125

On the distributions of some functions of the roots of a covariance matrix and non-central Wilks' Λ^*

by

K. C. Sreedharan Pillai and Sabri Al-Ani

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series No. 125

October, 1967

* This research was supported by the National Science Foundation, Grant No. GP-7663.

On the distributions of some functions of the roots of a covariance matrix and non-central Wilks' Λ*

by

K. C. Sreedharan Pillai and Sabri Al-Ani

Purdue University

1. Introduction and Summary. Let $\underset{\sim}{X}(p \times n)$ be a matrix variate with columns independently distributed as $N(\underset{\sim}{0},\underset{\sim}{\Sigma})$. Then the distribution of the latent roots, $0 \leq w_1 \leq \cdots \leq w_p < \infty$, of $\underset{\sim}{X}\,\underset{\sim}{X}'$ are first considered in this paper for deriving the distributions of the ratios of individual roots $w_i/w_j (i < j = 2,\ldots,p)$. In particular, the distributions of such ratios are derived for $p = 2,3$ and 4. The use of these ratios in testing the hypothesis $\delta \underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, $\delta > 0$ unknown, has been pointed out where $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ are the covariance matrices of two p-variate normal populations. Further, when $\underset{\sim}{\Sigma} = \underset{\sim}{I}_p$, the distribution of the sum of the two smallest roots is studied for $p = 3$, 4 and 5. This latter criterion is useful for various tests of hypotheses, for example, those regarding the number of indpendent linear equations satisfied by the means, μ_{it}, $i = 1,\ldots,p$, $t = 1,\ldots,N$ in N p-variate normal populations with a common covariance matrix. ([1],[10]).

Further, the non-central distribution of Wilks' Λ criterion has been obtained for $p = 2$, 3 and 4. In this connection a lemma has been proved using some results on Mellin transform.

2. Distribution of ratios of the roots of a covariance matrix. The distribution of the latent roots, $0 < w_1 \leq w_2 \leq \cdots \leq w_p < \infty$, of $\underset{\sim}{X}\,\underset{\sim}{X}'$ depends only upon the latent roots of $\underset{\sim}{\Sigma}$ and can be given in the form (James [6])

* This research was supported by the National Science Foundation Grant No. GP-7663.

$$(2.1) \qquad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \; |\underset{\sim}{W}|^{m} \prod_{i>j} (w_i - w_j) \int_{O(p)} \exp(-\tfrac{1}{2}\mathrm{tr}\; \underset{\sim}{\Sigma}^{-1}\underset{\sim}{H}\,\underset{\sim}{W}\,\underset{\sim}{H}')\; d(\underset{\sim}{H}) \; ,$$

$$0 < w_1 \leq w_2 \leq \ldots \leq w_p < \infty \quad ,$$

where the integral is taken over the orthogonal group of $(p \times p)$ orthogonal matrics $\underset{\sim}{H}$; $m = \frac{1}{2}(n-p-1)$ and $K(p,n) = \Pi^{\frac{1}{2}p^2}/2^{\frac{1}{2}pn}\, \Gamma_p(\frac{1}{2}n)\, \Gamma_p(\frac{1}{2}p)$ and $\underset{\sim}{W} = \mathrm{diag}(w_1, \ldots, w_p)$.

It may be shown that (2.1) can be written in the form James [6]

$$(2.2) \qquad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}|\underset{\sim}{W}|^{m} \{\exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{W})\} \prod_{i>j} (w_i - w_j)\, {}_0F_0(\tfrac{1}{2}(\underset{\sim}{I}_p - \Sigma^{-1}), \underset{\sim}{W}),$$

$$0 < w_1 \leq w_2 \leq \ldots \leq w_p < \infty \quad ,$$

where

$${}_pF_q(a_1, \ldots, a_p;\;\; b_1, \ldots, b_q;\; \underset{\sim}{S}, \underset{\sim}{T}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \ldots (a_p)_\kappa}{(b_1)_\kappa \ldots (b_q)_\kappa} \; \frac{C_\kappa(\underset{\sim}{S})\; C_\kappa(\underset{\sim}{T})}{C_\kappa(\underset{\sim}{I}_p) k!}$$

where $a_1, \ldots, a_p, b_1, \ldots, b_q$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is given by

$$(a)_\kappa \;\; = \;\; \prod_{i=1}^{p} \; (a - \tfrac{1}{2}(i-1))_{k_i} \qquad ,$$

where

$$(a)_k \;\; = \;\; a(a+1)\ldots(a+k-1) \quad ,$$

partition κ of k is such that $\kappa = (k_1, k_2, \ldots, k_p)$, $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$, $k_1 + k_2 + \ldots + k_p = k$ and the zonal polynomials, $C_\kappa(\underset{\sim}{S})$, are expressible in terms of elementary symmetric functions (esf) of the latent roots of $\underset{\sim}{S}$, James [6].

It may be pointed out that the form (2.2) can also be viewed as a limiting form of the non-central distribution of the latent roots Khatri [4] associated with the test of the hypothesis: $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, where $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ are the covariance matrices of two p-variate normal populations, when $n_2 \to \infty$, where n_2 is the size of the sample from the second population. Now, if we wish to test instead the null hypothesis $\delta \underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, $\delta > 0$ unknown, the ratios of the latent roots would be of interest as test criteria. In this context, in the limiting form (2.2), $\underset{\sim}{\Sigma}$ should be replaced by $\delta \underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1}$.

Now, let $\ell_i = w_i / w_p$, $i = 1, \ldots, p-1$, then the distribution of $\ell_1, \ldots, \ell_p, w_p$ can be written in the form

$$(2.3) \qquad K(p,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} w_p^{\frac{1}{2}pn-1} |\underset{\sim}{L}|^m |\underset{\sim}{I}-\underset{\sim}{L}| \prod_{i>j} (\ell_i - \ell_j) \exp -\tfrac{1}{2}(w_p \mathrm{tr} \underset{\sim}{L}_1) \cdot \left[\sum_{k=0}^{\infty} \frac{w_p^k}{2^k k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Sigma}^{-1})\, C_\kappa(\underset{\sim}{L}_1)}{C_\kappa(\underset{\sim}{I}_p)}\right]$$

where

$$\underset{\sim}{L} = \mathrm{diag}(\ell_1, \ldots, \ell_{p-1}) \quad \text{and} \quad \underset{\sim}{L}_1 = \mathrm{diag}(\ell_1, \ldots, \ell_{p-1}, 1) .$$

Integrating (2.3) with respect to w_p, then the distribution of $\ell_1, \ldots, \ell_{p-1}$ is of the form

$$(2.4) \qquad K_1(p,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\ |\underset{\sim}{L}|^m\ |\underset{\sim}{I} - \underset{\sim}{L}| \prod_{i>j} (\ell_i - \ell_j)$$

$$\left[\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}pn+k)}{k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Sigma}^{-1})\ C_\kappa(\underset{\sim}{L}_1)}{C_\kappa(\underset{\sim}{I}_p)(\mathrm{tr}\underset{\sim}{L}_1)^{\frac{1}{2}pn+k}}\right] ,$$

where $K_1(p,n) = 2^{\frac{1}{2}pn} K(p,n)$.

__Case i.__ Let $p = 2$ in (2.4), then the distribution of $\ell = w_1/ w_2$ is of the form

$$(2.5) \qquad K_1(2,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\ \ell^{\frac{1}{2}(n-3)}(1-\ell)\left[\sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!(1+\ell)^{n+k}} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_2 - \underset{\sim}{\Sigma}^{-1})\ C_\kappa\begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix}}{C_\kappa(\underset{\sim}{I}_p)}\right] .$$

__Case ii.__ Putting $p = 3$ in (2.4) and by the use of the results of Khatri and Pillai [5], the distribution of ℓ_1, ℓ_2 can be written in the form

$$(2.6) \qquad K_1(3,n)\ |\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\ (\ell_1\ \ell_2)^{\frac{1}{2}(n-4)}\ (\ell_2 - \ell_1)(1-\ell_1)(1-\ell_2)$$

$$\left[\sum_{k=0}^{\infty} \frac{\Gamma(a_k)}{k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Sigma}^{-1})}{C_\kappa(\underset{\sim}{I}_p)} \sum_{i=0}^{k} \sum_{\eta} b_{\eta,k}\ C_\eta\begin{pmatrix}\ell_1 & 0\\ 0 & \ell_2\end{pmatrix}\right.$$

$$\left.\sum_{r=0}^{\infty} \binom{-a_k}{r} \ell_1^r\ (1+\ell_2)^{-r-a_k}\right] .$$

where $a_k = (3n/2) + k$, $b_{\eta,\kappa}$ are the constants defined [7], and η is the partition of i into not more than p elements.

It may be noted that the distribution of ℓ_1 and of ℓ_2 can be found by writing $C_\eta\begin{pmatrix}\ell_1 & 0\\ 0 & \ell_2\end{pmatrix} = \sum_{i_1+i_2=i} a_{i_1,i_2}\, \ell_1^{i_1}\, \ell_2^{i_2}$ and expanding $(1+\ell_2)^{-r-a_k}$ then integrating ℓ_2 and ℓ_1 respectively.

Let $r_1 = \ell_1/\ell_2$ so the distribution of r_1,ℓ_2 can be written in the form

$$(2.7) \qquad K_1(3,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\, r_1^{\frac{1}{2}(n-4)}\, (1-r_1)\, \Big[\sum_{k=0}^{\infty} \frac{\Gamma(a_k)}{k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_3 - \underset{\sim}{\Sigma}^{-1})}{C_\kappa(\underset{\sim}{I}_p)} \,.$$

$$\sum_{i=0}^{k} \sum_{\eta} b_{\eta,\kappa}\, C_\eta\begin{pmatrix}r_1 & 0\\ 0 & 1\end{pmatrix} \sum_{r=0}^{\infty} \binom{-a_k}{r} r_1^r \sum_{h=0}^{\infty} \binom{-r-a_k}{h} \ell_2^{n-2+i+r+h} (1-\ell_2)(1-r_1\ell_2)\Big].$$

Integrating (2.7) with respect to ℓ_2, the distribution of r_1 can be written in the form

$$(2.8) \qquad K_1(3,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\, r_1^{\frac{1}{2}(n-4)}(1-r_1)\Big[\sum_{k=0}^{\infty} \frac{\Gamma(a_k)}{k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})}{C_\kappa(\underset{\sim}{I}_p)} \sum_{i=0}^{k} \sum_{\eta} b_{\eta,\kappa}$$

$$C_\eta\begin{pmatrix}r_1 & 0\\ 0 & 1\end{pmatrix} \sum_{r=0}^{\infty} \binom{-a_k}{r} r_1^r \sum_{h=0}^{\infty} \binom{-r-a_k}{h} \{\beta(a_1,2)-r_1\beta(a_1+1,2)\}\Big]$$

where $a_1 = n-1+i+r+h$.

<u>Case iii</u>. Let $p = 4$ in (2.4), then the distribution of ℓ_1,ℓ_2,ℓ_3 can be written in the form

$$(2.9) \qquad K_1(4,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} \prod_{i=1}^{3} \{\ell_i^{\frac{1}{2}(n-5)}(1-\ell_i)\} \prod_{i>j} (\ell_i - \ell_j) \,.$$

$$\Big[\sum_{k=0}^{\infty} \frac{\Gamma(2n+k)}{k!\,(1+\ell_1+\ell_2+\ell_3)^{2n+k}} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_4-\Sigma^{-1})}{C_\kappa(\underset{\sim}{I}_p)} \sum_{i=0}^{k} \sum_{\eta} b_{\kappa,\eta}\, C_\eta(\underset{\sim}{L})\Big] ,$$

where $\underset{\sim}{L} = \text{diag}(\ell_1, \ell_2, \ell_3)$.

Now, let $r_i = \ell_i/\ell_3$, $i = 1,2$ and integrate ℓ_3 from 0 to 1, then the distribution of r_1, r_2 can be written in the form

$$(2.10) \qquad K_1(4,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n} (r_1 r_2)^{\frac{1}{2}(n-5)} (1-r_1)(1-r_2)(r_2-r_1) \Big[\sum_{k=0}^{\infty} \frac{\Gamma(2n+k)}{k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Sigma}^{-1})}{C_\kappa(\underset{\sim}{I}_p)} \sum_{i=0}^{k} \sum_{\eta} b_{\kappa,\eta}\, C_\eta(\underset{\sim}{R}_1) \sum_{r=0}^{\infty} \binom{-2n-k}{r} (r_1+r_2)^r \sum_{h=0}^{\infty} \binom{-2n-k-r}{h}$$

$$\{\beta(b,2) - (r_1+r_2)\, \beta\,(b+1,2) + r_1 r_2\, \beta(b+2,2)\}\Big] \quad ,$$

where $b = \frac{3}{2}(n-1)+i+h+r$ and $\underset{\sim}{R}_1 = \text{diag}(r_1, r_2, 1)$. Now, we can find the distribution of r_1 or r_2 by expressing $(r_1+r_2)^r$ in terms of zonal polynomials of $R = \text{diag}(r_1, r_2)$ and using the method outlined in Pillai and Al-Ani [6] and integrating with respect to r_2 or r_1 such that $0 < r_1 \leq r_2 < 1$.

Now, let $r_1' = r_1/r_2$, then the distribution of r_1' can be written in the form

$$(2.11) \qquad K_1(4,n)|\underset{\sim}{\Sigma}|^{-\frac{1}{2}n}\, r_1'^{\frac{1}{2}(n-5)} (1-r_1') \sum_{k=0}^{\infty} \frac{\Gamma(2n+k)}{k!} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{I}_p - \underset{\sim}{\Sigma}^{-1})}{C_\kappa(\underset{\sim}{I}_p)} \sum_{i=0}^{k} \sum_{\eta}$$

$$b_{\kappa,\eta} \sum_{t=0}^{i} \sum_{\tau} b'_{i,\tau}\, C_\tau \begin{pmatrix} r_1' & 0 \\ 0 & 1 \end{pmatrix} \sum_{r=0}^{\infty} \binom{-2n-k}{r} (1+r_1')^r \sum_{h=0}^{\infty} \binom{-2n-k-r}{h}$$

$$\{\beta(b,2)\beta(C,2) + r_1'[\beta(C+2,2)\beta(b+2,2) - \beta(C+1,2)\beta(b,2)]$$

$$+ (1+r_1')\beta(b+1,2)(r_1'\beta(C+2,2) - \beta(C+1,2)) - r_1'^2\beta(b+2,2)\beta(C+3,2)\}$$

where $c = n-2+t+r$ and the constants $b'_{i,\tau}$ and τ are defined in [8].

3. The distribution of the sum of the two smallest roots. Let $\underset{\sim}{\Sigma} = \underset{\sim}{I}_p$ in (2.2) and transform $g_i = \frac{1}{2}w_i$, $i = 1,\ldots,p$, we get the joint density of $g_1,\ldots,g_p$ in the form

$$(3.1) \qquad K_1(p,n) \prod_{i=1}^{p} (g_i^m e^{-g_i}) \prod_{i>j} (g_i - g_j),\ 0 < g_1 \leq g_2 \leq \cdots \leq g_p < \infty .$$

In this section we will derive the distribution of $M_1 = g_1 + g_2$ for $p = 3$ and 4.

Case i. Put $p = 3$ in (3.1) and let $M = \ell_1 + \ell_2$, $G = \ell_1\ell_2$, where $\ell_i = g_i/g_3$, $i = 1,2$. Then the joint distribution of M and g_3 can be written in the form

$$(3.2) \qquad K_1(3,n)\, e^{-g_3(1+M)}\, g_3^{3m+5} \int_0^{M^2/4} G^m(1-M+G)dG,\quad 0 < M \leq 1 .$$

Further, transform $M_1 = g_3M$ and we get

$$(3.3) \qquad K_2(3,n)\ g_3^m\, M_1^{2m+2}\ \{(g_3 - M_1/2)^2 - M_1^2/(4(m+2))\}\, e^{-(g_3+M)} ,$$

where

$$K_2(p,n) = K_1(p,n) / \{(m+1)2^{2m+2}\} .$$

Now integrating g_3 from M_1 to ∞ we get for $0 < M \leq 1$

$$K_2(3,n)\ e^{-M_1}\ M_1^{2m+2}\ [a_o\ I(M_1,\infty;\ m+3) + a_1\ M_1\ I(M_1,\infty;\ m+2) + a_2\ M_1^2\ I(M_1,\infty;\ m+1)] \quad ,$$

where $a_o = 1$, $a_1 = -1$, $a_2 = (m+1)/\{4(m+2)\}$ and $I(x_1,x_2;q) = \int_{x_1}^{x_2} e^{-x}\ x^{q-1} dx$.

Now we consider the case when $1 \leq M \leq 2$. Let $\ell_i' = 1-\ell_i, i = 1,2$ such that $M' = 2-M$, $G' = (1-M+G)$, then the distribution of g_3 and M' can be written in the form

$$(3.4) \qquad K_1(3,n)e^{-g_3(3-M')}\ g_3^{3m+5} \left[\frac{(1-M'/2)^{2m+2}}{(m+1)} \left(\frac{M'^2}{4} - \frac{(1-M'/2)^2}{m+2}\right) + \frac{(1-M')^{m+2}}{(m+1)(m+2)}\right] .$$

Integrate (3.4) with respect to g_3, from $M_1/2$ to M_1 and combine the result with (3.3), then the distribution of M_1 can be written in the form

$$(3.5) \qquad K_2(3,n)\ e^{-M_1} \left[M_1^{2m+2} \sum_{i=0}^{2} a_i M^i\ I(M_1/2,\infty;m+3-i) + 2^{2m+2}(m+2)^{-1} \int_{M_1/2}^{M_1} g_3^{2m+2}(M_1 - g_3)^{m+2}\ e^{-g_3}\ dg_3\right] ,$$

$$0 < M_1 < \infty \quad .$$

<u>Case ii</u>. Put $p = 4$ in (3.1) and integrate g_4, then the distribution of g_3 and M is given by

$$(3.6) \qquad K_2(4,n)\ e^{-g_3(2+M)}\ M^{2m+2} \sum_{r=0}^{m+2} (r+1)\ g_3^{4m+7-r}$$

$$\left[(a-bM)\ \{(1-M/2)^2-M^2/4(m+2)\} + a_2CM^2\ \{(1-M/2)^2-M^2/4(m+3)\}\right],$$

where $a = (m+2)! / (m+2-r)!$, $b=(m+1)!/(m+1-r)!$ and $0<M\leq 1$, $C = m! / (m-r)!$. As before transform $M_1 = g_3M$, and integrate g_3, then the distribution of M_1, for $0 < M \leq 1$, takes the form

$$(3.7) \qquad 2^{-(2m+5)}\ K_2(4,n)\ e^{-M_1} \sum_{r=0}^{m+2} (r+1)\ \{M_1^{2m+2} \sum_{i=0}^{3} 2^{r+i}\ M_1^i$$

$$a_i'\ I(2M_1, \infty;\ 2m+5-r-i) + a_2CM_1^{2m+4} \sum_{i=0}^{2} 2^{r+i+2} M_1^i\ b_i I(2M_1, \infty;$$

$$2m+3-r-i)\},\quad 0 \leq M \leq 1\ ,$$

where $a_0' = a$, $a_1' = -(a+b)$, $a_2' = a(m+1)/\{(m+2)4\}+b$, $a_3' = -b(m+1)/4(m+2)$, $b_0=1$, $b_1 = -1$ and $b_2 = (m+2)/4(m+3)$. Now, when $1 \leq M \leq 2$, as before, transform to M' and G' and integrate out G', and further transform to $M = 2-M'$ and $M_1 = g_3M$ and integrate out g_3 between $M_1/2$ and M_1 and combining the result with (3.7) we get

$$(3.8) \qquad 2^{-m} K_2(4,n)\, e^{-M_1} \sum_{r=0}^{m+2} (r+1) \Big[M_1^{2m+2} \sum_{i=0}^{4} 2^{r+i-m-7} c_i$$

$$M_1^i\, I(M_1,\infty;\, 2m-r-i+5) + (m+2)^{-1} \{ (a-c) \sum_{i=0}^{m+2} \binom{m+2}{i} (-1)^i$$

$$g(r,i+1) + (c-b) \sum_{i=0}^{m+2} \binom{m+2}{i} (-1)^i\, g(r,i) - c(m+3)^{-1}$$

$$\sum_{i=0}^{m+3} \binom{m+3}{i} (-1)^i\, 2g(r,i) \quad . \qquad 0 < M_1 < \infty \quad ,$$

where

$$g(r,i) = 2^{r-i-2} M_1^{m+3-i}\, I(M_1,\, 2M_1;\, 3m+4+i-r) \;,$$

$$c_0 = 4a, \quad c_1 = -4(a+b), \quad c_2 = (C+a)(m+1)(m+2)^{-1} + 4b$$

$$c_3 = -(C+b)(m+1)(m+2)^{-1}, \quad \text{and} \quad c_4 = C(m+1)/\{4(m+3)\} \quad .$$

Case iii. Put $p = 5$ in (3.1) and integrate g_5 and g_4, then the distribution of g_3 and M is given by

$$(3.9) \qquad K_2(5,n)\, e^{-g_3(3+M)}\, g_3^{3m+5}\, M^{2m+2} \sum_{r=o}^{6} \eta_r M^r\, g_3^{2m+7-i-j}$$

where $\eta_o = K_{o,i,j}/(m+1)$, $\eta_1 = (K_{1,i,j} - K_{o,i,j})/(m+1)$,

$$\eta_2 = (K_{o,i,j} + K_{3,i,j})/4(m+2) + (K_{2,i,j} - K_{1,i,j})/(m+1),$$

$$\eta_3 = (K_{1,i,j} - K_{3,i,j} + K_{4,i,j})/4(m+2) - K_{2,i,j}/(m+1),$$

$$\eta_4 = (K_{2,i,j} - K_{4,i,j})/4(m+2) + (K_{3,i,j} + K_{5,i,j})/2^4(m+3),$$

$$\eta_5 = (K_{4,i,j} - K_{5,i,j})2^4(m+3) \text{ , and } \eta_6 = K_{5,i,j}/2^6(m+4)$$

and the $K_{\ell,i,j}$ are defined by

$$(3.10) \quad K_{\ell,i,j} = \sum_{j=o}^{2m+7-i-\ell_\delta} \sum_{i=o}^{m+k} \frac{i}{2^{j+1}} \Big[a_\ell^{(1)}(2m+7-i-\ell_{\delta-j}) - a_\ell^{(2)}(2m+6-i-\ell_{\delta-j}) + a_\ell^{(3)}(2m+5-i-\ell_{\delta-j}) \Big] ,$$

where

$$\ell_\delta = \begin{cases} \ell \text{ , for } \ell=0,1, \text{ and } 2, \\ \ell_{-1}, \text{ for } \ell=3,4, \text{ and } 5, \end{cases}$$

and

$$K = \begin{cases} 4 \text{ for } \ell=0,1,3 \\ 3 \text{ for } \ell=2,4 \\ 2 \text{ for } \ell=5 \end{cases}$$

and

$$
(3.11)\quad
\begin{array}{lll}
a_o^{(1)} = (m+3)_{-i+1}, & a_o^{(2)} = -a_i(m+2)_{-i+2}, & a_o^{(3)} = (m+2)_{-i+1} \\
a_1^{(1)} = a_o^{(2)}, & a_1^{(2)} = b_i(m+1)_{-i+3}, & a_1^{(3)} = -c_i(m+1)_{-i+1} \\
a_2^{(1)} = a_o^{(3)}, & a_2^{(2)} = -c_i(m+1)_{-i+2}, & a_2^{(3)} = (m+1)_{-i+1} \\
a_3^{(1)} = d_i(m+1)_{-i+3}, & a_3^{(2)} = -c_i(m)_{-i+4}, & a_3^{(3)} = g_i(m)_{-i+3} \\
a_4^{(1)} = a_2^{(2)}, & a_4^{(2)} = k_i(m)_{-i+3}, & a_4^{(3)} = -\ell_i(m)_{-i+2} \\
a_5^{(1)} = a_2^{(3)}, & a_5^{(2)} = a_4^{(3)}, & a_5^{(3)} = (m)_{-i+1}
\end{array}
$$

and $(a)_{-i+b} = a(a-1)$ ---- $(a-i+b+1)$; $a_1 = 2$, $a_i = 2M+7-i$, $i \geq 2$; $b_1 = 4$, $b_2 = 4m+8$ and $b_i = (2m+7-i)(2m+5-i) + i-1$ for $i \geq 3$; $c_1 = 2$, $c_i = 2m+5-i$ for $i \geq 2$; $d_1 = 2$, $d_2 = 2m+4$ and $d_i = (m+2)_2 + (m+3-i)_2$ for $i \geq 3$; $e_1 = 4$, $e_2 = 4m+6$, $e_3 = \sum_{i=o}^{3} (m+i)_{-2}$ and $e_i = \sum_{K=o}^{3} (m+2-i+K)_{3-K}(m+1)_K$ for $i \geq 4$; $g_1 = 2$, $g_2 = 2m+2$, $g_i = (m+1)_2 + (m+2-i)_2$ for $i \geq 3$; $\ell_1 = 2$, $\ell_i = 2m-i+3$, $i \geq 2$, $k_1 = 4$, $k_2 = 4m + 4$, $k_i = 4m^2 + 16m - 4im + i^2 - 7i + 14$ for $i \geq 3$.

As before transform $M_1 = g_3 M$, and integrate g_3, then the distribution of M_1, for $0 < M \leq 1$, takes the form

$$
(3.12)\quad K_2(5,n)\, M_1^{2m+2}\, e^{-M_1} \sum_{r=o}^{6} \eta_r M_1^r I(3M_1,\infty;3m+10-i-j-r)/3^{3m+10-i-j-r} .
$$

Now, when $1 \leq M \leq 2$, proceeding as before, and combining the result with (3.12) we get

$$(3.13) \quad K_3(5,n)M_1^{m+2} \; e^{-M_1}\Big[(3M_1)^m \sum_{r=0}^{6} 3^{i+j+r}\,\eta_r M_1^r \cdot I(3M_1/2,\infty;3m+10-i-j-r)$$

$$+ 2^{2m+2} \sum_{s=0}^{m+2} \binom{m+2}{s} (-1)^s \sum_{r=0}^{2} p_r M_1^{r-s}\; 3^{s+i+j+r} I(3M_1/2,3M_1;4m+10+s-i-j-r)\Big]$$

where $K_3(5,n) = K_2(5,n)/3^{4m+10}$,

$p_o = K_{o,i,j}/(m+1)(m+2) - K_{3,i,j}/(m+2)(m+3) + K_{5,i,j}/(m+3)(m+4),$

$p_1 = K_{1,i,j}/(m+1)(m+2) + (K_{3,i,j} - K_{4,i,j})/(m+2)(m+3) - 2K_{5,i,j}/(m+3)(m+4)$

$p_2 = K_{2,i,j}/(m+1)(m+2) + K_{4,i,j}/(m+2)(m+3) + K_{5,i,j}/(m+3)(m+4).$

4. The Non-Central distribution of Wilks' Criterion. In this section we shall derive the non-central distribution of Wilks' criterion, namely $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1-r_i)$ where $r_1,\ldots,r_p$ are the characteristic roots of the equation

$$|\underset{\sim}{S}_1 - r(\underset{\sim}{S}_1 + \underset{\sim}{S}_2)| = 0$$

where $\underset{\sim}{S}_1$ is a $(p \times p)$ matrix distributed non-central Wishart with s degrees of freedom and a matrix of non-centrality parameters $\underset{\sim}{\Omega}$ and $\underset{\sim}{S}_2$ has the Wishart distribution with t degrees of freedom, the covariance matrix in each case being $\underset{\sim}{\Sigma}$. For this, first we state below a few results on Mellin transform and then prove a lemma.

Theorem 1. If s is any complex variate and $f(x)$ is a function of a real variable x, such that

$$(4.1) \qquad F(s) = \int_0^\infty x^{s-1} f(x)dx$$

exists. Then, under certain conditions [3]

$$(4.2) \qquad f(x) = \frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)ds.$$

$F(s)$ in (4.1) is called the Mellin transform of $f(x)$ and $f(x)$ in (4.2) is called the inverse Mellin transform of $F(s)$. Now we state another theorem [3].

Theorem 2. If $f_1(x)$ and $f_2(x)$ are the inverse Mellin transform of $F_1(s)$ and $F_2(s)$ respectively, then the inverse Mellin transform of $F_1(s) F_2(s)$ is given by

$$(4.3) \qquad \frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F_1(s)F_2(s)ds = \int_0^\infty f_1(u)f_2(x/u) \cdot \frac{du}{u} .$$

Further we use theorem 2 to prove the following lemma.

Lemma 1. If s is a complex variate, a, b, c, d, m, n, p and ℓ are reals then

$$I = \frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} X^{-s} \frac{\Gamma(s+a)\,\Gamma(s+b)\,\Gamma(s+c)\,\Gamma(s+d)}{\Gamma(s+a+m)\,\Gamma(s+b+n)\,\Gamma(s+c+p)\,\Gamma(s+d+\ell)}\, ds$$

(4.4)
$$= \frac{X^d(1-X)^{m+n+p+\ell-1}}{\Gamma(m+n+p)} \sum_{k=0}^{\infty} \frac{(d+\ell-a)_k}{k!} \sum_{r=0}^{\infty} \frac{(p)_r(b+n-c)_r}{r!\,(m+n+p)_r} (1-X)^{k+r}$$

$$\frac{\Gamma(m+n+p+k+r)}{\Gamma(m+n+p+\ell+k+r)}\ {}_3F_2(a+m-b, n+p+r, m+n+p+k+r; m+n+p+r, m+n+p+\ell+k+r; 1-X).$$

<u>Proof</u>: Let $F_1(s) = \{\Gamma(s+a)\,\Gamma(s+b)\,\Gamma(s+c)/\Gamma(s+a+m)\,\Gamma(s+b+n)\,\Gamma(s+c+p)\}$, $F_2(s) = \Gamma(s+d)/\Gamma(s+d+\ell)$, then

(4.5)
$$f_1(x) = X^a(1-X)^{m+n+p-1}\,[\Gamma(m+n+p)]^{-1} \sum_{r=0}^{\infty} \frac{(p)_r(b+n-c)_r}{r!\,(m+n+p)_r} (1-X)^r$$
$${}_2F_1(a+m-b, n+p+r; m+n+p+r; 1-X),$$

and $f_2(x) = \dfrac{X^d(1-X)^{\ell-1}}{\Gamma(\ell)}$, $0 < X < 1$, [4].

Now by the use of Theorem 2 we get

(4.6)
$$I = \frac{X^d}{\Gamma(\ell)\,\Gamma(m+n+p)} \int_X^1 u^{a-d-\ell} (1-U)^{m+n+p-1} \sum_{r=0}^{\infty} \frac{(p)_r(b+n-c)_r}{r!\,(m+n+p)_r}$$
$$(1-U)^r\ {}_2F_1(a+m-b, n+p+r; m+n+p+r; 1-U)\,(U-X)^{\ell-1}\, du$$

Further, put $u = 1-(1-X)t$ in the above and by simplifying we have

$$I = \frac{X^d(1-X)^{m+n+p+\ell-1}}{\Gamma(\ell)\,\Gamma(m+p+n)} \int_0^1 \sum_{k=0}^{\infty} \frac{(d+\ell-a)_k}{k!} \sum_{r=0}^{\infty} \frac{(p)_r(b+n-c)_r}{r!(m+n+p)_r} \tag{4.7}$$

$$\sum_{i=0}^{\infty} \frac{(a+m-b)_i(m+p+r)_i}{i!(m+n+p+r)_i} (1-X)^{k+i+r}\, t^{m+n+p+k+i+r-1}(1-t)^{\ell-1}dt\ .$$

Now integrate (4.7) with respect to t, then the lemma follows immediately.

The moments of the Wilks' Criterion has been given [2] in the following form.

$$E\{W^{(h)}\} = \left[\Gamma_p(h+\tfrac{1}{2}t)\ \Gamma_p(\nu)/\Gamma_p(t/2)\ \Gamma_p(h+\nu)\right]\ {}_1F_1(h;h+\nu;-\underset{\sim}{\Omega})\Big] , \tag{4.8}$$

where $\nu = \frac{1}{2}(s+t)$, and $\Gamma_p(u) = \prod_{i=1}^{p} (u-\frac{1}{2}(i-1))$.

By using Kummar transformation then (4.8) can be written in the following form

$$E\{W^{(h)}\} = \left[\Gamma_p(h+\tfrac{1}{2}t)\ \Gamma_p(\nu)/\Gamma_p(t/2)\ \Gamma_p(h+\nu)\right] e^{-tr\underset{\sim}{\Omega}}\ {}_1F_1(\nu;h+\nu;\underset{\sim}{\Omega})\ . \tag{4.9}$$

<u>Case</u> <u>i</u>. Put $p = 2$ in (4.9), then

$$E\{W^{(h)}\} = \frac{\Gamma(2\nu-1)}{2^s\ \Gamma(t-1)}\ e^{-tr\underset{\sim}{\Omega}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa\ C_\kappa(\underset{\sim}{\Omega})}{k!} \cdot \frac{\Gamma(r)\ \Gamma(r+\frac{1}{2})}{\Gamma(r+\frac{1}{2}s+k_1+\frac{1}{2})\ \Gamma(r+\frac{1}{2}s+k_2)} , \tag{4.10}$$

where $r = h+\frac{1}{2}t-\frac{1}{2}$ and $k_1 \geq k_2 > 0$, $k_1 + k_2 = k$,

then

$$(4.11)\qquad f(W^{(2)}) = \frac{\Gamma(2\nu-1)}{2^s\Gamma(t-1)} \exp(\mathrm{tr}\text{-}\underset{\sim}{\Omega}) \sum_{k=o}^{\infty}\sum_{\kappa} \frac{(\nu)_\kappa\, C_\kappa(\underset{\sim}{\Omega})}{k!} \;.$$

$$\frac{1}{2\Pi i}\int_{c-i\infty}^{c+i\infty} \{W^{(2)}\}^{-h-1}\left[\Gamma(r)\ \Gamma(r+\tfrac{1}{2})/\Gamma(r+\tfrac{1}{2}s+k_2)\ \Gamma(r+\tfrac{1}{2}s+\tfrac{1}{2}+k_1)\right]dr \;.$$

Now, by the use of the results of Consul [4], we get the denisty function of $W^{(2)}$ in the following form

$$(4.12)\qquad f(W^{(2)}) = \frac{\Gamma(2\nu-1)}{2^s\Gamma(t-1)} \{W^{(2)}\}^{\frac{1}{2}(t-3)}\exp(\mathrm{tr}\text{-}\underset{\sim}{\Omega}) \cdot \sum_{k=o}^{\infty}\sum_{\kappa} \frac{(\nu)_\kappa\, C_\kappa(\underset{\sim}{\Omega})}{k!\,\Gamma(s+k)}$$

$$(1-W^{(2)})^{s+k-1}\ {}_2F_1(\tfrac{1}{2}s+k_1,\tfrac{1}{2}s+k_2-\tfrac{1}{2};s+k;1-W^{(2)}) \;.$$

Putting $\underset{\sim}{\Omega} = \underset{\sim}{0}$, then the central case can be written in the following form

$$(4.13)\quad f(W^{(2)}) = \frac{\Gamma(2\nu-1)}{2^s\ \Gamma(t-1)\ \Gamma(s)} \{W^{(2)}\}^{\frac{1}{2}} (1-W^{(2)})^{s-1}\ {}_2F_1(s/2,(s-1)/2;s;1-W^{(2)}) \;.$$

It may be pointed out that (4.13) can be reduced to

$$(4.14)\quad \frac{\Gamma(2\nu-1)}{2\Gamma(t-1)\ \Gamma(s)} \{W^{(2)}\}^{\frac{1}{2}(t-3)} (1-\sqrt{W^{(2)}})^{s-1} \;,$$

by observing that

$$(4.15)\qquad {}_2F_1(s/2,(s-1)/2;s;1-U) = 2^{s-1}/(1+\sqrt{U})^{s-1} \qquad ([11]) \;.$$

Also the density function of $W^{(2)}$ can be written in the following form by the use of the results in [3].

$$f(W^{(2)}) = \frac{\Gamma(2\nu-1)}{2\Gamma(t-1)} \{W^{(2)}\}^{\frac{1}{2}(t-3)} \exp(\mathrm{tr}-\underset{\sim}{\Omega}) \cdot \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!\Gamma(s+2k_2)}$$

$$(4.16) \qquad 4^{k_2}(1-W^{(2)})^{k_1-k_2} \sum_{r=o}^{s+2k_2-1} \binom{s+2k_2-1}{r} (-1)^r \{W^{(2)}\}^{r/2} \cdot$$

$$_2F_1(k_1-k_2;(r+1-s)/2 - k_2;k_1-k_2+1;1 - W^{(2)}) .$$

Setting $\underset{\sim}{r} = \underset{\sim}{0}$, then (4.16) reduces to (4.14).

Case ii. Put $p = 3$ in (4.9), and by the use of (4.2) the density function of $W^{(3)}$ can be written in the following form

$$(4.17) \quad f(W^{(3)}) = \frac{\Gamma_3(\nu)}{\Gamma_3(t/2)} \exp(\mathrm{tr}-\underset{\sim}{\Omega})\{W^{(2)}\}^{\frac{1}{2}(t-4)} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!} \cdot$$

$$\frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} \frac{\{W^{(3)}\}^{-r} \Gamma(r)\ \Gamma(r+\frac{1}{2})\ \Gamma(r+1)}{\Gamma(r+\frac{1}{2}s+k_3)\ \Gamma(r+\frac{1}{2}s+k_2+\frac{1}{2})\ \Gamma(r+\frac{1}{2}s+k_1+1)} dr$$

where $k_1 \geq k_2 \geq k_3 \geq 0, \quad k_1 + k_2 + k_3 = k$.

By (4.5), the density function of $W^{(3)}$ can be written in the form

$$(4.18) \quad f(W^{(3)}) = \frac{\Gamma_3(\nu)}{\Gamma_3(t/2)} \exp(\mathrm{tr}-\underset{\sim}{\Omega}) \{W^{(3)}\}^{\frac{1}{2}(t-1)} (1 - W^{(3)})^{\frac{3}{2}s-1} \cdot$$

$$\sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa C_\kappa(\underset{\sim}{\Omega})}{k!\ \Gamma(3s/2+k)} \sum_{r=o}^{\infty} \frac{(\frac{1}{2}s+k_1)_r(\frac{1}{2}(s-1)+k_2)_r}{r!\ (3s/2+k)_r} (1 - W^{(3)})^{r+k}$$

$$_2F_1(\tfrac{1}{2}(s-1)+k_3, s+k_1+k_2+r; 3s/2+k+r; 1 - W^{(3)}) .$$

Case iii. Put $p = 4$ in (4.9) and by the use of (4.2) the density function of $W^{(4)}$ can be written in the form

$$(4.19)\quad f(W^{(4)}) = \frac{\Gamma_4(\nu)}{\Gamma_4(\tfrac{1}{2}t)} \exp(\mathrm{tr}\text{-}\underset{\sim}{\Omega}) \{W^{(4)}\}^{\frac{1}{2}(t-5)} \sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa\, C_\kappa(\underset{\sim}{\Omega})}{k!} \cdot$$

$$\frac{1}{2\Pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(r)\, \Gamma(r+\frac{1}{2})\, \Gamma(r+1)\, \Gamma(r+\frac{3}{2})\, \{W^{(4)}\}^{-r} \qquad dr}{\Gamma(r+\frac{1}{2}s+k_4)\, \Gamma(r+\frac{1}{2}s+\frac{1}{2}+k_3)\, \Gamma(r+\frac{1}{2}s+1+k_2)\, \Gamma(r+\frac{1}{2}s+\frac{3}{2}+k_1)_r}$$

where $k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0$, and $\sum_{i=1}^{4} k_i = k$.

By using lemma 1, the density function of $W^{(4)}$ can be written in the form

$$(4.20)\quad f(w^{(4)}) = \frac{\Gamma_4(\nu)}{\Gamma_4(t/2)} \exp(\mathrm{tr}\text{-}\underset{\sim}{\Omega}) \{W^{(4)}\}^{\frac{1}{2}(t-2)} (1 - W^{(4)})^{2s-1}$$

$$\sum_{k=o}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa\, C_\kappa(\underset{\sim}{\Omega})}{k!} \sum_{=o}^{\infty} \frac{(\frac{1}{2}(s+3)+k_1)_j}{j!} \sum_{r=o}^{\infty}$$

$$\frac{(\frac{1}{2}(s+k_2)_r (\frac{1}{2}(s-1)+k_3)_r}{\Gamma(3s/2+k-k_1+r)r!} (1 - W^{(4)})^{k+j+r} \frac{\Gamma(3s/2+k+j-k_1+r)}{\Gamma(2s+k+j+r)}$$

$${}_3F_2(\tfrac{1}{2}(s-1)+k_4, s+k_2+k_3+r, 3s/2+k-k_1+j+r; 3s/2+k-k_1+r, 2s+j+k+r; 1 - W^{(4)}).$$

It may be pointed out that the non-central distribution of Wilks' criterion could be found for more than $p = 3$ by extending lemma 1. However the distribution would be complicated.

References

[1] Anderson, T.W. (1948). The Asymptotic Distribution of the Roots of Certain Determinantal Equations. Roy. Statist. Soc. 10, 132-139.

[2] Constantine, A.G. (1963). Some Non-Central Distribution Problems in Multivariate Analysis. Ann. Math. Statist. 34, 1270-1285.

[3] Consul, P.C. (1966). On Some Inverse Mellin Integral Transform. Bull. Class. Des Sci. 52, 547-561.

[4] Consul, P.C. (1967). On the Exact Distribution of the Criterion W for Testing Sphericity in a p-variate Normal Distribution. Ann. Math. Statist. 38, 1170-1174.

[5] James, A.T. (1960). The Distribution of the Latent Roots of the Covariance Matrix. Ann. Math. Statist. 31, 151-158.

[6] James, A.T. (1964). Distribution of Matrix Variates and Latent Roots Derived from Normal Samples. Ann. Math. Statist. 35, 475-501.

[7] Khatri, C.G. (1967). Some Distribution Problems Connected with the Characteristic roots of $S_1S_2^{-1}$. Ann. Math. Statist. 38, 944-948.

[8] Khatri, C.G. and Pillai, K.C.S. (1968). On the non-central distributions of two test criteria in multivariate analysis of variance. Ann. Math. Statist. 39, to appear.

[9] Pillai, K.C.S. and Al-Ani, S. (1967). On some distribution problems concerning characteristics roots and vectors in multivariate analysis. Mimeograph Series No. 123, Department of Statistics, Purdue University.

[10] Rao, C.R. (1965). Linear Statistical Inference and the Applications. John Wiley and Sons, Inc., New York.

[11] Snow, C. (1942). The Hypergeometric and Legendre functions with applications to integral equations of potential theory. National Bureau of Standards.