ON THE JOINT DISTRIBUTION OF THE LARGEST LATENT ROOT AND THE SMALLEST LATENT ROOT OF THE COVARIANCE MATRIX AND THE DISTRIBUTION OF THEIR RATIO[2]

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1. SUMMARY. The purpose of this paper is to find the joint distribution and the distribution of the ratio of the largest latent root and the smallest latent root of a sample covariance matrix when the population covariance matrix is a scalar matrix $\Sigma = \sigma^2 I$.

2. INTRODUCTION AND PRELIMINARY RESULTS. Let x be a p-component vector with mean vector μ and covariance matrix $E(x-\mu)(x-\mu)' = \Sigma$. In the hypothesis H that $\Sigma = \sigma^2 I$, all the roots of the equation

$$| \Sigma - \delta I | = 0$$

are equal, namely, the hypothesis implies that the ellipsoids of $(x-\mu)' \Sigma^{-1}(x-\mu)$ = Const. with center at μ are spheres since the roots $\delta_i^{\frac{1}{2}}$ are proportional to the length of the principal axes. Now we consider to test this null hypothesis H. Let $x_1,\ldots,x_N$ be observations on x. The usual unbiased estimate of Σ is $S = (1/(N-1))A$, where A is defined by

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$$A = \Sigma_{\alpha=1}^{N} (x_\alpha - \bar{x})(x_\alpha - \bar{x})'$$

and $\bar{x} = (1/N)\Sigma_{\alpha=1}^{N} x$. Then, as the criteria for testing sphericity in the p-variate normal distribution, we might consider to use the maximum likelihood ratio criterion of the geometric mean and the arithematic mean $w = \Pi\, \lambda_i^{1/p}/((\Sigma\lambda_i)/P)$, the likelihood ratio criterion $\nu = (e(N-1)/N)^{PN/2}\, \Pi\, \lambda_i^{N/2}\, e^{-\frac{1}{2}(N-1)\Sigma\lambda_i}$, and also may suggest the ratio of the largest latent root λ_1 and the smallest latent root λ_p, λ_p/λ_1 of the equation

$$| S - \lambda I | = 0 .$$

The distribution of the largest latent root λ_1, both in the null case and non-null case, may be found in Sugiyama (14), (15). The moments of λ_1 in the null-case is easily calculated from the formula given in (14). The distribution of the criterion W has been given by Consul (6), up to $p = 6$. And also the moments of the criterion W has been given in Anderson (1). Further, the moments of the likelihood ratio criterion ν has been described in Anderson (1).

The purpose of this paper is to find the distribution of the ratio (λ_p/λ_1) of the largest latent root λ_1 and the smallest latent root λ_p. The distribution and the moments of $1 - (\lambda_p/\lambda_1)$ is given as the power series of the zonal polynomials at the unit matrix. Further, we get the joint distribution of the largest latent root λ_1 and the smallest latent root λ_p.

Connected with the derivation of the distributions, we describe some known results. Let Λ_ℓ and ${}^1\Lambda_\ell$ be the matrices as follows

$$\Lambda_\ell = \begin{bmatrix} \ell_2 & 0 \\ 0 & \ell_m \end{bmatrix} \qquad {}^1\Lambda_\ell = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda_\ell \end{bmatrix}$$

and let κ be the usual partition of the degree k in the zonal polynomials. Then we have the following results on a beta-function integral

$$(1) \quad \int_{1>\ell_2>\ldots>\ell_m>0} |\Lambda/_\ell|^{t-(m+1)/2} C_\kappa({}^1\Lambda_\ell) \prod_{i=2}^{m} (1-\ell_i) \prod_{i<j} (\ell_i-\ell_j) \prod_{i=2}^{m} d\ell_i$$

$$= (mt+k)\left(\frac{\Gamma_m(m/2)}{\pi^{m^2/2}}\right)\left(\frac{\Gamma_m(t,\kappa)\,\Gamma_m((m+1)/2)}{\Gamma_m(t+(m+1)/2,\kappa}\right) C_\kappa(I_m)$$

where the value of the zonal polynomials at the unit matrix corresponding to the partition $\kappa = (k_1,\ldots,k_p)$, $k_1 \geq \ldots \geq k_p \geq 0$, of the degree k not more than p parts is given by Constantine (5) as follows

$$(2) \quad C_\kappa(I_m) = 2^{2k} k!\,(m/2)_\kappa \prod_{i<j}^{p} (2k_i-2k_j-i+j) / \prod_{i=1}^{p} (2k_i+p-i)!$$

The above equality (1) has been discussed by Sugiyama (14), (15). A detailed discussion of the zonal polynomials has been given by James (9), (10). Tumura (17) also discusses zonal polynomials, but in terms of rotation angles. We will use the following equality

$$C_\kappa(L)\, C_\sigma(L) = \sum_\delta g^\delta_{\kappa,\sigma}\, C_\delta(L)$$

where L is a $(m\times m)$ symmetric matrix, $C_\kappa(L)$ and $C_\sigma(L)$ are zonal polynomials corresponding to a partition κ of the degree k and a partition σ of the degree s, respectively, and the summation is over all partitions $\delta = (\delta_1,\ldots,\delta_m)$, $\delta_1 \geq \cdots \geq \delta_m \geq 0$, satisfying $\Sigma_{i=1}^m \delta_i = k + s = d$. Tables of $g_{\kappa,\sigma}^{\delta}$ are given by Hayakawa (8) and Khatri and Pillai (12).

3. JOINT DISTRIBUTION AND THE DISTRIBUTION OF THE RATIO OF THE LARGEST ROOT AND THE SMALLEST ROOT. Suppose the sample matrix $X(p\times n)$ consists of n observations (from normal distribution) with mean vector 0 and covariance matrix I. Then the joint density function of the latent roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ of the symmetric matrix $X X'$ is written as follows

$$(4) \qquad C|\Lambda|^{(n-p-1)/2} \exp(\mathrm{tr}-\tfrac{1}{2}\Lambda) \prod_{i<j} (\lambda_i-\lambda_j) \prod_{i=1}^{p} d\lambda_i$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \Lambda_1 \end{bmatrix} , \qquad \Lambda_1 = \begin{bmatrix} \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix}$$

with diagonal elements $\infty > \lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$, and

$$C = \Pi^{p^2/2}/2^{np/2} \quad \Gamma_p(p/2) \quad \Gamma_p(n/2) .$$

Let

$$\ell_i = (\lambda_1 - \lambda_i)/\lambda_i, \quad i = 2, \ldots, p .$$

Then, since

$$\exp(\mathrm{tr}-\tfrac{1}{2}\Lambda) = \exp(-\tfrac{p}{2}\lambda_1)\ \exp(\mathrm{tr}\tfrac{\lambda_1}{2}\ (I-\Lambda_1/\lambda_1))$$

$$= \exp(-\tfrac{p}{2}\lambda_1)\ \exp(\mathrm{tr}\tfrac{\lambda_1}{2}\Lambda_\ell) = \exp(-\tfrac{p}{2}\lambda_1) \sum_{k=o}^{\infty} \sum_{\kappa} (\lambda_1/2)^k\, C_\kappa(\Lambda_\ell)/k! \quad ,$$

and

$$|\Lambda| = \lambda_1^p |\Lambda_1/\lambda_i| = \lambda_1^p\ |I - \Lambda_\ell|\ ,$$

(4) reduces to

$$(5) \qquad C\ \exp(-\tfrac{p}{2}\ \lambda_1) \sum_{k=o}^{\infty} \sum_{\kappa} (\lambda_1{}^{pn/2+k-1}/2^k k!)d\lambda_1$$

$$\cdot\ |\Lambda_\ell|\ |I - \Lambda_\ell|^{(n-p-1)/2}\, C_\kappa(\Lambda_\ell) \prod_{i<j} (\ell_i-\ell_j) \prod_{i=2}^{p} d\ell_i$$

where $\infty > \lambda_1 > 0$, Λ_ℓ is a diagonal matrix with diagonal elements $1 > \ell_p > \ldots > \ell_2 > 0$.

Now in order to obtain the joint distribution of λ_1 and ℓ_p , we have to integrate out $\ell_{p-1},\ldots,\ell_2$ from (5). So we evaluate

$$(6) \quad \int_{\ell_p>\ell_{p-1}>\ldots>\ell_2>o} |\Lambda_\ell|\ |I - \Lambda_\ell|^{(n-p-1)/2} C_\kappa(\Lambda_\ell) \prod_{i<j} (\ell_i-\ell_j) \prod_{i=2}^{p-1} d\ell_i \quad .$$

Using (3), we get

$$|I - \Lambda_\ell|^{(n-p-1)/2} C_K(\Lambda_\ell) = \sum_{s=o}^{\infty} \sum_{\sigma} (p+1-n)\sigma \; C_\sigma(\Lambda_\ell)\; C_K(\Lambda_\ell)/s!$$

$$= \sum_{s=o}^{\infty} \sum_{\sigma} \sum_{\delta} (p+1-n)/2)_\sigma \; C_\delta(\Lambda_\ell)/s! \quad .$$

Setting $r_i = \ell_i/\ell_p$, $i = 2.,,,.p-1$, then since δ is a partition of the degree $k + s$, we can rewrite (6) as follows

$$(7) \quad \sum_{s=o}^{\infty} \sum_{\sigma} \sum_{\delta} (g^{\delta}_{K,\sigma}(p+1-n)/2)_\sigma \quad \ell_p^{(p-1)(p+2)/2+k+s-1}/s!)$$

$$\cdot \int_{1>r_{p-1}>\ldots>r_2>o} |\Lambda_r| \; C_\delta({}^1\Lambda_r) \prod_{i=2}^{p-1} (1-r_i) \prod_{i<j} (r_i - r_j) \prod_{i=2}^{p-1} dr_i \; .$$

In (1), let $m = p - 1$, and $t = (p + 2)/2$. Then, from (7) we have the following formula

$$(8) \quad \left(\frac{\Gamma_{(p-1)}((p-1)/2)}{\pi^{(p-1)^2/2}}\right) \cdot \sum_{s=o}^{\infty} \sum_{\sigma} \sum_{\delta} (g^{\delta}_{K,\sigma}((p+1-n)/2)_\sigma \ell_p^{(p-1)(p+2)/2+k+s-1}/s!)$$

$$\cdot \; ((p-1)(p+2)/2+k+s)(\Gamma_{(p-1)}((p+2)/2,\delta)\; \Gamma_{(p-1)}(p/2)/\Gamma_{(p-1)}(p+1,\delta))C_\delta(I_{p-1})$$

where σ is a partition of s and δ is a partition of $k + s$ into not more than $(p - 1)$ parts, and $g^{\delta}_{K,\sigma}$ is the same coefficients as defined in the formula (3), namely $C_K(\Lambda_\ell)\; C_\sigma(\Lambda_\ell) = \sum_{\delta} g^{\delta}_{K,\sigma}\; C_\delta(\Lambda_\ell)$.
Let $(a)_\delta = \prod_{i=1}^{p-1} (a-(i-1)/2)_{\delta_i}$, $\delta = (\delta_1,\ldots,\delta_p)$ such that $\delta_1 \geq \cdots \geq \delta_{p-1} \geq 0$ and $\sum_{i=1}^{p-1} \delta_i = k + s$. And, as usual, if a is such that the gamma functions are defined, then $(a)_K = \Gamma_p(a,K)/\Gamma_p(a)$ So (8) and (5) give the following joint distribution of λ_1 and ℓ_p

$$(9)\quad C(2)\cdot\exp(-\tfrac{p}{2}\lambda_1)\sum_{k=o}^{\infty}\sum_{\kappa}(\lambda_1^{\,pn/2+k-1}/2^k k!$$

$$\cdot\sum_{s=o}^{\infty}(((p-1)(p+2)/2+k+s)/s!)\quad \ell_p^{\,(p-1)(p+2)/2+k+s-1)}$$

$$\cdot\sum_{\sigma,\delta} g^{\delta}_{\kappa,\sigma}(((p+1-n)/2)\ ((p+2)/2)\delta/(p+1)_{\delta})\ C_{\delta}(I_{p-1})$$

where $\infty > \lambda_1 > 0$, $1 > \ell_p > 0$, and

$$C(2) = \pi^{p/2}\ B_{(p-1)}(p/2,(p+2)/2)/2^{pn/2}\ \Gamma(p/2)\quad \Gamma_p(n/2)\ .$$

Now integrating (9) with respect to λ_1, we obtain the distribution of $\ell_p = (\lambda_1 - \lambda_p)/\lambda_1 = 1 - \lambda_p/\lambda_1$, namely

$$(10)\quad C(p)\cdot\sum_{k=o}^{\infty}\sum_{\kappa}\ (\Gamma(pn/2+k)/p^k k!)$$

$$\cdot\sum_{s=o}^{\infty}\ (((p-1)(p+2)/2+k+s)/s!)\quad \ell_p^{\,(p-1)(p+2)/2+k+s-1}$$

$$\cdot\sum_{\sigma,\delta} g^{\delta}_{\kappa,\sigma}(((p+1-n)/2)_{\sigma}((p+2)\delta/(p+1)\delta)\ C_{\delta}(I_{p-1})$$

where $1 > \ell_p > 0$, and

$$C(p) = \pi^{p/2}\ B_{(p-1)}(p/2,(p+2)/2)/P^{pn/2}\Gamma(p/2)\ \Gamma_p(n/2)\ .$$

The cdf of ℓ_p is

$$(11)\quad P(\ell_p < x) = C(p)\ x^{(p-1)(p+2)/2}\sum_{s=o}^{\infty}\sum_{\kappa}(\Gamma(pn/2+k)/P^k k!)\cdot\sum_{s=o}^{\infty}(x^{k+s}/s!)$$

$$\cdot\sum_{\sigma,\delta} g^{\delta}_{\kappa,\sigma}\ (((p+1-n)/2)_{\sigma}((p+2)/2)_{\delta}/(p+1)_{\delta})\ C_{\delta}(I_{p-1})\ .$$

Further, $E(\ell_p^h)$, the h-th moment of h, is given by

$$(12)\quad E(\ell_p^h) = C(p) \sum_{k=o}^{\infty} \sum_{\kappa} (\Gamma(pn/2 + k)/P^k k!)$$

$$\cdot \sum_{s=o}^{\infty} (((p-1)(p+2)/2+k+s)/(p-1)(p+2)/2+k+s+h)s!)$$

$$\cdot \sum_{\sigma,\delta} g^{\delta}_{\kappa,\sigma} (((p+1-n)/2)_{\sigma}(p+2)/2)_{\delta}/(p+1)_{\delta}) \, C_{\delta}(I_{p-1}) \; .$$

Integrating (9) with respect to ℓ_p, we obtain the distribution of the largest latent root. In that distribution, let $n = p + 1$. Then since $g^{\delta}_{o,\sigma} = 1$ and $\sigma = \delta$, we have the following formula

$$(13)\quad C' \cdot \exp(-\tfrac{p}{2}\lambda_1) \sum_{k=o}^{\infty} \sum_{\kappa} (((p+2)/2)\kappa/(p+1)\kappa)(C_{\kappa}(\tfrac{\lambda_1}{2} I_{p-1})/k!)\lambda_1^{P(p+1)/2-1}$$

$$= C' \cdot \exp(-\tfrac{p}{2}\lambda_1) \; {}_1F_1((p+2)/2;\ p+1;\ \tfrac{\lambda_1}{2} I_{p-1}) \; \lambda_1^{P(p+1)/2-1}$$

where

$$C' = \pi^{p/2} \, B_{(p-1)}(p/2,(p+2)/2)/2^{P(p+1)/2} \; \Gamma(p/2) \; \Gamma_p((p+1)/2)$$

Applying the Kummer transformation formula given by Herz (6), namely

$${}_1F_1(a;\ b;\ \Omega) = \exp(\mathrm{tr}\Omega) \; {}_1F_1(b-a;\ b;\ -\Omega) \;,$$

from (11) we obtain the following formula

$$(14)\quad C' \cdot \exp(-\tfrac{1}{2}\lambda_1) \; {}_1F_1(p/2;\ (p+1);\ -\tfrac{\lambda_1}{2} I_{p-1}) \; \lambda_1^{P(p+1)/2-1} \quad .$$

In the distribution of the largest latent root given by Sugiyama (14), let $n = p + 1$. Then we have the same formula as (14).

Since $\ell_p = (\lambda_1 - \lambda_p)/\lambda_1$, from (9) we have the joint distribution of λ_1 and λ_p, namely

$$(15)\qquad C(2) \cdot \sum_{k=o}^{\infty} \sum_{\kappa} (\lambda_1^{pn/2+k-2}/2^k k!)$$

$$\cdot \sum_{s=o}^{\infty} (((p-1)(p+2)/2+k+s)/s\,!)\ (1 - \lambda_p/\lambda_1)^{(p-1)(p+2)/2+k+s-1}$$

$$\cdot \sum_{\sigma,\delta} g^{\delta}_{\kappa,\sigma}\ (((p+1-n)/2)\delta/(p+1)_\delta)\ C_\delta(I_{p-1})$$

where $\infty > \lambda_1 > \lambda_p > 0$, κ is a partition $(k_1,\ldots,k_p)$, $k_1 \geq \ldots. \geq k_p \geq 0$, of k not more than p parts, as defined in (3), σ is a partition $(\sigma_1,\ldots,\sigma_p)$, $\sigma_1 \geq \ldots. \geq \sigma_{p-1} \geq 0$ of s into not more than $(p-1)$ parts, δ is a partition $(\delta_1,\ldots,\delta_{p-1})$, $\delta_1 \geq \ldots \geq \delta_{p-1} \geq 0$ of $k + s$ into not more than $(p-1)$ parts, and the summation $\sum_{\sigma,\delta}$ is over all combinations of these partitions. We note that if $(p+1-n)/2$ is an integer, the summation of s will be terminated in finite number of terms.

So summarizing the results we have the

THEOREM: Let U has the Wishart distribution $W(p, n, I)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the latent root of the symmetric matrix U. Then the distribution of $1 - \lambda_p/\lambda_1$, cdf, and h-th moment are given by (10), (11), and (12), respectively. The joint distribution of λ_1 and λ_p is given by (15).

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