\kappa(\underset{\sim}{I}_{n-1})}\int_B|\underset{\sim}{B}'(\underset{\sim}{I}_p-\underset{\sim}{b}\underset{\sim}{b}'/(\underset{\sim}{b}'\underset{\sim}{b}))\underset{\sim}{B}|^{\frac{1}{2}(m+n-p)}C_\kappa(\underset{\sim}{B}'\underset{\sim}{Z}\underset{\sim}{B})e^{-\frac{1}{2}\mathrm{tr}\underset{\sim}{B}'\underset{\sim}{B}}d\underset{\sim}{B}.$$

Let us make the transformation $\underset{\sim}{B} = \underset{\sim}{L}_1\underset{\sim}{D}_\delta\underset{\sim}{M}$ where $\underset{\sim}{M}:(n-1) \times (n-1)$ is a random orthogonal matrix, $\delta_i \neq 0$, $\underset{\sim}{D}_\delta = \mathrm{diag}(\delta_1,\ldots,\delta_{n-1})$ and $\underset{\sim}{L}_1:p \times (n-1)$ has random elements such that $\underset{\sim}{L}_1'\underset{\sim}{L}_1 = \underset{\sim}{I}_{n-1}$. Let $\underset{\sim}{L}:p \times p$ be a complete orthogonal matrix due to $\underset{\sim}{L}_1$. Then the jacobian of the transformation is

$$J(\underset{\sim}{B};\underset{\sim}{L},\underset{\sim}{D}_\delta,\underset{\sim}{M}) = \frac{\Pi^{\frac{1}{2}(n-1)(p+n-1)}}{\Gamma_{n-1}(\frac{1}{2}p)\Gamma_{n-1}(\frac{n-1}{2})}\,d\underset{\sim}{L}^{(p,n-1)}d\underset{\sim}{M}^{(n-1,n-1)}(\prod_{i=1}^{n-1}|\delta_i|)^{p-n+1}$$

$$(\prod_{i<j=1}^{n-1}(\delta_i^2-\delta_j^2)) \quad .$$

Use the transformation $\delta_i^2 = \omega_i, i = 1,2,\ldots,n-1$ and $\underset{\sim}{D}_\omega = \mathrm{diag}(\omega_1,\ldots,\omega_{n-1})$. Using this in (4.13) we get

$$(4.14)\quad h(\underset{\sim}{b},\underset{\sim}{D}_1) = \frac{\Pi^{\frac{1}{2}(n-1)(p+n-1)}}{\Gamma_{n-1}(\frac{1}{2}p)\Gamma_{n-1}(\frac{n-1}{2})}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(-\frac{1}{2}\underset{\sim}{D}_1)}{k!C_\kappa(\underset{\sim}{I}_{n-1})}\int_{\underset{\sim}{D}_\omega}\int_{\underset{\sim}{L}}|\underset{\sim}{D}_\omega|^{\frac{1}{2}m}$$

$$[1-\underset{\sim}{b}'\underset{\sim}{L}_1\underset{\sim}{L}_1'\underset{\sim}{b}/\underset{\sim}{b}'\underset{\sim}{b}]^{\frac{1}{2}(m+n-p)}(\prod_{i<j=1}^{n-1}(\omega_i-\omega_j))C_\kappa(\underset{\sim}{L}_1'\underset{\sim}{Z}\underset{\sim}{L}_1\underset{\sim}{D}_\omega)\exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{D}_\omega)d\underset{\sim}{L}^{(p,n-1)}d\underset{\sim}{D}_\omega \quad .$$

For the integration over $\underset{\sim}{D}_{\omega}$, we introduce a random orthogonal matrix $\underset{\sim}{H}:(n-1) \times (n-1)$ with normalized measure $d\underset{\sim}{H}^{(n-1,n-1)}$. Let us transform $\underset{\sim}{L}$ by $\underset{\sim}{L} \rightarrow \underset{\sim}{L}\begin{pmatrix} \underset{\sim}{H} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I}_{p-n+1} \end{pmatrix}$ and then integrating over $\underset{\sim}{H}$ and then $\underset{\sim}{D}_{\omega}$, we can write (4.14) as

$$(4.15) \quad h(\underset{\sim}{b},\underset{\sim}{D}_1) = \frac{2^{\frac{1}{2}(n-1)(m+n)}\pi^{\frac{1}{2}(n-1)p}}{\Gamma_{n-1}(\frac{1}{2}p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma_{n-1}(\frac{m+n}{2},\kappa)C_{\kappa}(-\underset{\sim}{D}_1)}{k!C_{\kappa}(\underset{\sim}{I}_{n-1})}$$

$$\int_{\underset{\sim}{L}} [1-\underset{\sim}{b}'\underset{\sim}{L}_1\underset{\sim}{L}_1'\underset{\sim}{b}/(\underset{\sim}{b}'b)]^{\frac{1}{2}(m+n-p)}C_{\kappa}(\underset{\sim}{L}_1'\underset{\sim}{Z}\underset{\sim}{L}_1)d\underset{\sim}{L}^{(p,n-1)} \quad .$$

Let us write

$$(4.16) \quad h_1(b,\kappa) = \int_{O(p)} [\underset{\sim}{b}'(\underset{\sim}{I}-\underset{\sim}{L}_1\underset{\sim}{L}_1')\underset{\sim}{b}]^{\frac{1}{2}(m+n-p)}C_{\kappa}(\underset{\sim}{L}_1'\underset{\sim}{Z}\underset{\sim}{L}_1)d\underset{\sim}{L}^{(p,n-1)} \quad ,$$

Transform $\lambda_i/\lambda_1 = y_{i-1}$, $i = 2,3,\ldots,n$ in (4.11) and then integrate over λ_1 and $y_1,\ldots,y_{n-1}$. This gives us the density function of $\underset{\sim}{a}$ for $n \leq p$ as

$$(4.17) \quad c_8 \sum_{k=0}^{\infty} (-1)^k \binom{\frac{1}{2}pn+k-1}{k} \left(\frac{2}{\underset{\sim}{b}'Zb}\right)^{\frac{1}{2}pn+k} e^{-\frac{1}{2}\underset{\sim}{b}'\underset{\sim}{b}} \sum_{\kappa} \frac{(\frac{m+n}{2})_{\kappa}(\frac{p-1}{2})_{\kappa}}{(\frac{n+p+1}{2})_{\kappa}} h_1(\underset{\sim}{b},\kappa) ,$$

where

$$c_8 = \Gamma(\tfrac{1}{2}pn)\Gamma_{n-1}(\tfrac{n+2}{2})\Gamma_{n-1}(\tfrac{m+n}{2})\{2^{\frac{1}{2}(m+n)}\pi^{\frac{1}{2}(p-1)}\Gamma(\tfrac{m-p}{2})\Gamma(\tfrac{n}{2})\Gamma_{n-1}(\tfrac{n+p+1}{2})$$

$$\Gamma_{n-1}(\tfrac{m+n-p}{2})\}^{-1}|\underset{\sim}{Z}|^{\frac{1}{2}n}|\underset{\sim}{\Sigma}_2|^{-\frac{1}{2}}$$

and $h_1(\underset{\sim}{b},\kappa)$ is defined in (4.16).

Similarly, the density function of $\underset{\sim}{a}$ for $n \geq p$ can be given by

$$(4.18) \quad c_9 \sum_{k=0}^{\infty} (-1)^k \binom{\frac{1}{2}pn+k-1}{k} \left(\frac{2}{\underset{\sim}{b}'Zb}\right)^{\frac{1}{2}pn+k} e^{-\frac{1}{2}\underset{\sim}{b}'\underset{\sim}{b}} \sum_{\kappa} \frac{(\frac{m+n}{2})_{\kappa}(\frac{n-1}{2})_{\kappa}}{(\frac{n+p+1}{2})_{\kappa}} h_2(\underset{\sim}{b},\kappa) ,$$

where

$$c_9 = \Gamma(\tfrac{1}{2}pn)\Gamma_{p-1}(\tfrac{m+n}{2})\Gamma_{p-1}(\tfrac{p+2}{2})\{2^{\frac{1}{2}(m+n)}\Gamma_p(\tfrac{1}{2}m)\Gamma_{p-1}(\tfrac{n+p+1}{2})\Gamma(\tfrac{n}{2})\Gamma_{p-1}(\tfrac{1}{2}p)\}^{-1}|\underset{\sim}{Z}|^{\frac{1}{2}n}|\underset{\sim}{\Sigma}_2|^{-\frac{1}{2}}$$

and

$$(4.19) \quad h_2(\underset{\sim}{b},\kappa) = \int_{O(p)} [\underset{\sim}{b}'(\underset{\sim}{I}-\underset{\sim}{L}_1\underset{\sim}{L}_1')\underset{\sim}{b}]^{\frac{1}{2}(m+n-p)}C_{\kappa}(\underset{\sim}{L}_1'\underset{\sim}{Z}\underset{\sim}{L}_1)d\underset{\sim}{L}^{(p,p-1)}, \quad \underset{\sim}{L}_1:p \times (p-1)$$

and $\kappa = \{k_1,\ldots,k_{p-1}\}$.

Case 2. Let $\underset{\sim}{\mu} \neq \underset{\sim}{0}$ and the rank of $\underset{\sim}{\mu}$ be one. The density function of $\underset{\sim}{\Delta}$ is extremely difficult and hence we try to obtain the density function of $\underset{\sim}{a}$. For this purpose, we integrate from (4.19) and (4.10) $\underset{\sim}{\Delta}_1$ and $\underset{\sim}{\Delta}$ and transform $\alpha_i^2 = \lambda_i$ $i = 1,2,\ldots, p$ or n. Then, the joint density function of $\underset{\sim}{A}_1 : p \times n$ and $\underset{\sim}{D}_\lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ for $n \leq p$ is given by

$$(4.20) \quad c_6 |\underset{\sim}{A}_1' \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{A}_1|^{\frac{1}{2}(m+n-p)} |\underset{\sim}{D}_\lambda|^{\frac{1}{2}(p-n-1)} \Big(\prod_{i<j=1}^{n} (\lambda_i - \lambda_j)\Big) \exp[-\tfrac{1}{2}\operatorname{tr}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{A}_1\underset{\sim}{A}_1' - \tfrac{1}{2}\operatorname{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{A}_1\underset{\sim}{D}_\lambda\underset{\sim}{A}_1'$$

$$-\tfrac{1}{2}\operatorname{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{\mu}\underset{\sim}{\mu}'] \sum_{k=0}^{\infty} \{k!(\tfrac{n}{2})_k\}^{-1} \{\tfrac{1}{4}\operatorname{tr}\underset{\sim}{\mu}'\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{A}_1\underset{\sim}{D}_\lambda\underset{\sim}{A}_1'\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{\mu}\}^k$$

and the joint density function of $\underset{\sim}{A} : p \times p$ and $\underset{\sim}{D}_\lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_p)$ for $n \geq p$ is given by

$$(4.21) \quad c_7 |\underset{\sim}{A}'\underset{\sim}{A}|^{\frac{1}{2}(m+n-p)} |\underset{\sim}{D}_\lambda|^{\frac{1}{2}(n-p-1)} \Big(\prod_{i<j=1}^{p} (\lambda_i - \lambda_j)\exp[-\tfrac{1}{2}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{A}\underset{\sim}{A}' - \tfrac{1}{2}\operatorname{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{A}\underset{\sim}{D}_\lambda\underset{\sim}{A}' - \tfrac{1}{2}\operatorname{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{\mu}\underset{\sim}{\mu}']$$

$$\sum_{k=0}^{\infty} \{k!\ (\tfrac{1}{2}n)_k\}^{-1} [\operatorname{tr}(\tfrac{1}{4}\underset{\sim}{\Sigma}_1^{-1}\ \underset{\sim}{A}\underset{\sim}{D}_\lambda\underset{\sim}{A}'\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{\mu}\underset{\sim}{\mu}')]^k.$$

Case (i). Let us write $\underset{\sim}{Z}_1 = \underset{\sim}{\Sigma}_2^{\frac{1}{2}}\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$ and $\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}\underset{\sim}{\mu}\underset{\sim}{\mu}'\underset{\sim}{\Sigma}_1^{-\frac{1}{2}} = \underset{\sim}{x}\ \underset{\sim}{x}'$, where $\underset{\sim}{x} : p \times 1$ is a vector. Note that $\underset{\sim}{Z} = \underset{\sim}{Z}_1\underset{\sim}{Z}_1'$. First of all, we consider $n \leq p$. Let us write $\underset{\sim}{A}_1 = (\underset{\sim}{a}\ \underset{\sim}{A}_3)$, $\underset{\sim}{b} = \underset{\sim}{\Sigma}_2^{-\frac{1}{2}}\underset{\sim}{a}$, Using the transformation $\underset{\sim}{B} = \underset{\sim}{\Sigma}_2^{-\frac{1}{2}}\underset{\sim}{A}_3$ and then integrating over $\underset{\sim}{B}$, we get the joint density function of $\underset{\sim}{a}$ and $\underset{\sim}{D}_\lambda = \operatorname{diag}(\lambda_1, \underset{\sim}{D}_1)$, $\underset{\sim}{D}_1 = \operatorname{diag}(\lambda_2,\ldots,\lambda_n)$ as

$$(4.22) \quad c_6 |\underset{\sim}{\Sigma}_2|^{\frac{1}{2}(n-1)} |\underset{\sim}{D}_\lambda|^{\frac{1}{2}(p-n-1)} \Big(\prod_{i<j=1}^{n} (\lambda_i - \lambda_j)\Big) \exp[-\tfrac{1}{2}(\underset{\sim}{x}'\underset{\sim}{x} + \underset{\sim}{b}'\underset{\sim}{b})] \sum_{k=0}^{\infty} \{(\tfrac{n}{2})_k\}^{-1} n(\underset{\sim}{b}, \underset{\sim}{D}_\lambda, k)$$

$$\text{for } n \leq p,$$

where

$$(4.23) \quad n(\underset{\sim}{b}, \underset{\sim}{D}_\lambda, k) = (k!)^{-1} \int_{\underset{\sim}{B}} \left| \begin{pmatrix} \underset{\sim}{b}' \\ \underset{\sim}{B}' \end{pmatrix} (\underset{\sim}{b}\ \underset{\sim}{B}) \right|^{\frac{1}{2}(m+n-p)} [\tfrac{1}{4}\underset{\sim}{x}'\underset{\sim}{Z}_1'(\underset{\sim}{b}\ \underset{\sim}{B})\underset{\sim}{D}_\lambda \begin{pmatrix} \underset{\sim}{b}' \\ \underset{\sim}{B}' \end{pmatrix} \underset{\sim}{Z}_1\underset{\sim}{x}]^k$$

$$\exp[-\tfrac{1}{2}\operatorname{tr}\underset{\sim}{B}\underset{\sim}{B}' - \tfrac{1}{2}\operatorname{tr}\underset{\sim}{Z}_1'(\underset{\sim}{b}\ \underset{\sim}{B})\underset{\sim}{D}_\lambda \begin{pmatrix} \underset{\sim}{b}' \\ \underset{\sim}{B}' \end{pmatrix} \underset{\sim}{Z}_1] d\underset{\sim}{B} \quad .$$

Hence

$$(4.24)\quad h_\theta(b,\underset{\sim}{D}_\lambda) = \sum_{k=0}^{\infty} \theta^k\, h(\underset{\sim}{b},\underset{\sim}{D}_\lambda,k) = \int_{\underset{\sim}{B}} \left| \begin{pmatrix} \underset{\sim}{b}' \\ \underset{\sim}{B}' \end{pmatrix} (\underset{\sim}{b}\ \underset{\sim}{B}) \right|^{\frac{1}{2}(m+n-p)}$$

$$\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{BB}'-\tfrac{1}{2}\mathrm{tr}Q(\theta)\underset{\sim}{B}\ \underset{\sim}{D}_1\underset{\sim}{B}']d\underset{\sim}{B}\ \exp[-\frac{\lambda_1}{2}\underset{\sim}{b}'Q(\theta)\underset{\sim}{b}]$$

where $Q(\theta) = \underset{\sim}{Z}_1(\underset{\sim}{I}_p-\frac{1}{2}\theta\ \underset{\sim}{x}\ \underset{\sim}{x}')\underset{\sim}{Z}_1'$. Employing the technique used in case 1 of this section and integrating over $\underset{\sim}{D}_\lambda$, we get the density function of $\underset{\sim}{a}$ as

$$(4.25)\quad \exp[-\tfrac{1}{2}\mathrm{tr}\Sigma_1^{-1}\mu\mu'-\tfrac{1}{2}b'b] \sum_{k=0}^{\infty} \{(\tfrac{n}{2})_k\}^{-1}\, g_1(k,\underset{\sim}{b}) \quad \text{for} \quad n \le p\,,$$

where $g_1(k,b)$ is the coefficient of θ^k in the expansion of

$$(4.26)\quad g_1(\theta,b) = c_8 \sum_{j=0}^{\infty} (-1)^j \binom{\frac{1}{2}pn+j-1}{j} \left(\frac{2}{\underset{\sim}{b}'Q(\theta)\underset{\sim}{b}}\right)^{\frac{1}{2}pn+j} \sum_J \frac{(\frac{m+n}{2})_J(\frac{p-1}{2})_J}{(\frac{n+p+1}{2})_J}\, h_1(\underset{\sim}{b},\kappa,\theta)\,,$$

$$(4.27)\quad h_1(\underset{\sim}{b},\kappa,\theta) = \int_{O(p)} [\underset{\sim}{b}'(\underset{\sim}{I}-\underset{\sim}{L}_1\underset{\sim}{L}_1')\underset{\sim}{b}]^{\frac{1}{2}(m+n-p)} C_\kappa(\underset{\sim}{L}_1'Q(\theta)\underset{\sim}{L}_1)d\underset{\sim}{L}^{(p,n-1)},\ \underset{\sim}{L}_1{:}p \times (n-1)$$

and $Q(\theta) = \underset{\sim}{Z}_1[\underset{\sim}{I}-\frac{1}{2}\theta\ \underset{\sim}{x}\ \underset{\sim}{x}']\underset{\sim}{Z}_1'$.

Similarly, the density function of $\underset{\sim}{a}$ for $n \ge p$ is given by

$$(4.28)\quad \exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{\mu\mu}'-\tfrac{1}{2}\underset{\sim}{bb}') \sum_{k=0}^{\infty} \{(\tfrac{n}{2})_k\}^{-1}\, g_2(\underset{\sim}{k},\underset{\sim}{b})\,,$$

where $g_2(k,\underset{\sim}{b})$ is the coefficient of θ^k in the expansion of

$$(4.29)\quad g_2(\theta,b) = c_9 \sum_{j=0}^{\infty} (-1)^j \binom{\frac{1}{2}pn+j-1}{j} \left(\frac{2}{\underset{\sim}{b}'Q(\theta)\underset{\sim}{b}}\right)^{\frac{1}{2}pn+j} \sum_J \frac{(\frac{m+q}{2})_J(\frac{n-1}{2})_J}{(\frac{n+p+1}{2})_J}\, h_2(\underset{\sim}{b},x,\theta)\,,$$

$$(4.30)\quad h_2(b,x,\theta) = \int_{O(p)} [\underset{\sim}{b}'(\underset{\sim}{I}-\underset{\sim}{L}_1\underset{\sim}{L}_1')\underset{\sim}{b}]^{\frac{1}{2}(m+n-p)} C_\kappa(\underset{\sim}{L}_1'Q(\theta)\underset{\sim}{L}_1)d\underset{\sim}{L}^{(p,p-1)},\ \underset{\sim}{L}_1{:}p \times (p-1) \text{ and}$$

$$Q(\theta) = \underset{\sim}{Z}_1[\underset{\sim}{I}_p-\tfrac{1}{2}\theta\ \underset{\sim}{x}\ \underset{\sim}{x}']\underset{\sim}{Z}_1' \ .$$

Case (ii). In the above case, we have assumed that $\underset{\sim}{\mu}$ is fixed. Now suppose that $\underset{\sim}{\mu} = \underset{\sim}{\beta Y}$, $\underset{\sim}{\beta}{:}p \times q$ is fixed having its rank equal to one, and the density function of $\underset{\sim}{YY}'{:}q \times q$ is given by

$$(4.31) \quad \{2^{\frac{1}{2}qn_1}\Gamma_q(\tfrac{1}{2}n_1)|\underset{\sim}{\Sigma}_3|^{\frac{1}{2}n_1}\}^{-1}|(\underset{\sim}{YY}')|^{\frac{1}{2}(n_1-q-1)}\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_3^{-1}(\underset{\sim}{YY}')]\, d(\underset{\sim}{YY}') \quad .$$

Then for $n \leq p$, the joint density function of $\underset{\sim}{A}_1$ and $\underset{\sim}{D}_\lambda$ is obtained from (4.20) by integrating over $\underset{\sim}{YY}'$. This gives us the density function of $\underset{\sim}{A}_1$ and $\underset{\sim}{D}_\lambda$ as

$$(4.32) \quad c_6|\underset{\sim}{I}_q+\underset{\sim}{\Sigma}_3\underset{\sim}{\beta}'\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{\beta}|^{-\frac{1}{2}n_1}|\underset{\sim}{A}_1'\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{A}_1|^{\frac{1}{2}(m+n-p)}|\underset{\sim}{D}_\lambda|^{\frac{1}{2}(p-n-1)}(\prod_{i<j=1}^{n}(\lambda_i-\lambda_j))$$

$$\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{A}_1\underset{\sim}{A}_1'] \exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{A}_1\underset{\sim}{D}_\lambda\underset{\sim}{A}_1')\sum_{k=0}^{\infty}(\tfrac{n_1}{2})_k\{k!(\tfrac{n}{2})_k\}^{-1}$$

$$\{\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}\underset{\sim}{A}_1\underset{\sim}{D}_\lambda\underset{\sim}{A}_1'\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}\underset{\sim}{yy}'\}^k$$

where $yy' = \Sigma_1^{-\frac{1}{2}}\beta(\Sigma_3^{-1}+\beta'\Sigma_1^{-1}\beta)^{-1}\beta'\Sigma_1^{-\frac{1}{2}}$, because β is of rank one. Similarly for $n \geq p$, the joint density function of $\underset{\sim}{A}$ and $\underset{\sim}{D}_\lambda$ is given by

$$(4.33) \quad c_7|\underset{\sim}{I}_q+\underset{\sim}{\Sigma}_3\beta'\underset{\sim}{\Sigma}_1^{-1}\beta|^{-\frac{1}{2}n_1}|\underset{\sim}{A}'\underset{\sim}{A}|^{\frac{1}{2}(m+n-p)}|\underset{\sim}{D}_\lambda|^{\frac{1}{2}(n-p-1)}(\prod_{i<j=1}^{p}(\lambda_i-\lambda_j))\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{AA}']$$

$$\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{AD}_\lambda\underset{\sim}{A}']\sum_{k=0}^{\infty}(\tfrac{n_1}{2})_k\{k!(\tfrac{n}{2})_k\}^{-1}\{\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{AD}_\lambda\underset{\sim}{A}'\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}\,\underset{\sim}{yy}'\}^k$$

where $\underset{\sim}{yy}' = \underset{\sim}{\Sigma}_1^{-\frac{1}{2}}\underset{\sim}{\beta}(\underset{\sim}{\Sigma}_3^{-1}+\underset{\sim}{\beta}'\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{\beta})^{-1}\underset{\sim}{\beta}'\underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$.

Note that the method for obtaining the density function of $\underset{\sim}{a}$ is the same as that of case (i). Hence, we write down the density function of $\underset{\sim}{a}$ as

$$(4.34) \quad (1-\underset{\sim}{yy}')^{-\frac{1}{2}n_1}\exp(-\tfrac{1}{2}\underset{\sim}{b}'\underset{\sim}{b})\sum_{k=0}^{\infty}\{(\tfrac{n_1}{2})_k\}\{(\tfrac{n}{2})_k\}^{-1} g_3(k,\underset{\sim}{b}) \quad \text{for} \quad n \leq p$$

where $g_3(k,\underset{\sim}{b})$ is the coefficient of θ^k in the expansion of $g_3(\theta,\underset{\sim}{b})$ which is the same as $g_1(\theta,\underset{\sim}{b})$ after replacing $Q(\theta)$ by $\underset{\sim}{Z}_1(\underset{\sim}{I}_p-\theta\underset{\sim}{yy}')\underset{\sim}{Z}_1'$ ~~and b =~~ $\underset{\sim}{\Sigma}_2^{-\frac{1}{2}}\underset{\sim}{a}$, while for $n \geq p$

$$(4.35) \quad (1-\underset{\sim}{y}'\underset{\sim}{y})^{-\frac{1}{2}n_1}\exp(-\tfrac{1}{2}\underset{\sim}{b}'\underset{\sim}{b})\sum_{k=0}^{\infty}(\tfrac{n_1}{2})_k\{(\tfrac{n}{2})_k\}^{-1} g_4(k,\underset{\sim}{b}),\ \underset{\sim}{b} = \underset{\sim}{\Sigma}_2^{-\frac{1}{2}}\underset{\sim}{a},$$

where $g_4(k,\underset{\sim}{b})$ is the coefficient of θ^k in the expansion of $g_4(\theta,\underset{\sim}{b})$ which is the same as $g_2(\theta,\underset{\sim}{b})$ after replacing $Q(\theta)$ by $\underset{\sim}{Z}_1(\underset{\sim}{I}_p-\theta\,\underset{\sim}{yy}')\underset{\sim}{Z}_1'$.

Case 3. Let $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2 = \underset{\sim}{\Sigma}$, (say) and $\underset{\sim}{\mu}$ is fixed and of rank one.

Here we shall try to obtain the density function of $\underset{\sim}{\Delta}$ from (4.9) and (4.10)

Let $n \leq p$. Using the transformation $\alpha_i^2 = \lambda_i, i = 1,2,\ldots,n$, $\underset{\sim}{\Sigma}^{-\frac{1}{2}}\underset{\sim}{A}_1(\underset{\sim}{I}+\underset{\sim}{D}_{\alpha}^2)^{\frac{1}{2}} \to \underset{\sim}{A}_1$ and then integrating over A_1, we get the joint density function of $\underset{\sim}{D}_{\lambda}$, $\underset{\sim}{\Delta}_1:(n-1) \times (n-1)$ and $\underset{\sim}{\Delta}:n \times n$ as

$$(4.36) \quad c_{10}|\underset{\sim}{D}_{\lambda}|^{\frac{1}{2}(p-n-1)}|\underset{\sim}{I}+\underset{\sim}{D}_{\lambda}|^{-\frac{1}{2}(m+n)}(\prod_{i<j=1}^{n}(\lambda_i-\lambda_j))\exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\underset{\sim}{\mu}')\sum_{k=0}^{\infty}\{k!(\tfrac{p}{2})_k\}^{-1}$$

$$(\tfrac{m+n}{2})_k\,\{\mathrm{tr}\tfrac{1}{2}[\underset{\sim}{D}_{\lambda}(\underset{\sim}{I}+\underset{\sim}{D}_{\lambda})^{-1}\begin{pmatrix}1 & \underset{\sim}{0} \\ 0 & \underset{\sim}{\Delta}_1'\end{pmatrix}\underset{\sim}{\Delta}'\underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\underset{\sim}{\Delta}\begin{pmatrix}1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Delta}_1\end{pmatrix}]\}^k d\underset{\sim}{\Delta}_1^{(n-1,n-1)}d\underset{\sim}{\Delta}^{(n,1)}$$

where $c_{10} = \Pi^{\frac{1}{2}n^2}\Gamma_n(\frac{m+n}{2})/\{\Gamma_n(\frac{p}{2})\Gamma_n(\frac{n}{2})\Gamma_n(\frac{m+n-p}{2})\}$.

Integrating with respect to $\underset{\sim}{D}_{\lambda}$ and $\underset{\sim}{\Delta}_1$, we get the density function of $\underset{\sim}{\Delta}$ as

$$(4.37) \quad \exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\,\underset{\sim}{\mu}')\sum_{k=0}^{\infty}\{(\tfrac{p}{2})_k\}^{-1}(\tfrac{m+n}{2})_k\; g_5(k,\underset{\sim}{\Delta})d\underset{\sim}{\Delta}^{(n,1)},$$

where $g_5(k,\Delta)$ is the coefficient of θ^k in the expansion of

$$(4.38) \quad g_5(\theta,\underset{\sim}{\Delta}) = c_{10}\int_{\underset{\sim}{D}_{\lambda}}\int_{\underset{\sim}{\Delta}_1}\frac{|\underset{\sim}{D}_{\lambda}|^{\frac{1}{2}(p-n-1)}\exp[\mathrm{tr}\frac{\theta}{2}(\underset{\sim}{I}_n+\underset{\sim}{D}_{\lambda}^{-1})\begin{pmatrix}1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Delta}_1'\end{pmatrix}\underset{\sim}{\Delta}'\underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\underset{\sim}{\Delta}\begin{pmatrix}1 & 0 \\ 0 & \underset{\sim}{\Delta}_1\end{pmatrix})|}{|\underset{\sim}{I}_n+\underset{\sim}{D}_{\lambda}|^{\frac{1}{2}(m+n)}\{\prod_{i<j}(\lambda_i-\lambda_j)\}}$$

$$d\underset{\sim}{\Delta}_1^{(n-1,n-1)}\;d\underset{\sim}{D}_{\lambda} \;.$$

Let us write $(\underset{\sim}{I}_n+\underset{\sim}{D}_{\lambda}^{-1})^{-1} = \underset{\sim}{W} = \mathrm{diag}(\omega_1,\ldots,\omega_n)$, and if $\underset{\sim}{\Delta} = (\underset{\sim}{\delta}_1,\underset{\sim}{\Delta}_2)$, then

$$\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{I}_n+\underset{\sim}{D}_{\lambda}^{-1})^{-1}\begin{pmatrix}1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Delta}_1'\end{pmatrix}\underset{\sim}{\Delta}'\underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\underset{\sim}{\Delta}\begin{pmatrix}1 & 0 \\ 0 & \underset{\sim}{\Delta}_2\end{pmatrix} = \omega_1\phi + \mathrm{tr}\underset{\sim}{W}_1\underset{\sim}{\Delta}_1'\underset{\sim}{R}\underset{\sim}{\Delta}_1,$$

where $\phi = \dfrac{\underset{\sim}{\delta}_1'\underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\underset{\sim}{\delta}_1}{2}$, $\underset{\sim}{R} = \frac{1}{2}\underset{\sim}{\Delta}_2'\underset{\sim}{\mu}'\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\mu}\underset{\sim}{\Delta}_2$ and $\underset{\sim}{W}_1 = \mathrm{diag}\,(\omega_2,\ldots,\omega_n)$. ~~Then~~ (4.38) can be written as

$$(4.39) \quad g_5(\theta,\underset{\sim}{\Delta}) = c_{10}\int_{\underset{\sim}{W}}\int_{\underset{\sim}{\Delta}_1}|\underset{\sim}{W}|^{\frac{1}{2}(p-n-1)}|\underset{\sim}{I}_n-\underset{\sim}{W}|^{\frac{1}{2}(m-p-1)}(\prod_{i<j=1}^{n}(\omega_i-\omega_j))\exp[\theta\omega_1\phi+$$

$$\theta\mathrm{tr}\underset{\sim}{W}_1\underset{\sim}{\Delta}_1'\underset{\sim}{R}\underset{\sim}{\Delta}_1]d\underset{\sim}{\Delta}_1^{(n-1,n-1)}d\underset{\sim}{W} = c_{10}\sum_{k=0}^{\infty}\sum_{\kappa}C_{\kappa}(\theta\underset{\sim}{R})\{k!C_{\kappa}(\underset{\sim}{I}_{n-1})\}^{-1}$$

$$\int_{\underset{\sim}{W}}|\underset{\sim}{W}|^{\frac{1}{2}(p-n-1)}|\underset{\sim}{I}-\underset{\sim}{W}|^{\frac{1}{2}(m-p-1)}(\prod_{i<j=1}^{n}(\omega_i-\omega_j))C_{\kappa}(\underset{\sim}{W}_1)\exp(\theta\omega_1\phi)d\underset{\sim}{W} \;.$$

Making the transformation $\omega_i/\omega_1 = y_{i-1}$ for $i = 1,2,\ldots,n-1$ and letting $\underset{\sim}{Y} = \operatorname{diag}(y_1,\ldots,y_{n-1})$, we get (4.39) as

$$(4.40)\quad g_5(\theta,\underset{\sim}{\Delta}) = c_{10} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{j=0}^{\infty} \sum_{J} \sum_{i=0}^{\infty} \theta^{k+i} C_\kappa(\underset{\sim}{R})\phi^i\{i!k!j!C_\kappa(\underset{\sim}{I}_{n-1})\}^{-1}(-1)^j$$

$$\frac{\Gamma(\frac{1}{2}pn+i+k+j)\Gamma(\frac{m-p+1}{2})}{\Gamma(\frac{m+pn-p+1}{2}+i+k+j)} \sum_{\eta} b^{\eta}_{\kappa,J} \frac{\Gamma_{n-1}(\frac{n-1}{2})\Gamma_{n-1}(\frac{p-1}{2},\kappa)\Gamma_{n-1}(\frac{n+2}{2})}{\Pi^{\frac{1}{2}(n-1)^2}\Gamma_{n-1}(\frac{n+p+1}{2},\kappa)} C_\eta(\underset{\sim}{I}_{n-1}),$$

where $\sum_{\eta} b^{\eta}_{\kappa,J}\, C_\eta(\underset{\sim}{X}) = C_\kappa(\underset{\sim}{X})C_J(\underset{\sim}{X})$, [6] . Similarly when $n \geq p$, the density function of $\underset{\sim}{\Delta}$ can be given in the same way, but we are not giving this explicitly here.

5. Testing of hypothetical principal vectors of $\underset{\sim}{\Sigma}_1$ in the field of $\underset{\sim}{\Sigma}_2$. Let us suppose that $b_1,\ldots,b_r$ are hypothetical principal vectors of $\underset{\sim}{\Sigma}_1$ in the field of $\underset{\sim}{\Sigma}_2$ i.e.

$$\underset{\sim}{\Sigma}_1\underset{\sim}{b}_i = \lambda_i\underset{\sim}{\Sigma}_2\underset{\sim}{b}_i \quad \text{for} \quad i = 1,2,\ldots, r ,$$

where λ_i's are r of the characteristic roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$. Let $\underset{\sim}{B}_1 = (\underset{\sim}{b}_1,\ldots,\underset{\sim}{b}_r):p \times r\ (r \leq p)$ be of rank r and $\underset{\sim}{B}_2:p \times (p-r)$ be a completion of $\underset{\sim}{B}_1$ such that $\underset{\sim}{B} = (\underset{\sim}{B}_1, \underset{\sim}{B}_2):p \times p$ is nonsingular and $\underset{\sim}{B}_1'\underset{\sim}{B}_2 = \underset{\sim}{0}$. Then, we can write

$$(5.1)\qquad \underset{\sim}{B}'\underset{\sim}{\Sigma}_1\underset{\sim}{B} = \begin{pmatrix} \underset{\sim}{D}_\omega & \underset{\sim}{D}_\lambda\underset{\sim}{A}_3 \\ \underset{\sim}{A}_3'\underset{\sim}{D}_\lambda & \underset{\sim}{A}_1 \end{pmatrix},\quad \underset{\sim}{B}'\underset{\sim}{\Sigma}_2\underset{\sim}{B} = \begin{pmatrix} \underset{\sim}{D}_\eta & \underset{\sim}{A}_3 \\ \underset{\sim}{A}_3' & \underset{\sim}{A}_2 \end{pmatrix} \text{ if } \lambda_i \text{ are all distinct,}$$

where $\underset{\sim}{D}_\lambda:r \times r$, $\underset{\sim}{D}_\omega:r \times r$ and $\underset{\sim}{D}_\eta:r \times r$ are diagonal matrices with positive diagonal elements such that $\omega_i/\eta_i = \lambda_i$, $i = 1,2,\ldots,r$, $\underset{\sim}{A}_j:(p-r) \times (p-r)$ for $j = 1,2$ are symmetric positive definite matrices and $\underset{\sim}{A}_3:r \times (p-r)$.

Now, let us consider the joint density function of $\underset{\sim}{S}_1:p \times p$ and $\underset{\sim}{S}_2:p \times p$ be given by

$$(5.2)\qquad \{|2\underset{\sim}{\Sigma}_1|^{\frac{1}{2}\nu_1}|2\underset{\sim}{\Sigma}_2|^{\frac{1}{2}\nu_2}\Gamma_p(\tfrac{1}{2}\nu_1)\Gamma_p(\tfrac{1}{2}\nu_2)\}^{-1}|\underset{\sim}{S}_1|^{\frac{1}{2}(\nu_1-p-1)}|\underset{\sim}{S}_2|^{\frac{1}{2}(\nu_2-p-1)}$$

$$\operatorname{etr}(-\tfrac{1}{2}\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{S}_1-\tfrac{1}{2}\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{S}_2) .$$

We note that $H_0(\utilde{\Sigma}_1 = \utilde{\Sigma}_2)$ against $H_1(\utilde{\Sigma}_1 \neq \utilde{\Sigma}_2)$ can be tested by the statistic

$$\Lambda = \frac{|\utilde{S}_2|}{|\utilde{S}_1 + \utilde{S}_2|} \quad . \tag{5.3}$$

Suppose the structure of $\utilde{\Sigma}_1$ and $\utilde{\Sigma}_2$ be given by (5.1). Under this structure, the joint density function of

$$\utilde{V} = \utilde{B}'\utilde{S}_1\utilde{B} = \begin{array}{cc} \left(\begin{array}{cc} \utilde{V}_{11} & \utilde{V}_{12} \\ \utilde{V}'_{12} & \utilde{V}_{22} \end{array}\right) & \begin{array}{l} r \\ p-r \end{array} \\ \begin{array}{cc} r & p-r \end{array} & \end{array} \quad \text{and} \quad \utilde{W} = \utilde{B}'\utilde{S}_2\utilde{B} = \begin{array}{cc} \left(\begin{array}{cc} \utilde{W}_{11} & \utilde{W}_{12} \\ \utilde{W}'_{12} & \utilde{W}_{22} \end{array}\right) & \begin{array}{l} r \\ p-r \end{array} \\ \begin{array}{cc} r & p-r \end{array} & \end{array}$$

is given by

$$\{|2\utilde{D}_\omega|^{\frac{1}{2}\nu_1}|2(\utilde{A}_2-\utilde{A}'_3\,\utilde{D}_\eta^{-1}\utilde{A}_3)|^{\frac{1}{2}\nu_2}|2\utilde{D}_\eta|^{\frac{1}{2}\nu_2}|2(\utilde{A}_1-\utilde{A}'_3\utilde{D}_\lambda\utilde{D}_\omega^{-1}\utilde{D}_\lambda\utilde{A}_3)|^{\frac{1}{2}\nu_1}\ \Gamma_p(\tfrac{1}{2}\nu_1)\Gamma_p(\tfrac{1}{2}\nu_2)\}^{-1}$$
$$|\utilde{V}|^{\frac{1}{2}(\nu_1-p-1)}|\utilde{W}|^{\frac{1}{2}(\nu_2-p-1)}\operatorname{etr}[-\tfrac{1}{2}\begin{pmatrix} \utilde{D}_\omega & \utilde{D}_\lambda\utilde{A}_3 \\ \utilde{A}'_3\utilde{D}_\lambda & \utilde{A}_1 \end{pmatrix}^{-1}\utilde{V}-\tfrac{1}{2}\begin{pmatrix} \utilde{D}_\eta & \utilde{A}_3 \\ \utilde{A}'_3 & \utilde{A}_2 \end{pmatrix}^{-1}\utilde{W}] \ . \tag{5.4}$$

Noting the distributions of (5.2) and (5.4), $H'_0 = H_{0,1}(\utilde{A}_3 = \utilde{0}) \cap H_{0,2}(\utilde{A}_1 = \utilde{A}_2)$ against $H'_1 \neq H'_0$ can be tested by the statistic

$$\Lambda_2 = \frac{|\utilde{W}_{22}|}{|\utilde{V}_{11} + \utilde{W}_{22}|} \tag{5.5}$$

in the same way as (5.3) is defined. Suppose that the hypothetical vector $\utilde{b}_i$'s are given i.e. $\utilde{B}_1$ is ~~given.~~ Then, we ~~shall~~ write Λ_2 in terms of $\utilde{B}_1$ as under:

$$\Lambda_2 = \frac{|\utilde{S}_2|}{|\utilde{S}_1 + \utilde{S}_2|}\ \frac{(\utilde{B}'_1\utilde{S}_2^{-1}\utilde{B}_1)|}{|\utilde{B}'_1(\utilde{S}_1+\utilde{S}_2)^{-1}\utilde{B}_1|} \tag{5.6}$$

because

$$\begin{aligned} \Lambda_2^{-1} &= |\utilde{I}_{p-r}+\utilde{W}_{22}^{-1}\utilde{V}_{22}| = |\utilde{I}_p+\utilde{B}_2(\utilde{B}'_2\utilde{S}_2\utilde{B}_2)^{-1}\utilde{B}'_2\utilde{S}_1| \\ &= |\utilde{I}_p+\{\utilde{S}_2^{-1}-\utilde{S}_2^{-1}\utilde{B}_1(\utilde{B}'_1\utilde{S}_2^{-1}\utilde{B}_1)^{-1}\utilde{B}'_1\utilde{S}_2^{-1}\}\utilde{S}_1| \\ &= |\utilde{S}_2|^{-1}|\utilde{S}_2+\utilde{S}_1|\ |\utilde{I}_p-(\utilde{S}_1+\utilde{S}_2)^{-1}\utilde{B}_1(\utilde{B}'_1\utilde{S}_2^{-1}\utilde{B}_1)^{-1}\utilde{B}'_1\utilde{S}_2^{-1}\utilde{S}_1| \\ &= \Lambda^{-1}|\utilde{I}_p-\utilde{B}'_1\utilde{S}_2^{-1}\utilde{S}_1(\utilde{S}_1+\utilde{S}_2)^{-1}\utilde{B}_1(\utilde{B}'_1\utilde{S}_2^{-1}\utilde{B}_1)^{-1}| \\ &= \Lambda^{-1}|(\utilde{B}'_1\utilde{S}_2^{-1}\utilde{B}_1)|^{-1}\ |\utilde{B}'_1\{\utilde{S}_2^{-1}-\utilde{S}_2^{-1}\utilde{S}_1(\utilde{S}_1+\utilde{S}_2)^{-1}\}\utilde{B}_1| \\ &= \Lambda^{-1}|(\utilde{B}'_1\utilde{S}_2^{-1}\utilde{B}_1)|^{-1}\ |\utilde{B}'_1(\utilde{S}_1+\utilde{S}_2)^{-1}\utilde{B}_1| \ . \end{aligned}$$

In the above proof we have used the following results:

$$(5.7)\qquad |\underset{\sim}{I} + \underset{\sim}{X}\,\underset{\sim}{Y}| = |\underset{\sim}{I} + \underset{\sim}{Y}\,\underset{\sim}{X}|$$

and

$$(5.8)\qquad \underset{\sim}{B}_2(\underset{\sim}{B}_2'\underset{\sim}{S}_2\underset{\sim}{B}_2)^{-1}\underset{\sim}{B}_2'=\underset{\sim}{S}_2^{-1}-\underset{\sim}{S}_2^{-1}\underset{\sim}{B}_1(\underset{\sim}{B}_1'\underset{\sim}{S}_2^{-1}\underset{\sim}{B}_1)^{-1}\underset{\sim}{B}_1'\underset{\sim}{S}_2^{-1} \quad \text{if} \quad \underset{\sim}{B}_1'\underset{\sim}{B}_2 = \underset{\sim}{0} \;.$$

The result (5.8) is proved by Khatri [4], while (5.7) is well known. Hence, noting (5.6) and (5.3), we get

$$(5.9)\qquad \Lambda = \Lambda_1 \Lambda_2 \;,$$

where

$$(5.10)\qquad \Lambda_1 = |\underset{\sim}{B}_1'(\underset{\sim}{S}_1+\underset{\sim}{S}_2)^{-1}\underset{\sim}{B}_1| \;/\; |(\underset{\sim}{B}_1'\underset{\sim}{S}_2^{-1}\underset{\sim}{B}_1)| \;.$$

$$= |(\underset{\sim}{W}_{11}-\underset{\sim}{W}_{12}\underset{\sim}{W}_{22}^{-1}\underset{\sim}{W}_{12}')|/|\underset{\sim}{W}_{11}+\underset{\sim}{V}_{11}-(\underset{\sim}{W}_{12}+\underset{\sim}{V}_{12})(\underset{\sim}{W}_{22}+\underset{\sim}{V}_{11})^{-1}(\underset{\sim}{W}_{12}+\underset{\sim}{V}_{12})'|.$$

We may note that Λ_1 can be used for testing $H_0''(\lambda_i = 1, i = 1,2,\ldots,r)$ (i.e. ch. roots λ_i corresponding to hypothetical vector $\underset{\sim}{b}_i$ in $\Sigma_2^{-1}\Sigma_1 = 1$ for $i=1,2,\ldots,r$) against H_1''(at least one $\lambda_i \neq 1$) when $H_{01}(\underset{\sim}{A}_3 = \underset{\sim}{0})$ is given. It can be shown that under $H_{01}(\underset{\sim}{A}_3 = \underset{\sim}{0})$ and $H_0''\ (\lambda_i = 1, i = 1,2,\ldots,r)$, Λ_1 and Λ_2 are independently distributed and the distribution of Λ_1 is similar to that of Λ, under H_0(or Λ_2 under H_0').

For the proof of the above result, let us put $\underset{\sim}{D}_{\omega} = \underset{\sim}{D}_{\eta} = \underset{\sim}{D}$ (say) in (5.4) and let us use the transformation:

$$\underset{\sim}{V} + \underset{\sim}{W} = \underset{\sim}{R} = \begin{pmatrix} \underset{\sim}{R}_{11} & \underset{\sim}{R}_{12} \\ \underset{\sim}{R}_{12}' & \underset{\sim}{R}_{22} \end{pmatrix}, \quad \underset{\sim}{H} = \begin{pmatrix} \underset{\sim}{H}_{11} & \underset{\sim}{H}_{12} \\ \underset{\sim}{H}_{12}' & \underset{\sim}{H}_{22} \end{pmatrix} = \underset{\sim}{T}'^{-1}\,\underset{\sim}{W}\,\underset{\sim}{T}^{-1}$$

where $\underset{\sim}{T}'\underset{\sim}{T} = \underset{\sim}{R}$ and $\underset{\sim}{T} = \begin{pmatrix} \underset{\sim}{T}_{11} & \underset{\sim}{0} \\ \underset{\sim}{T}_{21} & \underset{\sim}{T}_{22} \end{pmatrix}$ is a lower triangular matrix. Then $\underset{\sim}{W}_{22} = \underset{\sim}{T}_{22}'\underset{\sim}{H}_{22}\underset{\sim}{T}_{22}$, $\underset{\sim}{T}_{22}'\underset{\sim}{T}_{22} = \underset{\sim}{V}_{22} + \underset{\sim}{W}_{22}$, and if $\underset{\sim}{H}_{11}-\underset{\sim}{H}_{12}\underset{\sim}{H}_{22}^{-1}\underset{\sim}{H}_{12}' = \underset{\sim}{H}_{11}^{(1)}$, then $|\underset{\sim}{H}_{11}^{(1)}| = \Lambda_1$. Using the jacobian of the transformation $|R|^{\frac{1}{2}(p+1)}$ in(5.4) we get the joint density function of $\underset{\sim}{R}$ and $\underset{\sim}{H}$ under H_0'' as

$$(5.11)\qquad c|\underset{\sim}{R}|^{\frac{1}{2}(\nu_1+\nu_2-p-1)}|\underset{\sim}{H}|^{\frac{1}{2}(\nu_2-p-1)}|\underset{\sim}{I}-\underset{\sim}{H}|^{\frac{1}{2}(\nu_1-p-1)}\operatorname{etr}(-\tfrac{1}{2}\underset{\sim}{D}^{-1}\underset{\sim}{R}_{11})$$

$$\operatorname{etr}(-\tfrac{1}{2}\underset{\sim}{A}_1^{-1}\underset{\sim}{R}_{22}-\tfrac{1}{2}(\underset{\sim}{A}_2^{-1}-\underset{\sim}{A}_1^{-1})\underset{\sim}{T}_{22}'\underset{\sim}{H}_{22}\underset{\sim}{T}_{22})$$

where c is the constant term of the density given in (5.4).

Now, we note that

$$|\underset{\sim}{H}| = |\underset{\sim}{H}_{22}|\ |\underset{\sim}{H}_{11}^{(1)}| \quad \text{and} \quad |\underset{\sim}{I}_p-\underset{\sim}{H}|=|\underset{\sim}{I}_{p-r}-\underset{\sim}{H}_{22}|\ |\underset{\sim}{I}_r-\underset{\sim}{H}_{11}^{(1)}|\ |\underset{\sim}{I}_r-\underset{\sim}{G}\underset{\sim}{G}'|\ ,$$

where $\underset{\sim}{H}_{11}^{(1)} = \underset{\sim}{H}_{11} - \underset{\sim}{H}_{12}\underset{\sim}{H}_{22}^{-1}\underset{\sim}{H}_{12}'$ and $\underset{\sim}{G} = (\underset{\sim}{I}-\underset{\sim}{H}_{11}^{(1)})^{\frac{1}{2}}\underset{\sim}{H}_{12}\{\underset{\sim}{H}_{22}^{-1}+(\underset{\sim}{I}-\underset{\sim}{H}_{22})^{-1}\}^{\frac{1}{2}}$. Using the above transformations and their jacobian in (5.11), we see that $\underset{\sim}{H}_{11}^{(1)}$, $\underset{\sim}{G}$ and $(\underset{\sim}{H}_{22},\underset{\sim}{R})$ are independently distributed. Note that $|\underset{\sim}{H}_{11}^{(1)}| = \Lambda_1$ and $|\underset{\sim}{H}_{22}| = \Lambda_2$ and hence Λ_1 and Λ_2 are independently distributed. The density function of $\underset{\sim}{H}_{11}^{(1)}$ is given by

$$(5.12)\quad \{\Pi^{-r(r-1)/4}\prod_{i=1}^{r}\Gamma\tfrac{1}{2}(\nu_1+\nu_2-p+r-i+1)/ \Gamma\tfrac{1}{2}(\nu_1-p+r-i+1)\Gamma\tfrac{1}{2}(\nu_2-i+1)\}|\underset{\sim}{H}_{11}^{(1)\frac{1}{2}(\nu_2-r-1)}|\underset{\sim}{I}_r-\underset{\sim}{H}_{11}^{(1)}|^{\frac{1}{2}(\nu_1-p+r-r-1)} \ .$$

Hence, the distribution of Λ_1 under H_0'' is the same as that of Λ under H_0 by changing the parameters p, ν_1 and ν_2 to r, ν_1-p+r and ν_2 respectively.

We note that the hypothesis H_0' is the intersection of the three hypotheses, namely, H_{03}:($b_1,b_2,\ldots,b_r$) are the true principal vectors of $\underset{\sim}{\Sigma}_1$ in the field of Σ_2), H_{01}: all the orthogonal vectors to the vector space generated by $(\underset{\sim}{b}_1,\ldots,\underset{\sim}{b}_r)$ lie in the vector space generated by the ch. vectors due to ch. roots other than λ_i, $i = 1,2,\ldots,r$) and H_{02}:(the ch. roots of $\Sigma_2\Sigma_1^{-1}$, other than λ_i, $i = 1,2,\ldots,r$, are equal to unity). Thus, Λ_2 is an appropriate over-all test for H_{01}, H_{02} and H_{03} .

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