Non-central Distributions of the Largest Latent Roots of Three Matrices in Multivariate Analysis*

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1. Introduction and Summary. The cdf of the largest latent root of the generalized B statistic in multivariate analysis in the central case is given by Pillai [9],[10],[11], [13], and also useful formulae [12] approximating at the upper end the cdf of the largest latent root. Further, the above c d f has been obtained by Pillai as a series of incomplete beta functions [9],[10], [14] and also independently by Sugiyama and Fukutomi [17]. Recently, Sugiyama [19] has obtained the cdf of the same, as power series. In the non-central MANOVA case, Hayakawa [7] and Khatri and Pillai [15] have obtained the density in a beta function series form. The purpose of this paper is to find simpler power series expressions than these obtained by the above authors for the non-central density function and the cdf of the largest latent root in the MANOVA situation, both in the generalized beta case and by usual transformation in the generalized F case. We will also obtain similar formulae for the non-central density functions of the largest roots for canonical correlation and equality of two covariance matrices.

2. Non-central distribution of the largest latent root in the MANOVA case. Let $\underset{\sim}{X}$ be a $p \times n_1$ matrix variate $(p \le n_1)$ and $\underset{\sim}{Y}$ a $p \times n_2$ matrix variate $(p \le n_2)$ and the columns be all independently normally distributed with covariance matrix $\underset{\sim}{\Sigma}$, $E(\underset{\sim}{X}) = \underset{\sim}{M}$ and $E(\underset{\sim}{Y}) = 0$. Then it is well known that $\underset{\sim}{X}\underset{\sim}{X}' = \underset{\sim}{U}_1$ is non-central Wishart with n_1 degrees of freedom and $\underset{\sim}{Y}\underset{\sim}{Y}' = \underset{\sim}{U}_2$ is central Wishart with

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n_2 degrees of freedom and the covariance matrix Σ, respectively. The generalized non-central statistics L be defined as the latent roots of

$$L = (U_1 + U_2)^{-\frac{1}{2}} U_1 (U_1 + U_2)^{-\frac{1}{2}}.$$

Let $1 > \ell_1 > \cdots > \ell_p > 0$ be the ordered latent roots of the matrix L, namely the roots of the following determinantal equation

$$|U_1 - \ell(U_1 + U_2)| = 0,$$

then the joint density function of $\ell_1, \ldots, \ell_p$ is given by Constantine [2], James [6].

$$C(p,n_1,n_2)\exp(\mathrm{tr}-\Omega)\,|L|^{\frac{1}{2}(n_1-p-1)}\,|I-L|^{\frac{1}{2}(n_2-p-1)} \cdot \prod_{i<j}(\ell_i-\ell_j) \tag{1}$$
$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{(n_1/2)_\kappa}\frac{C_\kappa(\Omega)C_\kappa(L)}{C_\kappa(I)k!},$$

where Ω is the non-centrality matrix, $\frac{1}{2} M' \Sigma^{-1} M$, determinants $|L|$ and $|I-L|$ expressed as products of the latent roots of their matrices,

$C(p,n_1,n_2) = \pi^{p^2/2}\,\Gamma_p((n_1+n_2)/2)/\Gamma_p(p/2)\Gamma_p(n_1/2)\Gamma_p(n_2/2)$ and $C_\kappa(L)$ are zonal polynomials defined in [4], [5]. In this section, we obtain first the density and c.d.f. of ℓ_1. In this connection, we state below two lemmas:

Lemma 1. Let D_ℓ be a diagonal matrix with diagonal elements $1 > \ell_2 > \cdots > \ell_m > 0$, and let κ be a partition of k. Then

$$\int\limits_{1>\ell_2>\cdots>\ell_m>0} |\underset{\sim}{D}_\ell|^{t-(m+1)/2} \, C_\kappa(\underset{\sim}{D}_\ell) \prod_{i=2}^{m} (1-\ell_i) \prod_{i<j} (\ell_i-\ell_j) \prod_{i=2}^{m} d\ell_i$$

$$= (mt + k) \left(\frac{\Gamma_m(m/2)}{\pi^{m^2/2}}\right) \left(\frac{\Gamma_m(t,\kappa)\Gamma_m((m+1)/2)}{\Gamma_m(t+(m+1)/2,\kappa)}\right) C_\kappa(\underset{\sim}{I}_m).$$

Lemma 2. Let $\underset{\sim}{S}(p\times p)$ be a symmetric matrix, and $C_\kappa(\underset{\sim}{S})$ and $C_\sigma(\underset{\sim}{S})$ be zonal polynomials of degree k and s respectively corresponding to the partition $\kappa = (\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_p \geq 0)$ and $\sigma = (s_1 \geq s_2 \geq \cdots s_p \geq 0)$. Then

$$C_\kappa(\underset{\sim}{S}) \, C_\sigma(\underset{\sim}{S}) = \sum_j g^\delta_{\kappa,\sigma} \, C_\delta(\underset{\sim}{S}) \, ,$$

where

$$\delta = (\delta_1 \geq \delta_2 \geq \cdots \geq \delta_p \geq 0) \, , \sum_{i=1}^{p} \delta_i = k+s \text{ and } g^\delta_{\kappa,\delta} \text{ are constants.}$$

Lemma 1 has been discussed by Sugiyama [18] and [19]. Tables of the coefficients $g^\delta_{\kappa,\sigma}$ of Lemma 2 are given by Hayakawa [7] and Khatri and Pillai [15] for various values of k and s .

Now using Lemma 2

$$|I-L|^{(n_2-p-1)/2} \, C_\kappa(\underset{\sim}{L})$$

$$= \sum_{s=0}^{\infty} \sum_{\sigma} ((p+1-n_2)/2)_\sigma \, C_\sigma(\underset{\sim}{L}) \, C_\kappa(\underset{\sim}{L})/s!$$

$$= \sum_{s=0}^{\infty} \sum_{\sigma} \sum_{\delta} ((p+1-n_2)/2)_\sigma \, g^\delta_{\kappa,\sigma} \, C_\delta(\underset{\sim}{L})/s! \; ,$$

and from (1), we get

$$(2)\quad C(p,n_1,n_2)\exp(\mathrm{tr}-\underset{\sim}{\Omega})\ |\underset{\sim}{L}|^{(n_1-p-1)/2}\prod_{i<j}(\ell_i-\ell_j)\sum_{k=o}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{(n_1/2)_\kappa}\ \frac{C_\kappa(\underset{\sim}{\Omega})}{C_\kappa(\underset{\sim}{I})k!}$$

$$\sum_{s=o}^{\infty}\sum_{\sigma,\delta} g_{\kappa,\sigma}^{\delta}\ ((p+1-n_2)/2)_\sigma\ C_\delta(\underset{\sim}{L})/s!.$$

Now consider the integral

$$(3)\quad \int_{\ell_1>\ell_2>\cdots>\ell_p>0} |\underset{\sim}{L}|^{(n_1-p-1)/2} C_\delta(\underset{\sim}{L})\prod_{i<j}(\ell_i-\ell_j)\prod_{i=2}^{p} d\ell_i \quad .$$

In (3) transform $q_i = \ell_i/\ell_1$, $i = 2, \ldots, p$ and integrate with respect to $q_2,\ldots,q_p$, we get

$$(4)\quad \ell_1^{pn_1/2+k+s-1}\int_{1>q_2>\cdots>q_p>0}|\underset{\sim}{D}_q|^{n_1/2-(p+1)/2}\ C_\kappa(\,^1\underset{\sim}{D}_q)\prod_{i=2}(1-q_i)\cdot\prod_{i<j}(q_i-q_j)\prod_{i=2}^{p} d\ell_i$$

$$= \ell_1^{pn_1/2+k+s-1}(pn_1/2+k+s)\left(\frac{\Gamma_p(p/2)}{\pi^{p^2/2}}\right)\cdot\frac{\Gamma_p(n_1/2,\delta)\Gamma_p((p+1)/2)}{\Gamma_p((n_1+p+1)/2,\delta)}\ C_\delta(\underset{\sim}{I}_p)\ .$$

Hence, from (2) and (4) we have the following formula

$$C(p,n_1,n_2)\exp(\mathrm{tr}-\Omega)\left(\frac{\Gamma_p(p/2)\Gamma_p((p+1)/2)}{\pi^{p^2/2}}\right)$$

$$(5)\qquad \sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{(n_1/2)_\kappa}\ \frac{C_\kappa(\Omega)}{C_\kappa(I)k!}\ \sum_{s=0}^{\infty}\ell_1^{pn_1/2+k+s-1}$$

$$\cdot((pn_1/2+k+s)/s!)\sum_{\sigma,\delta}g_{\kappa,\sigma}^{\delta}\ \frac{((p+1-n_2)/2)_\sigma(n_1/2)_\delta}{((n_1+p+1)/2)_\delta}\ C_\delta(I_p).$$

Further, noting that $\Gamma_p(a,\delta)=\Gamma_p(a)(a)_\delta$, we obtain the density of the largest latent root in the following form

$$(6)\quad C_1(p,n_1,n_2)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{(n_1/2)_\kappa}\frac{C_\kappa(\underset{\sim}{\Omega})}{C_\kappa(I)k!}\sum_{s=0}^{\infty}((pn_1/2+k+s)/s!)\sum_{\sigma,\delta}g^{\delta}_{\kappa,\sigma}\frac{((p+1-n_2)/2)_\sigma(n_1/2)_\delta}{((n_1+p+1)/2)_\delta} C_\delta(I_p)\cdot \ell_1^{pn_1/2+k+s-1}$$

where $1 > \ell_1 > 0$, and

$$C_1(p,n_1,n_2) = \frac{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2)}{\Gamma_p(n_2/2)\Gamma_p((n_1+p+1)/2)}\ e^{tr-\underset{\sim}{\Omega}}$$

Further, the c.d.f. of the largest latent root is given by

$$(7)\quad P(\ell_1 < x) = C_1(p,n_1,n_2)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{((n_1+n_2)/2)_\kappa}{(n_1/2)_\kappa}\frac{C_\kappa(\underset{\sim}{\Omega})}{C_\kappa(\underset{\sim}{I})k!} \sum_{s=0}^{\infty}\sum_{\sigma,\delta}g^{\delta}_{\kappa,\sigma}\frac{((p+1-n_2)/2)_\sigma(n_1/2)_\delta}{((n_1+p+1)/2)_\delta\ s!}C_\delta(\underset{\sim}{I}_p)\ x^{pn_1/2+k+s}.$$

Let $\underset{\sim}{\Omega} = 0$ in (9). Then, since $g^{\delta}_{0,\sigma} = 1$ and $\delta = \sigma$, we obtain the following formula given by Sugiyama [19]

$$(8)\quad C_1(p,n_1,n_2)\ {}_2F_1((-n_2+p+1)/2,\ n_1/2;(n_1+p+1)/2;\ x\underset{\sim}{I}_p)\ x^{pn_1/2}\quad .$$

And also, in (7), let $n_2=p+1$, and $x=1$. Then we have ${}_0F_0(\underset{\sim}{\Omega})= e^{tr\ \underset{\sim}{\Omega}}$.

Since the roots $\ell_1,\ldots,\ell_p$ of the generalized beta case are related to the roots $f_1,\ldots,f_p$ of the generalized F case in the following manner:

$$\ell_1 = \frac{f_1}{(1+f_1)}\ , \ \ldots,\ \ell_p = \frac{f_p}{1+f_p}\ ,$$

we obtain from (9), the c.d.f. of the largest latent root in the non-central generalized F case in the form

$$P(f_1 < y) = C_1(p,n_1,n_2) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{((n_1+n_2)/2)_\kappa}{(n_1/2)_\kappa}$$

(9) $$\cdot \frac{C_\kappa(\underset{\sim}{\Omega})}{C_\kappa(\underset{\sim}{I})k!} \sum_{s=0}^{\infty} \sum_{\sigma,\delta} g^{\delta}_{\kappa,\sigma} \frac{((p+1-n_2)/2)_\sigma (n_1/2)_\delta}{s!((n_1+p+1)/2)_\delta}$$

$$\cdot\ C_\delta(\underset{\sim}{I}_p)\ \ (y/(1+y))^{pn_1/2+k+s} \quad .$$

THEOREM 1. Let $\underset{\sim}{U}_1$ be the matrix having non-central Wishart distribution with n_1 degrees of freedom and matrix of non-centrality parameter $\underset{\sim}{\Omega}$, and $\underset{\sim}{U}_2$ be the matrix having the Wishart distribution with n_2 degrees of freedom. Then the pdf and the cdf of the largest latent root ℓ_1 of the equation

$$|\underset{\sim}{U}_1 - (\underset{\sim}{U}_1 + \underset{\sim}{U}_2)\ \ell| = 0$$

is given by (6) and (7). And the cdf of the largest latent root f_1 of the equation

$$|\underset{\sim}{U}_1 - \underset{\sim}{U}_2\ f| = 0$$

is given by (9).

3. <u>Distribution of the largest latent root in the canonical correlation case</u>. Let the columns of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be n independent normal (p+q)-dimensional variates ($p \le q$) with zero means and covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix} .$$

Let R be the diagonal matrix with diagonal element $r_1, r_2, \ldots, r_p$, where $r_1^2, r_2^2, \ldots, r_p^2$ are the latent roots of the equation

$$|X_1 X_2' (X_2 X_2')^{-1} X_2 X_1' - r^2 X_1 X_1'| = 0$$

and also P be the diagonal matrix with diagonal elements $\rho_1, \rho_2, \ldots, \rho_p$, where $\rho_1^2, \rho_2^2, \ldots, \rho_p^2$ are the latent roots of the equation

$$|\Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}' - \rho^2 \Sigma_{11}| = 0 .$$

Then, the distribution of $r_1^2, r_2^2, \ldots, r_p^2$, is given by Constantine [2] in the following form

$$C(n,p,q)\, |I-P^2|^{n/2}\, |R^2|^{(q-p-1)/2}\, |I-R^2|^{(n-p-q-1)/2}$$

$$\prod_{i<j} (r_i^2 - r_j^2) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa (n/2)_\kappa}{(q/2)_\kappa} \frac{C_\kappa(R^2)\, C_\kappa(P^2)}{C_\kappa(I_p)\, k!} , \tag{10}$$

where $$C(n,p,q) = \frac{\Gamma_p(n/2)\pi^{p^2/2}}{\Gamma_p(\frac{1}{2}q)\Gamma_p(\frac{1}{2}(n-q))\Gamma_p(\frac{1}{2}p)} .$$

By the same method as before, namely using lemmas (1) and (2), we have the following

$$\int_{r_1^2>r_2^2>\cdots>r_p^2>0} |\underset{\sim}{R}^2|^{(q-p-1)/2} |\underset{\sim}{I}-\underset{\sim}{R}^2|^{(n-q-p-1)/2} \Pi(r_i^2-r_j^2) \prod_{i=2}^{p} dr_i^2$$

(11)

$$=\sum_{s=0}^{\infty} \sum_{\sigma,\delta} g_{\kappa,\sigma}^{\delta} \left(((p+q+1-n)/2)_\sigma/s!\right) \cdot (pq/2+k+s) \left(\frac{\Gamma_p(p/2)}{\pi^{p^2/2}}\right)$$

$$\cdot \frac{\Gamma_p(q/2,\delta)\Gamma_p((p+1)/2)}{\Gamma_p((q+p+1)/2,\delta)} C_\delta(\underset{\sim}{I}_p)\cdot(r_1^2)^{pq/2+k+s-1}$$

Hence, from (10) and (11), we have the following formula

$$C_2(n,p,q) \; |\underset{\sim}{I}-\underset{\sim}{P}^2|^{n/2} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa (n/2)_\kappa}{(q/2)_\kappa} \frac{C_\kappa(\underset{\sim}{P}^2)}{C_\kappa(\underset{\sim}{I}_p)k!}$$

(12)

$$\sum_{s=0}^{\infty} ((pq/2+k+s)/s!) \sum_{\sigma,\delta} g_{\kappa,\sigma}^{\delta} ((p+q+1-n)/2)_\sigma$$

$$\cdot \frac{(q/2)_\delta}{((q+p+1)/2)_\delta} \; C_\delta(\underset{\sim}{I}_p) \; (r_1^2)^{pq/2+k+s-1},$$

where

$$C_2(n,p,q) = \frac{\Gamma_p((p+1)/2)\;\Gamma_p(n/2)}{\Gamma_p((n-q)/2)\;\Gamma_p((q+p+1)/2}\quad .$$

Integrating (12) from 0 to x with respect to r_1^2, we have the following c.d.f of the largest latent root in the canonical correlation case

$$P(r_1^2 < x) = C_2(n,p,q)\ |\underset{\sim}{I}-\underset{\sim}{P}^2|^{\frac{n}{2}} \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(n/2)_\kappa (n/2)_\kappa}{(q/2)_\kappa}\ \frac{C_\kappa(\underset{\sim}{P}^2)}{C_\kappa(\underset{\sim}{I}_p)k!}$$

$$(13)\qquad \sum_{s=0}^{\infty} g^{\delta}_{\kappa,\sigma}((p+q+1-n)/2)_\sigma \frac{(q/2)_\delta}{((q+p+1)/2)_\delta} \cdot \frac{C_\delta(\underset{\sim}{I}_p)}{s!} \cdot x^{pq/2+k+s} \quad .$$

THEOREM 2. Let $\begin{pmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \end{pmatrix}$ be n independent normal (p+q) - dimensional variates ($p \le q$) with zero means and covariance matrix, $\underset{\sim}{\Sigma}$. Then the pdf and the cdf of the largest latent root r_1^2 of the equation

$$|\underset{\sim}{X}_1\underset{\sim}{X}_2'(\underset{\sim}{X}_2\underset{\sim}{X}_2')^{-1}\ \underset{\sim}{X}_2\underset{\sim}{X}_1' - r^2\underset{\sim}{X}_1\underset{\sim}{X}_1'| = 0$$

is given by (12) and (13).

4. <u>Non-central distribution of the largest latent root for test of equality of two covariance matrices.</u> Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be independently distributed as Wishart $W(n_1,p,\underset{\sim}{\Sigma}_1)$ and $W(n_2,p,\underset{\sim}{\Sigma}_2)$, respectively. Let the latent roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ and $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ be denoted $g_1,\ldots, g_p$ and $\delta_1',\ldots, \delta_p'$ respectively such that $\infty > g_1 > \ldots > g_p > 0$ and $\infty > \delta_1' \ge \ldots \ge \delta_p' > 0$. Let

$$\omega_i = \lambda g_i/(1+\lambda g_i),\ i = 1,\ \ldots,\ p\ ,$$

where λ is a given positive constant in the test of the null-hypothesis H that $\lambda \underset{\sim}{\Delta}' = I$ and $\underset{\sim}{\Delta}' = \text{diag.}\ (\delta_1', \ldots, \delta_p')$. Then the joint distribution of ω_i is given by Khatri [8] in the following form

$$C(p,n_1,n_2)\ |\lambda\underset{\sim}{\Delta}'|^{-n_1/2}\ |\underset{\sim}{W}|^{(n_1-p-1)/2}\ |\underset{\sim}{I}-\underset{\sim}{W}|^{(n_2-p-1)/2} \prod_{i<j} (\omega_i - \omega_j)$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} ((n_1+n_2)/2)_\kappa \frac{C_\kappa(\underset{\sim}{I} - (\lambda\underset{\sim}{\Delta}')^{-1})\ C_\kappa(\underset{\sim}{W})}{C_\kappa(\underset{\sim}{I})\ k!}$$

where $\underset{\sim}{W} = \text{diag}\ (\omega_1, \ldots, \omega_p)$. Then, by the same method as before, we can obtain the density function of the largest latent root ω_1 in the following form

$$C_3(p,n_1,n_2)\ |\lambda\underset{\sim}{\Delta}'|^{-n_1/2} \sum_{k=0}^{\infty} \sum_{\kappa} ((n_1+n_2)/2)_\kappa \frac{C_\kappa(\underset{\sim}{I} - (\lambda\underset{\sim}{\Delta}')^{-1})}{C_\kappa(\underset{\sim}{I})\ k!}$$

(14)

$$\sum_{s=0}^{\infty} \sum_{\sigma,\delta} (pn_1/2+k+s)\ g^{\delta}_{\kappa,\sigma} \frac{((p+1-n_2)/2)_\sigma (n_1/2)_\delta}{s!((n_1+p+1)/2)_\delta}\ C_\delta(\underset{\sim}{I}_p)\ \omega_1^{pn_1/2+k+s-1}$$

where $1 > \omega_1 > 0$, and $C_3(p,n_1,n_2) = \dfrac{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2)}{\Gamma_p(n_2/2)\Gamma_p((n_1+p+1)/2)}$.

Let $\lambda\underset{\sim}{\Delta}' = \underset{\sim}{I}$, namely the central case. Then, since $g^{\delta}_{0,\sigma} = 1$ and $\sigma = \delta$, the cdf of ω_1 is

$$P(\omega_1 < x) = C_3(p,n_1,n_2)\,{}_2F_1((p+1-n_2)/2,\ n_1/2;\ (n_1+p+1)/2;\ \omega_1 \underset{\sim}{I}_p)\ \omega_1^{pn_1/2} .$$

This is the same formula given by Sugiyama [18]. We note that if $(p+1-n)/2$ is an integer, the summation of s will be terminated in finite terms.

Further, $g^{\delta}_{\kappa,\sigma}$'s are constants which do not exceed unity [14]. Again when $n_2 = p + 1$ we get

$$P(\omega_1 < x) = |\lambda\underset{\sim}{\Delta}'|^{-\frac{1}{2}n_1} {}_1F_0(\tfrac{1}{2}n_1;\, x(\underset{\sim}{I} - (\lambda\underset{\sim}{\Delta}')^{-1}))x^{\frac{1}{2}pn_1} .$$

Let $x=1$, $a=\frac{1}{2}n_1$, and $\underset{\sim}{\Omega}=I-(\lambda\underset{\sim}{\Delta}')^{-1}$. Then we have ${}_1F_0(a;\, \underset{\sim}{\Omega}) = |I-\underset{\sim}{\Omega}|^{-a}$.

THEOREM 3. Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ are the matrices having Wishart distributions $W(n_1, p, \underset{\sim}{\Sigma}_1)$ and $W(n_2, p, \underset{\sim}{\Sigma}_2)$, respectively. Then the pdf of $\omega_1 = \lambda g_1/(1+\lambda g_1)$, where g_1 is the largest latent root of the equation

$$|\underset{\sim}{S}_1 - g\ \underset{\sim}{S}_2| = 0$$

is given by (14).

It may be pointed out that Khatri [8] has given the density of g_1 but (14) does not follow from his result by transformation.

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