On the distribution of the ith latent root under null hypothesis concerning complex multivariate normal populations

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1. Introduction and summary: Khatri [2], has pointed out that one can handle all the classical problems of point estimation and testing hypotheses concerning the parameters of complex multivariate normal populations much as one handles those for multivariate normal populations in real variates. Further Khatri [3], has suggested the maximum latent root statistic for testing the reality of a covariance matrix. The joint distribution of the latent roots under certain null hypotheses can be written as, [1], [2],

$$(1) \qquad c_1\{\prod_{j=1}^{q} w_i^m (1-w_j)^n\} \prod_{i>j} (w_i-w_j)^2$$

where $c_1 = \prod_{j=1}^{q} \Gamma(n+m+q+j)/\{\Gamma(n+j)\,\Gamma(m+j)\,\Gamma(j)\}$ and $0 \le w_1 \le w_2 \le \cdots \le w_q \le 1$.

We may also note that when n is large, the joint distribution of $nw_j = f_j$, $j = 1,\ldots,q$, $0 \le f_1 \le \cdots \le f_q < \infty$, can be written as

$$(2) \qquad c_2 \prod_{j=1}^{q} f_j^m \exp(-\sum_{j=1}^{q} f_j)\{\prod_{i>j} (f_i - f_j)^2\}$$

where $c_2 = 1/\{\prod_{j=1}^{q} [\Gamma(m+j)\,\Gamma(j)]\}$.

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Khatri [1], has derived the distribution of w_q (or w_1) and f_q in a determinant form. In this paper we first derive the distribution of w_{q-1} and f_{q-1} and then the distribution of w_i and f_i. In this connection a lemma has been proved.

2. Preliminary results. In this section, we first state two lemmas in order to obtain a third lemma.

Lemma 1.

$$\sum \int_{\mathcal{D}} \prod_{j=1}^{s} [x_j^{m'_j}(1-x_j)^{n'_j}\, dx_j] = \prod_{j=1}^{s} [\int_0^x x_j^{m_j}(1-x_j)^{n_j}\, dx_j]$$

where $\mathcal{D}:(0 \le x_1 \le \cdots \le x_s \le x)$, $(x \le 1)$; and on the left hand side $(m'_s, n'_s),\ldots,(m'_1, n'_1)$ is any permutation of $(m_s,n_s),\ldots,(m_1,n_1)$ and the summation is taken over all such permutations.

For Proof, see Roy([5],(A. 9.3), p. 203).

Lemma 2.

$$\prod_{i>j} (w_i - w_j)^2 = \sum \begin{vmatrix} w_{j1}^{2q-2} & w_{j2}^{2q-3} & w_{jq}^{q-1} \\ w_{j1}^{2q-3} & w_{j2}^{2q-4} & w_{jq}^{q-2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ w_{j1}^{q-1} & w_{j2}^{q-2} & w_{jq}^{0} \end{vmatrix} ,$$

where Σ means summation over all permutations $(j_1,j_2,\ldots,j_q)$ of $(1,2,\ldots,q)$, and $|A|$ means the determinant of A.

For Proof, see Khatri [1] .

Lemma 3.

$$\sum \int_{\mathcal{D}'} \prod_{j=1}^{s} [x_j^{m'_j}(1-x_j)^{n'_j}\, dx_j] = \prod_{j=1}^{s} [\int_x^1 x_j^{m_j}(1-x_j)^{n_j}\, dx_j] ,$$

where $\wp': (x \leq x_1 \leq x_2 \leq \cdots \leq x_s \leq 1)$, and on the left hand side $(m_s', n_s'), \ldots, (m_1', n_1')$ is any permutation of $(m_s, n_s), \ldots, (m_1, n_1)$ and the summation is taken over all such permutations.

Proof is similar to Lemma 1.

3. The distribution of w_{q-1}. In this section we obtain first the cdf's of w_{q-1} and f_{q-1} and in the next those of w_i and f_i. Note that

$$(3) \quad \Pr\{w_{q-1} \leq x\} = \Pr\{w_q \leq x\} + \Pr\{w_{q-1} \leq x < w_q \leq 1\}$$

Khatri [1], showed that

$$(4) \qquad \Pr\{w_q \leq x\} = c_1 |(\beta_{i+j-2})| = c_1 \begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{q-1} \\ \beta_1 & \beta_2 & & \beta_q \\ \vdots & \vdots & & \vdots \\ \beta_{q-1} & \beta_q & & \beta_{2q-2} \end{vmatrix} ,$$

where c_1 is defined in (1), $\beta_{i+j-2} = \int_0^x w^{m+i+j-2} (1-w)^n \, dw$ for $i,j = 1,2,\ldots, q$ and (β_{i+j-2}) is a $q \times q$ matrix. Now the determinant in Lemma 2, can be written as

$$(5) \qquad \sum\nolimits_1 \operatorname{sign}(t_1,\ldots,t_q)\, w_{j1}^{q-1+t_1}\, w_{j2}^{q-2+t_2} \cdots\cdots w_{jq}^{tq} ,$$

where $(t_1,\ldots,t_q)$ is a permutation of $(0,1,\ldots,q-1)$, sign $(t_1,\ldots,t_q)$ is positive if the permutation is even and negative if the permutation is odd, and Σ_1 means the summation over all such permutations. Then (1) can be written as

$$(6)\quad c_1\{\prod_{j=1}^{q} w_j^m (1-w_j)^n \sum_{j_1,\ldots,j_{q-1}} \sum_1 \text{sign}\,(t_1,\ldots,t_q)[w_q^{q-1+t_1} w_{j1}^{q-2+t_2} w_{j2}^{q-3+t_3} \cdots\cdots$$

$$w_{jq-1}^{t_q} + w_q^{q-2+t_2} w_{j1}^{q-1+t_1} w_{j2}^{q-3+t_3} \cdots\cdot w_{jq-1}^{t_q} + \cdots\cdots + w_q^{t_q} w_{j1}^{q-1+t_1} w_{j2}^{q-2+t_2} \cdots$$

$$w_{jq-1}^{1+t_q}]\,.$$

First taking summation over $(j_1,\ldots,j_{q-1})$, the permutation of $(1,2,\ldots,q-1)$ and integrate w_q over $x < w_q < 1$, and apply lemma, we get

$$(7)\quad \Pr(w_{q-1} \leq x \leq w_q < 1) = c_1 \sum_1 \text{sign}\,(t_1,\ldots,t_q)\,[\beta'_{q-1+t_1} \beta_{q-2+t_2} \cdots \beta_{t_q}$$

$$+ \beta_{q-1+t_1} \beta'_{q-2+t_2} \cdots \beta_{t_q} + \cdots \beta_{q-1+t_1} \beta_{q-2+t_2} \cdots \beta'_{t_q}]\,,$$

where $\beta'_{i+j-2} = \int_x^1 w^{m+i+j-2} (1-w)^n\,dw$, then (7) can be written as

$$(8)\qquad c_1 \sum_{k=1}^{q} |(\beta^{(k)}_{i+j-2})|\,,$$

where $|(\beta^{(k)}_{i+j-2})|$ is the determinant obtained from $|(\beta_{i+j-2})|$ by replacing, the kth column of $|(\beta_{i+j-2})|$, β_α, by the corresponding β'_α's. So we proved the following theorem.

Theorem 1. If the joint distribution of $w_1,\ldots,w_q$ is given by (1), then

$$\Pr\{w_{q-1} \leq x\} = c_1 \sum_{k=0}^{q} |(\beta^{(k)}_{i+j-2})| \tag{9}$$

where $|(\beta^{(0)}_{i+j-2})| = |(\beta_{i+j-2})|$, and $|(\beta^{(k)}_{i+j-2})|$ is defined in (8), and c_1 is defined in (1) .

Theorem 2. If the distribution of $f_1,\ldots,f_q$ is given by (2) then

$$\Pr\{f_{q-1} \leq x\} = c_2 \sum_{k=0}^{q} |(\gamma^{(k)}_{i+j-2})| \,, \tag{10}$$

where $\gamma_{i+j-2} = \int_0^x w^{m+i+j-2} \exp(-w)dw$, (γ_{i+j-2}) is a $q \times q$ matrix and $(\gamma^{(k)}_{i+j-2})$ is defined similar to that of (9), and c_2 is defined in (2). Proof is similar to that of theorem 1.

4. The distribution of w_i. It may be noted here that

$$\Pr\{w_i \leq x\} = \Pr\{w_{i+1} \leq x\} + \Pr\{w_i \leq x < w_{i+1}\} \,, \; i = 1,\ldots,q-1 \,. \tag{11}$$

To evaluate the second term of (11), we may write

$$\prod_{i>j} (w_i - w_j)^2 = \sum_1 \text{sign}\,(t_1,\ldots,t_q) \sum_2 \sum_{i_1,\ldots,i_{q-i}} w_{i_1}^{\alpha_1} w_{i_2}^{\alpha_2} \cdots \tag{12}$$

$$\cdots w_{i_{q-i}}^{\alpha_{q-1}} \sum_{j_1,\ldots,j_i} w_{j1}^{\alpha_{q-i+1}} w_{j2}^{\alpha_{q-i+2}} \cdots w_{ji}^{\alpha_q}$$

where $(i_1,\ldots,i_{q-i})$ is permutation of $(i + 1,\ldots, q)$ and $\sum_{i_1,\ldots,i_{q-i}}$ runs over all such permutations; $(j_1,\ldots,j_i)$ is a permutation of $(1,\ldots,i)$, and $\sum_{j1,\ldots,ji}$ runs over all such permutations; Σ_2 is the summation over the terms $\binom{q}{q-i}$ terms of obtained by taking $q-i$, $(\alpha_1,\ldots,\alpha_{q-i})$, at a time of $q-1+t_1$, $q-2+t_2,\ldots,t_q$.

Substituting (12) in (1) and using Lemma 1, and Lemma 3, and as in section (3), we get

$$(13) \qquad \Pr(w_i \leq x < w_{i+1}) = c_1 \sum_2 |(\beta_{i+j-2}^{(i_0)})| ,$$

where $(\beta_{i+j-2}^{(i_0)})$ is a $q \times q$ matrix obtained from (β_{i+j-2}) by replacing i columns of (β_{i+j-2}) by the corresponding β'_α 's . Therefore by (10), (14) and Theorem 1 and reduction process, we can get the distribution of w_i .

It may be pointed out that, [4],

$$(13) \qquad \Pr\{w_i \leq x;\ m,n) = 1 - \Pr(w_{q-i+1} \leq 1 - x;\ n,m)$$

where on the right side of (13) the parameters m and n are interchanged, hence the distribution of w_1, [1], can be written as

$$(14) \qquad \Pr\{w_1 \leq x\} = 1 - c_1 \ |(\delta_{i+j-2})| ,$$

where $\delta_{i+j-2} = \int_0^{1-x} z^{n+i+j-2} (1-z)^m \, dz$, and (δ_{i+j-2}) is a $q \times q$ matrix, similarly, if we define $\delta'_{i+j-2} = \int_{x-1}^{1} z^{n+i+j-2} (1-z)^m \, dz$, the distribution of w_2 can be written as

$$\Pr\{w_2 \leq x\} = 1 - c_1 \sum_{k=0}^{q} |(\delta^{(k)}_{i+j-2})| \ , \tag{15}$$

where, as before, $|(\delta^{(k)}_{i+j-2})|$ is the determinant obtained from $|(\delta_{i+j-2})|$ by replacing the kth column of $|(\delta_{i+j-2})|$ by the corresponding δ'_α 's, and $(\delta^{(0)}_{i+j-2}) = (\delta_{i+j-2})$. A similar method gives

$$\Pr\{f_i \leq x\} = \Pr\{f_{i+1} \leq x\} + \Pr\{f_i \leq x < f_{i+1}\} \ , \tag{16}$$

$i = 1,2,\ldots,q-1$, and

$$\Pr\{f_i \leq x < f_{i+1}\} = c_2 \sum\nolimits_2 |(\gamma^{(i_0)}_{i+j-2})| \ , \tag{17}$$

where c_2 is defined in (2), and also $(\gamma^{(i_0)}_{i+j-2})$ is a $q \times q$ matrix obtained from (γ_{i+j-2}) by replacing i columns of (γ_{i+j-2}) by the corresponding γ_α's .

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