A priority rule based on the ranking of the service times for the $M|G|1$ queue (*)

by

S. Sreekantan Nair and Marcel F. Neuts

Department of Statistics

Division of Mathematical Sciences

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1. Introduction

The time dependence of a queue in which priorities are based on the ranking of the service times of the waiting customers is difficult to study. The reason for this is the complete breakdown of Markovian features, even in the $M|G|1$ queue. In this paper we propose a priority rule which is of some practical interest and may be studied rigorously.

We first recall a branching process description of the $M|G|1$ queue suggested by Kendall [3] and investigated by Neuts [5]. Suppose that at time $t = 0$ there are $i \geq 1$ customers in the queue and that one of them is just entering service at that time. These customers form the first generation and their total service time is the lifetime of the first generation. Customers arriving during the lifetime of the first generation, if any, make up the second generation, with its lifetime and so on. If at the end of the first or a subsequent generation's lifetime there are no customers in the queue, then there is an idle period at the end of which a customer arrives who makes up the first generation of a busy period.

It is clear that the lifetime of a generation does not depend on the order in which customers have been served during it. We will study the virtual

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waitingtime for the $M|G|1$ queue under the assumption that <u>within each generation customers are served in the order of shortest (or longest) service times first</u>. We will refer to this policy as the shortest processing time (SPT) and the longest processing time (LPT) disciplines respectively and compare them to the first come, first served discipline (FCFS).

It would be informative to make a numerical comparison between our results and those obtained heuristically by Conway and Maxwell [2] and Phipps [7].

We will assume throughout that at time $t = 0$ there are $i > 0$ customers in the queue and that the one with shortest service time enters service at that time. A sequence of random variables $T_0, T_1, T_2, \ldots$ is defined as follows:

$T_0 = 0$ and T_n is the time instant in which all customers, if any, present at T_{n-1} complete service. If there are no customers at T_{n-1}, then T_n is the instant in which the first customer who arrives after T_{n-1} completes service. That is, T_n is the instant of service completion of n-th generation, if the n-th generation is not empty. On the other hand, if the n-th generation is empty then T_n is the instant of the service completion of the first customer who initiates the first busy period after time T_{n-1}.

Let $\xi(t)$ denote the queue length at time $t + 0$ and denote by ${}_0Q_{ij}^{(n)}(x)$ the probability that in a branching process of the type described above the n-th transition occurs before time x, that there are j customers present at the time of the n-th transition and that the population has not become extinct before, given that there are $i > 0$ customers in the first generation.

That is,

$${}_0Q_{ij}^{(n)}(x) = P\{T_n \leq x,\ \xi(T_n) = j,\ \xi(T_\nu) \neq 0,\ \nu = 1,\ldots,n-1 \mid \xi(T_0) = i\}$$

and

$$ {}_oQ_{ij}^{(o)}(x) = \delta_{ij}\ U(x) $$

where $U(\cdot)$ is the degenerate distribution.

2. The Virtual Waitingtime Process

Let us consider an $M|G|1$ queue which has a Poisson input with parameter λ and a continuous service time distribution function $H(\cdot)$ with finite mean α. We denote by $\underline{\eta}(t,x)$ the waiting time of a virtual customer arriving at time t whose service time is $x > 0$ where the $M|G|1$ queue observes SPT discipline, and $\overline{\eta}(t,x)$ the corresponding virtual waiting time in an $M|G|1$ queue with LPT discipline.

Denote

$$ W_i(t,x,y) = P\{0 \le \underline{\eta}\ (t,x) \le y |\ \xi(0) = i\} $$

$$ \widetilde{W}_i(t,x,y) = P\{0 < \underline{\eta}\ (t,x) \le y,\ \underline{\eta}(\tau,x) \neq 0 \quad \text{for} \quad \tau \in (o,t] | \xi(0) = i\} \quad . $$

Then,

$$ W_i(t,x,y) = \int_{\substack{o\\(u)}}^{t} \int_{\substack{t\\(v)}}^{y+t} \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \sum_{\nu=0}^{\infty} d\ {}_oQ_{ij}^{(n)}(u)\ e^{-\lambda(v-u)} \frac{[\lambda(v-u)]^{\nu}}{\nu!} $$

$$ \cdot \sum_{k=0}^{\nu} \binom{\nu}{k} H^k(x)\ [1-H(x)]^{\nu-k}\ \widetilde{H}^{(k)}(y-v+t)\ d_v\ H^{(j)}(v-u) $$

where $\overset{(m)}{H}(\cdot)$ and $\overset{(m)}{\widetilde{H}}(\cdot)$ are the m-fold convolutions of $H(\cdot)$ and $\widetilde{H}(\cdot)$ respectively; and $\widetilde{H}(\cdot)$ is the truncated distribution of $H(\bullet)$ given by,

$$\tilde{H}(Z) = \frac{H(Z)}{H(x)} \quad \text{if} \quad 0 \leq Z \leq x$$

$$= 0 \quad \text{otherwise} \quad .$$

The probabilistic argument for this is the following:

If the queue has never become empty in $(0,t]$, let the last beginning of the life of a generation occur between u and $u + du$ and let there be j individuals in that generation. Let the end of the lifetime of that generation be between v and $v + dv$ $(v > t)$. In the interval (u,v), $\nu \geq 0$ customers arrive and they have priority over the virtual customer if and only if their service time does not exceed x. If there are k such customers, $0 \leq k \leq \nu$ then the distribution of their total service time is $\tilde{H}^{(k)}(\cdot)$. The formula is obtained by using the independence properties and by summing over all allowable values of n, j, ν, k, u and v.

Next we shall introduce the following transforms,

$$\tilde{w}_i(t,x,s) = \int_0^\infty e^{-sy} \, d_y \tilde{W}_i(t,x,y)$$

$$\tilde{w}_i^*(\xi,x,s) = \int_0^\infty e^{-\xi t} \, \tilde{w}_i(t,x,s) \, dt$$

$$w_i(t,x,s) = \int_0^\infty e^{-sy} \, d_y \, W_i(t,x,y)$$

$$w_i^*(\xi,x,s) = \int_0^\infty e^{-\xi t} \, w_i(t,x,s) \, dt$$

$$h(s) = \int_0^\infty e^{-sy} \, d \, H(y)$$

$$\tilde{h}(s) = \int_0^\infty e^{-sy} \, d \, \tilde{H}(y)$$

$$ {}_{o}q_{ij}^{(n)}(\xi) = \int_{0}^{\infty} e^{-\xi u}\, d\,{}_{o}Q_{ij}^{(n)}(u) $$

$$ {}_{o}q_{i}^{(n)}(\xi,z) = \sum_{j=o}^{\infty} {}_{o}q_{ij}^{(n)}(\xi)\, z^{j} \quad . $$

Now,

$$ \tilde{w}_{i}(t,x,s) = e^{st} \int_{0}^{t}\!\!{}_{(u)} \int_{t}^{\infty}\!\!{}_{(v)} \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} d\,{}_{o}Q_{ij}^{(n)}(u)\, e^{-(v-u)\{s+\lambda\, H(x)[1-\tilde{h}(s)]\}}\, d_{v}H^{(j)}(v-u) $$

and

$$ \tilde{w}_{i}^{*}(\xi,x,s) = \frac{1}{\xi-s} \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} {}_{o}q_{ij}^{(n)}(\xi)\, \{h^{j}[s+\lambda\, H(x)(1-\tilde{h}(s))] - h^{j}[\xi+\lambda\, H(x)(1-\tilde{h}(s))]\} $$

$$ = \frac{1}{\xi-s} \sum_{n=0}^{\infty} [{}_{o}q_{i}^{(n)}(\xi,Z) - {}_{o}q_{i}^{(n)}(\xi,Z')] $$

where

$$ Z = h\{s+\lambda\, H(x)\, [1-\tilde{h}(s)]\} $$

$$ Z' = h\{\xi+\lambda\, H(x)\, [1-\tilde{h}(s)]\} \quad . $$

By a standard renewal argument we get

$$ W_{i}(t,x,y) = \tilde{W}_{i}(t,x,y) + \int_{0}^{t} \tilde{W}_{1}(t-\tau,x,y)\, d\, M_{1}(\tau) + P\,\{\underline{\eta}(t,x) = 0 \,|\, \xi(0) = i\} \quad . $$

$M_1(\cdot)$ is the renewal function of the general renewal process formed by the beginnings of busy periods. We have:

$$m_1(\xi) = \int_0^\infty e^{-\xi t}\, dM_1(t) = \frac{\lambda\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)}$$

$$= \lambda \int_0^\infty e^{-\xi t}\, P\{\underline{\eta}(t,x) = 0 | \xi(0) = i\}\, dt$$

from the classical theory of the $M|G|1$ queue. $\gamma(\xi)$ is the Laplace-Stieltjes transform of the distribution of the busy period.

Upon taking transforms, we find:

$$w_i^*(\xi,x,s) = \frac{1}{\xi-s} \sum_{n=0}^{\infty} [{}_oq_i^{(n)}(\xi,Z) - {}_oq_i^{(n)}(\xi,Z')]$$

$$+ \frac{\gamma^i(\xi)}{\xi+\lambda-\lambda\gamma(\xi)} \{1 + \frac{\lambda}{\xi-s} \sum_{n=0}^{\infty} [{}_oq_1^{(n)}(\xi,Z) - {}_oq_1^{(n)}(\xi,Z')]\} \quad .$$

3. <u>The Asymptotic Behaviour of the Virtual Waitingtime Process.</u>

The existence of a limiting distribution as $t \to \infty$ may be demonstrated by applying general properties of the imbedded semi-Markov process discussed in [5]. A limiting distribution

$$W(x,y) = \lim_{t \to \infty} W_i(t,x,y)$$

exists if and only if $1 - \lambda\alpha > 0$, otherwise this limit is zero for all x and y.

If $w^o(x,s)$ denotes the Laplace-Stieltjes transform of $W(x,y)$, then we obtain:

$$w^{o}(x,s) = \lim_{\xi \to 0+} \xi \, w_{i}^{*}(\xi,x,s)$$

$$= (1-\lambda\alpha)\,\{1-\frac{\lambda}{s}\sum_{n=0}^{\infty}[{}_{o}q_{1}^{(n)}(0,Z) - {}_{o}q_{1}^{(n)}(0,\widetilde{Z})]\} \quad \text{if} \quad 1-\lambda\alpha > 0 \; .$$

$$= 0 \quad \text{otherwise}$$

where

$$Z = h\{s + \lambda\, H(x)\, [1-\widetilde{h}(s)]\}$$

$$\widetilde{Z} = h\{\lambda\, H(x)\, [1-\widetilde{h}(s)]\}$$

$$\alpha = \int_{0}^{\infty} u \, dH(u) \quad .$$

It can be easily verified that $w^{o}(x,0) = 1$, if $1 - \lambda\alpha > 0$.

To find the moments of this limiting distribution we shall introduce the following functions.

$$h_{o}(\xi,z) = z$$

$$h_{n+1}(\xi,z) = h[\xi + \lambda - \lambda\, h_{n}(\xi,z)], \quad n \geq 0 \quad .$$

Then it is easily seen from the appendix that

$$w^{o}(x,s) = (1-\lambda\alpha)\,\{1-\frac{\lambda}{s}\sum_{n=0}^{\infty}[h_{n}(0,Z) - h_{n}(0,\widetilde{Z})]\}$$

$$= (1-\lambda\alpha)\,\{1-\lambda\sum_{n=0}^{\infty}\frac{\Psi_{n}(x,s)}{s}\}$$

where

$$\Psi_n(x,s) = h_n[0,\ Z(x,s)] - h_n[0,\ \tilde{Z}(x,s)] \quad .$$

Analogous to the arguments given in Neuts [5] (Lemma 4), it can be shown that the series

$$\sum_{n=1}^{\infty} [h_n(o,z) - h_n(o,o)]$$

is dominated by the series

$$\sum_{n=1}^{\infty} [1-h_n(o,o)] < \sum_{n=1}^{\infty} \lambda^n \alpha^n$$

$$= \frac{\lambda\alpha}{1-\lambda\alpha} \quad \text{if} \quad 1 - \lambda\alpha > 0 \quad .$$

Hence the series $\sum_{n=1}^{\infty} \Psi_n(x,s)$ is dominated by a convergent series $2 \sum_{n=1}^{\infty} \lambda^n \alpha^n$ if $1 - \lambda\alpha > 0$. By applying Lebesgue dominated convergence theorem, term by term differentiation gives

$$\frac{\partial\ w^o(x,s)}{\partial s} = -\lambda(1-\lambda\alpha) \sum_{n=0}^{\infty} \frac{s\ \Psi_n'(s,x) - \Psi_n(s,x)}{s^2}$$

$$\left[\frac{\partial\ w^o(x,s)}{\partial s}\right]_{s=0} = \frac{-\lambda\ (1-\lambda\alpha)}{2} \sum_{n=0}^{\infty} \Psi_n''\ (x,0)$$

where the number of primes denotes the number of derivatives taken in succession with respect to s.

Let us denote,

$$\beta = \int_0^{\infty} u^2\ d\ H(u)$$

$$\gamma = \int_0^\infty u^3 \, d\, H(u)$$

$$\alpha_x = \int_0^x u \, d\, H(u)$$

$$\beta_x = \int_0^x u^2 \, d\, H(u) \qquad \text{and}$$

$$\gamma_x = \int_0^x u^3 \, d\, H(u) \quad .$$

Then we get the following recurrence relations:

Where the notation used is $\frac{\partial h_n(0,z)}{\partial z}\Big]_{z=1} = h_n' \,(0,1)$ etc. Starting with

$$h_{n+1}(0,z) = h[\lambda - \lambda h_n(0,z)], \; n \geq 0$$

$$h_{n+1}'(0,1) = -\lambda \; h'(o) \; h_n'(0,1)$$

$$h_{n+1}''(0,1) = \lambda^2 \; h''(o)[h_n'(0,1)]^2 - \lambda \; h'(o) \; h_n''(0,1)$$

$$h_{n+1}'''(0,1) = -\lambda^3 \; h'''(o) \; [h_n'(0,1)]^3 + 3\lambda^2 h''(o) \; h_n'(0,1) \; h_n''(0,1)$$

$$+ \lambda\alpha \; h_n'''(0,1) \quad .$$

Further simplification gives:

$$h_{n+1}'(0,1) = \lambda\alpha \; h_n'(0,1)$$

$$= (\lambda\alpha)^{n+1}, \quad n \geq 0$$

$$h_0'(0,1) = 1$$

$$h''_{n+1}(0,1) = \beta\lambda^2(\lambda\alpha)^{2n} + \lambda\alpha\, h''_n(0,1)$$

$$= \beta\lambda^2 \left[\frac{(\lambda\alpha)^n - (\lambda\alpha)^{2n+1}}{1-\lambda\alpha}\right] \quad , \quad n \geq 0$$

$$h''_0(0,1) = 0$$

$$h'''_{n+1}(0,1) = \gamma\lambda^3(\lambda\alpha)^{3n} + 3(\lambda^2\beta)^2\left[\frac{(\lambda\alpha)^{2n-1} - (\lambda\alpha)^{3n-1}}{1-\lambda\alpha}\right]$$

$$+ \lambda\alpha\, h'''_n(0,1), \quad n \geq 0$$

$$h'''_0(0,1) = 0 \quad .$$

From these recurrence relations the following quantities may be obtained.

$$\sum_{n=0}^{\infty} h'_n(0,1) = \frac{1}{1-\lambda\alpha}$$

$$\sum_{n=0}^{\infty} h''_n(0,1) = \frac{\beta\lambda^2}{(1-\lambda\alpha)(1-\lambda^2\alpha^2)}$$

$$\sum_{n=0}^{\infty} h'''_n(0,1) = \frac{1}{1-\lambda\alpha}\left[\frac{\gamma\,\lambda^3}{1-\lambda^3\alpha^3} + \frac{3\alpha\,\beta^2\,\lambda^5}{(1-\lambda^2\alpha^2)(1-\lambda^3\alpha^3)}\right] \quad .$$

Also we may obtain,

$$\frac{\partial Z}{\partial s}\Big]_{s=0} = -\alpha\ (1+\lambda\alpha_x)$$

$$\frac{\partial \tilde{Z}}{\partial s}\Big]_{s=0} = -\lambda\ \alpha\alpha_x$$

$$\frac{\partial^2 Z}{\partial s^2}\Big]_{s=0} = \beta\ (1+\lambda\alpha_x)^2 + \lambda\ \alpha\ \beta_x$$

$$\frac{\partial^2 \tilde{Z}^2}{\partial s^2}\Big]_{s=0} = \beta \lambda^2 \alpha_x^2 + \lambda \alpha \beta_x$$

$$\frac{\partial^3 Z}{\partial s^3}\Big]_{s=0} = -\gamma(1 + \lambda\alpha_x)^3 - 3\lambda\beta\, \beta_x(1 + \lambda \alpha_x) - \lambda \alpha \gamma_x$$

$$\frac{\partial^3 \tilde{Z}}{\partial s^3}\Big]_{s=0} = -\gamma\, (\lambda \alpha_x)^3 - 3\lambda^2\beta\, \beta_x \alpha_x - \lambda \alpha \gamma_x$$

$$\Psi_n''(x,0) = h_n''(0,1)\, \Big[\Big(\frac{\partial Z}{\partial s}\Big)^2 - \Big(\frac{\partial \tilde{Z}}{\partial s}\Big)^2\Big]_{s=0} + h_n'(0,1)\, \Big[\frac{\partial^2 Z}{\partial s^2} - \frac{\partial^2 \tilde{Z}}{\partial s^2}\Big]_{s=0} \quad \text{for} \quad n \geq 0$$

$$\Psi_n'''(x,0) = h_n'''(0,1)\, \Big[\Big(\frac{\partial Z}{\partial s}\Big)^3 - \Big(\frac{\partial \tilde{Z}}{\partial s}\Big)^3\Big]_{s=0} + 3h_n''(0,1)\Big[\frac{\partial Z}{\partial s}\frac{\partial^2 Z}{\partial s^2} - \frac{\partial \tilde{Z}}{\partial s}\frac{\partial^2 \tilde{Z}}{\partial s^2}\Big]$$

$$+ h_n'(0,1)\, \Big[\frac{\partial^3 Z}{\partial s^3} - \frac{\partial^3 \tilde{Z}}{\partial s^3}\Big] \quad \text{for} \quad n \geq 0 \quad .$$

We may use the above calculations to obtain,

$$\sum_{n=0}^{\infty} \Psi_n''(x,0) = \frac{\beta(1 + 2\lambda\, \alpha_x)}{(1-\lambda\alpha)(1-\lambda^2\alpha^2)}$$

and

$$\sum_{n=0}^{\infty} \Psi_n'''(x,0) = \frac{-1}{(1-\lambda\alpha)(1-\lambda^2\alpha^2)}\, \Big\{3\lambda\beta\, \beta_x +$$

$$+ (1 + 3\lambda\alpha_x + 3\lambda^2\alpha_x^2)\, \frac{[\gamma(1-\lambda^2\alpha^2) + 3\alpha\lambda^2\beta^2]}{1 - \lambda^3\alpha^3}\Big\} \quad .$$

Let us denote by $\eta\,(\infty,x)$ the limiting value of $\eta(t,x)$ as $t \to \infty$. Then the first two moments of $\eta\,(\infty,x)$ are given by,

$$E\,\underline{\eta}(\infty,x) = \frac{-\partial w^{o}(x,s)}{\partial s}\Big]_{s=0}$$

$$= \frac{\lambda(1-\lambda\alpha)}{2} \sum_{n=0}^{\infty} \Psi''_n(x,0)$$

$$= \frac{\lambda\ \beta(1+2\lambda\alpha_x)}{2\ (1-\lambda^2\alpha^2)}$$

$$E\,\underline{\eta}^2(\infty,x) = \frac{\partial^2\ w^{o}(x,s)}{\partial s^2}\Big]_{s=0}$$

$$= \frac{-\lambda(1-\lambda\alpha)}{2} \sum_{n=0}^{\infty} \Psi'''_n\ (x,0)$$

$$= \frac{\lambda}{2(1-\lambda^2\alpha^2)}\{3\lambda\beta\ \beta_x + (1+3\lambda\alpha_x+3\lambda^2\alpha_x^2)\ \frac{[\gamma(1-\lambda^2\alpha^2)+3\alpha\lambda^2\beta^2]}{1-\lambda^3\alpha^3}\}\ .$$

4. Longest Processing Time (LPT) Discipline

In the case of longest processing time discipline, within each generation the customers are ordered according to their length of service times with highest priority going to the customer with longest demanded service time. The virtual waitingtime process of the present case can be treated as in the case of SPT discipline. As we have denoted, $\bar{\eta}(t,x)$ is the virtual waitingtime of a customer arriving at time t whose service time is $x > 0$, in an $M|G|1$ queue with LPT discipline. The Laplace-Stieltjes transform of the limiting distribution of $\bar{\eta}(t,x)$ can be obtained as

$$(1-\lambda\alpha)\ \{1-\frac{\lambda}{s} \sum_{n=0}^{\infty} [{}_oq_1^{(n)}(0,\zeta) - {}_oq_1^{(n)}(0,\tilde{\zeta})]\} \qquad \text{for} \qquad 1-\lambda\alpha > 0\ .$$

where,

$$\zeta = h\{s+\lambda[1 - H(x)]\,[1 - \tilde{h}(s)]\}$$

$$\tilde{\zeta} = h\{\lambda[1 - H(x)]\,[1 - \tilde{h}(s)]\} \quad .$$

And the first moment is observed to be

$$E\,\bar{\eta}(\infty,x) = \frac{\lambda\;\beta(1 + 2\lambda\;\alpha_x^*)}{2(1-\lambda^2\alpha^2)}$$

where,

$$\alpha_x^* = \int_x^\infty u\;d\;H(u)$$

$$= \alpha - \alpha_x \quad .$$

Similar expression can be obtained for the second moment.

5. Comparison of the SPT, LPT and FCFS disciplines.

Let us denote by $\eta(t)$ the virtual waiting time of a customer arriving at time t in an $M|G|1$ queue with FCFS discipline and let $\eta(\infty)$ be the corresponding limiting value. Then it is known that

$$E\;\eta(\infty) = \frac{\lambda\beta}{2(1-\lambda\alpha)} \quad .$$

Hence we observe an interesting relationship among $E\;\underline{\eta}\;(\infty,x)$, $E\;\bar{\eta}\;(\infty,x)$ and $E\;\eta(\infty)$, namely

$$E\,\eta(\infty) = \frac{E\,\underline{\eta}(\infty,x) + E\,\overline{\eta}(\infty,x)}{2} \quad .$$

Also, $E\,\underline{\eta}(\infty,x) \leq E\,\eta(\infty) \leq E\,\overline{\eta}(\infty,x)$ iff $\alpha_x \leq \frac{\alpha}{2}$.

Again $E\,\underline{\eta}(\infty,X)$ and $E\,\overline{\eta}(\infty,X)$ are random variables with respect to X which has a distribution function $H(\cdot)$. If we denote by E_x the expectation with respect to the random variable X, then

$$E_x\,E\,\underline{\eta}(\infty,X) = \int_0^\infty E\,\underline{\eta}(\infty,x)\, d\,H(x)$$

$$= \frac{\lambda\beta}{2(1-\lambda^2\alpha^2)}\left[1+2\lambda\alpha - 2\lambda\int_0^\infty u\,H(u)\,d\,H(u)\right]$$

and

$$E_x E\overline{\eta}(\infty,X) = \frac{\lambda\beta}{2(1-\lambda^2\alpha^2)}\left[1+2\lambda\int_0^\infty u\,H(u)\,d\,H(u)\right] \quad .$$

In particular if we take $H(x) = 1-e^{-\mu x}$, then

$$E_x\,E\,\underline{\eta}(\infty,X) = \frac{\rho\,(2+\rho)}{2\mu(1-\rho^2)}$$

$$E_x\,E\,\overline{\eta}(\infty,x) = \frac{\rho(2+3\rho)}{2\mu(1-\rho^2)}$$

$$E\,\eta(\infty) = \frac{\rho}{\mu(1-\rho)}$$

where ρ is the traffic intensity $\frac{\lambda}{\mu}$. Hence it is easily seen that in the case of an $M|M|1$ queue,

$$E_x\,E\,\underline{\eta}(\infty,X) \leq E\,\eta(\infty) \leq E_x\,E\,\overline{\eta}(\infty,X) \quad .$$

Appendix

For an easy reading of the text we may state and prove the following theorem which is essentially given in Neuts [5].

Theorem

If $h_n(\xi,z)$, $n \geq 0$ are the sequence of functions and ${}_oq_i^{(n)}(\xi,z)$ $n \geq 1$, $i > 0$ are the transforms, defined in the text then they satisfy the following relation:

$$ {}_oq_i^{(n)}(\xi,z) = h_n^i(\xi,z) - h_{n-1}^i(\xi,0) \;, \quad n \geq 1 \;. $$

Proof

By definition,

$$ {}_oQ_{ij}^{(1)}(x) = P\{T_1 \leq x \;,\; \xi(T_1) = j \mid \xi(T_0) = i\} $$

$$ = \int_0^x e^{-\lambda y} \frac{(\lambda y)^j}{j!} \, dH^{(i)}(y) $$

and

$$ {}_oq_{ij}^{(1)}(\xi) = \int_0^\infty e^{-\xi y} \; d\,{}_oQ_{ij}^{(1)}(y) $$

$$ = \int_0^\infty e^{-(\lambda+\xi)y} \frac{(\lambda y)^j}{j!} \, dH^{(i)}(y) \quad . $$

Hence

$$ {}_oq_i^{(1)}(\xi,z) = \sum_{j=0}^\infty {}_oq_{ij}^{(1)}(\xi) \, z^j $$

$$ = h^i(\xi + \lambda - \lambda z) $$

$$ = h_1^i(\xi,z) \quad . $$

It can be easily verified that the transforms ${}_{o}q_{ij}^{(n)}(\xi)$ satisfy the Chapman-Kolmogorov type of equations:

$$ {}_{o}q_{ij}^{(n+1)}(\xi) = \sum_{\nu=1}^{\infty} {}_{o}q_{i\nu}^{(n)}(\xi)\ {}_{o}q_{\nu j}^{(1)}(\xi) \ , \ n \geq 1 $$

Therefore

$$ \begin{aligned} {}_{o}q_{i}^{(n+1)}(\xi,z) &= \sum_{j=0}^{\infty} {}_{o}q_{ij}^{(n+1)}(\xi)\ z^{j} \\ &= \sum_{\nu=1}^{\infty} {}_{o}q_{i\nu}^{(n)}(\xi)\ h^{\nu}(\xi+\lambda-\lambda z) \\ &= {}_{o}q_{i}^{(n)}[\xi, h(\xi+\lambda-\lambda z)] - {}_{o}q_{i}^{(n)}(\xi,0) \\ &= {}_{o}q_{i}^{(n)}[\xi, h_{1}(\xi,z)] - {}_{o}q_{i}^{(n)}(\xi,0) \ , \ n \geq 1 \ . \end{aligned} $$

Successive substitution yields

$$ \begin{aligned} {}_{o}q_{i}^{(n)}[\xi,h_{1}(\xi,z)] &= {}_{o}q_{i}^{(n-1)}\{\xi, h[\xi+\lambda-\lambda h_{1}(\xi,z)]\} - {}_{o}q_{i}^{(n-1)}(\xi,0) \\ &= {}_{o}q_{i}^{(n-1)}[\xi,h_{2}(\xi,z)] - {}_{o}q_{i}^{(n-1)}(\xi,0) \end{aligned} $$

and

$$ {}_{o}q_{i}^{(n)}(\xi,0) = {}_{o}q_{i}^{(n-1)}[\xi,h_{1}(\xi,0)] - {}_{o}q_{i}^{(n-1)}(\xi,0) \ . $$

Now

$$ {}_{o}q_{i}^{(n+1)}(\xi,z) = {}_{o}q_{i}^{(n-1)}[\xi,h_{2}(\xi,z)] - {}_{o}q_{i}^{(n-1)}[\xi,h_{1}(\xi,0)] $$

Proceeding like this one gets

$$
\begin{aligned}
{}_{o}q_{i}^{(n+1)}(\xi,z) &= {}_{o}q_{i}^{(1)}[\xi,h_{n}(\xi,z)] - {}_{o}q_{i}^{(1)}[\xi,h_{n-1}(\xi,0)] \\
&= h^{i}[\xi+\lambda-\lambda h_{n}(\xi,z)] - h^{i}[\xi+\lambda-\lambda h_{n-1}(\xi,0)] \\
&= h_{n+1}^{i}(\xi,z) - h_{n}^{i}(\xi,0) \ , \quad n \geq 1
\end{aligned}
$$

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13. ABSTRACT

The virtual waitingtime for the M|G|1 queue is studied under the priority rule under which within each generation, customers are served in the order of shortest (or longest) service times. These policies are named here the shortest processing time (SPT) and the longest processing time (LPT) disciplines respectively. The limiting behaviour of the virtual waitingtime is studied and the asymptotic means are compared with that of the first come, first served discipline (FCFS).

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