On Some Distribution Problems Concerning Characteristic Roots and Vectors in Multivariate Analysis*

by

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0. Introduction and Summary. In this paper, exact non-central distributions of individual characteristic roots have been obtained first in two and three roots cases in connection with tests of the hypothesis $\delta \underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, where $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ are covariance matrices of two normal populations and $\delta > 0$, known. Powers of tests using individual roots are tabulated for the test of this hypothesis against various one-sided simple alternatives and comparisons of powers made.

Further, the central distribution of the second largest (smallest) of s non-zero roots following the Fisher-Girshick-Hsu-Roy distribution under certain null-hypotheses has been derived in series form. The distribution of the characteristic vectors is obtained next corresponding to the largest and second largest root of a sample covariance matrix. The three roots-case is dealt with in more detail.

While the earlier sections deal with the studies of individual roots, the last two sections present the distributions of differences and ratios respectively of characteristic roots which again follow the Fisher-Hsu-Girshick-Roy distribution. In regard to differences, the study has been carried out up to (including) the four-roots case while for the ratios, results have been obtained up to five roots.

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1. <u>Non-Central cdf of the Largest Root For Testing $\delta\, \Sigma_1 = \Sigma_2$</u>: Let $S_i(p\times p)$, $(i=1,2)$ be independently distributed as Wishart (f_i, p, Σ_i). Let the characteristics (Ch.) roots of $S_1 S_2^{-1}$ and $\Sigma_1 \Sigma_2^{-1}$ be denoted by c_i and λ_i, $i=1,\ldots,p$ respectively such that $0 < c_1 < c_2 < \ldots < c_p < \infty$ and $0 < \lambda_1 < \ldots < \lambda_p < \infty$. Let $g_i = \delta\, c_i/1+\delta\, c_i$, $i=1,\ldots,p$; $\delta > 0$ and $G = \mathrm{diag}\,(g_1,\ldots,g_p)$ and $\Lambda = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_p)$, then the distribution of $g_1,\ldots,g_p$ is given by Khatri [4] in the following form

$$(1.1)\quad c(p,m,n)\,|\delta\Lambda|^{-\frac{1}{2}f_1}\,|G|^m\,|I-G|^n \prod_{i>j} (g_i - g_j)\; {}_1F_0\left(\tfrac{1}{2}\nu;\, \Lambda_1, G\right),$$

where

$$c(p,m,n)=[\pi^{p/2}\prod_{i=1}^{p} \Gamma\{\tfrac{1}{2}(2m+2n+p+i+2)\}]/[\prod_{i=1}^{p} \Gamma\{\tfrac{1}{2}(2m+i+1)\}\Gamma\{\tfrac{1}{2}(2n+i+1)\}\Gamma(\tfrac{i}{2})],$$

$\Lambda_1 = I - (\delta\Lambda)^{-1}$, $m = \frac{1}{2}(f_1-p-1)$, $n = \frac{1}{2}(f_2-p-1)$, $f_1 + f_2 = \nu$ and ${}_1F_0$ is the hypergeometric function of matrix argument defined by James [3] as

$$(1.2)\quad {}_sF_t(a_1,\ldots,a_s;\, b_1,\ldots,b_t;\, S, T) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots\cdots (a_t)_\kappa\, C_\kappa(S) C_\kappa(T)}{(b_1)_\kappa \cdots\cdots (b_t)_\kappa\, C_\kappa(I_p) k!},$$

where

$a_1,\ldots,a_s$, $b_1,\ldots,b_t$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is given by

$$(1.3)\quad (a)_\kappa = \prod_{i=1}^{p} (a-\tfrac{1}{2}(i-1))_{k_i},$$

where

$$(1.4) \qquad (a)_k = a(a+1)\ldots(a+k-1)$$

and κ is the partition of k such that $\kappa = (k_1,\ldots,k_p)$, $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$ and the zonal polynomials $C_\kappa(\underset{\sim}{S})$ are expressible in terms of elementary symmetric functions (esf) of the characteristic roots of $\underset{\sim}{S}$ [3] .

Now define by $V(q_p,n;\ldots\ldots;\ x',x'',\ q_j,\ n;\ldots\ldots;\ q_1,n)$ the determinant

$$(1.5) \quad \begin{vmatrix} \int_{x_{p-1}}^{1} x_p^{q_p}(1-x_p)^n\,dx_p \int_{x_{p-2}}^{1} x_{p-1}^{q_p}(1-x_{p-1})^n dx_{p-1}\cdots\int_{x'}^{x''} x_j^{q_p}(1-x_j)^n dx_j\cdots\int_0^{x_3} x_2^{q_p}(1-x_2)^n dx_2\int_0^{x_2} x_1^{q_p}(1-x_1)^n dx_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \int_{x_{p-1}}^{1} x_p^{q_1}(1-x_p)^n dx_p \int_{x_{p-2}}^{1} x_{p-1}^{q_1}(1-x_{p-1})^n dx_{p-1}\cdots \int_{x'}^{x''} x_j^{q_1}(1-x_j)^n dx_j\cdots \int_0^{x_3} x_2^{q_1}(1-x_2)^n dx_2 \int_0^{x_2} x_1^{q_1}(1-x_1)^n dx_1 \end{vmatrix}$$

It may be observed that the cdf of the largest root from (1.1) under the null hypothesis $\delta\,\underset{\sim}{\Sigma_1} = \underset{\sim}{\Sigma_2}$ can be thrown into the form $V(0,x;q_p,n;\ldots,q_1;n)$, which for simplicity of notation will be written here after $V(0,x;q_p,\ldots,q_1;n)$, multiplied by $C(p,m,n)$ [6], [7], [9]. Further, in view of the fact that the zonal polynomials $C_\kappa(\underset{\sim}{S})$ in (1.2) can be expressed in terms of the esf's of ch-roots of $\underset{\sim}{S}$, by the use of Pillai's lemma on the multiplication of the basic Vandermonde type determinant by powers of esf's, [9], it is easy to see that the non-central distribution of the cdf of g_p in (1.1) can be expressed as a series whose terms are linear compounds of determinants of type $V(0,x;q_p',\ldots,q_1';n)$, where $(q_p',\ldots,q_1')$ may differ from term to term.

Further, it has been shown that [6], [7]

$$(1.6)\quad V(0,x;q_s,q_{s-1},\ldots,q_1;n) = (q_s+n+1)^{-1}\,(A^{(s)}+B^{(s)}+q_s\,C^{(s)})\ ,$$

where

$$A^{(s)}= -I_o(0,x;q_s,n+1)V(0,x;q_{s-1},\ldots,q_1;n),\ B^{(s)}=2\sum_{j=s-1}^{1}(-1)^{s-j-1}I(0,x;q_s+q_j;2n+1)$$

$$V(0,x;q_{s-1},\ldots,q_{j+1},q_{j-1},\ldots,q_1;n),\ C^{(s)}= V(0,x;q_s-1,q_{s-1},\ldots,q_1;n),\ I_o(x',x'';q_s,n+1) =$$

$$x^{q_s}(1-x)^{n+1}\Big|_{x'}^{x''},\quad \text{and}\quad I(x',x'';q,r) = \int_{x'}^{x''} x^q(1-x)^r\,dx\ .$$

It may be noted that $C^{(s)}$ vanishes if $q_s= q_{s-1}+1$. Using (1.6) in each of the determinants of the linear compounds involved in the series obtainable from (1.2), after the necessary number of reductions, the cdf of the largest root (g_p) can be ultimately reduced in terms of simple incomplete beta functions.

2. Non-Central cdf's of individual roots. In this section we give the non-central cdf's of individual roots, associated power function tabulations and comparisons of powers for testing $\delta\ \Sigma_1 = \Sigma_2$ against various simple hypotheses.

a) Non-Central cdf of g_2. Now putting $p = 2$ in (1.1) and using the method outlined in the preceding section the cdf of the largest root is obtained in the following form:

$$(2.1)\quad \Pr\{g_2 \le x\} = k\Big\{-I_o(0,x;m+1,n+1)\Big[\Big(\sum_{i=0}^{6} B_i x^i\Big) I(0,x;m,n) + \Big(\sum_{i=2}^{6} C_i x^{i-1}\Big) I(0,x;m+1,n)$$

$$+ \Big(\sum_{i=4}^{6} D_i x^{i-2}\Big) I\,(0,x;m+2,n) + E_6 x^3 I(0,x;m+3,n)\Big]$$

$$+ 2\Big[(B_6 + C_6 + D_6 + E_6)\, I\,(0,x;2m+7,2n+1)$$

$$+ (B_5 + C_5 + D_5)\, I\,(0,x;2m+6,2n+1) + (B_4 + C_4 + D_4)\, I\,(0,x,2m+5,2n+1)$$

$$+ (B_3 + C_3)\, I\,(0,x;2m+4,2n+1) + (B_2 + C_2)\, I\,(0,x;2m+3,2n+1)$$

$$+ B_1 I\,(0,x;2m+2,2n+1) + B_0\, I\,(0,x;2m+1,2n+1)\Big]\Big\}$$

where $k = (\delta^2\lambda_1\lambda_2)^{-\frac{f_1}{2}} c(2,m,n)$, B's, C's, D's and E_6 are obtained from Pillai [10] by making the following changes:

In the A_{ij} coefficients in [10], delete each linear factor involving f_2 in the denominator, each linear factor involving ν in the numerator should be raised only to a single power instead of two and b_1 and b_2 should be changed to $2 - (1/\lambda_1 + 1/\lambda_2)/\delta$ and $[1-1/(\delta\lambda_1)][1-1/(\delta\lambda_2)]$ respectively.

In obtaining the cdf of g_2 in (2.1), zonal polynomials of degree 1 to 6 were used. The expression for the cdf of g_2 in (2.1) has been used to compute the power of test H_o: $\delta\, \Sigma_1 = \Sigma_2$, $\delta > 0$, known, against $\delta\, \lambda_i \ge 1$, $i = 1,\ldots,p$, $\sum_{i=1}^{p} (\delta\lambda_i) > p$, for various pairs of values $(\delta\lambda_1, \delta\lambda_2)$ and the results are presented in table 1.

$$(5.12)\ k\ (\omega_1\ \omega_2\ \omega_3)^{\frac{1}{2}(n-4)} \prod_{i>j} (\omega_i - \omega_j) \sin\theta_{33} \exp(-b_3\ \omega_3)$$

$$\left[\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=o}^{k} \binom{k}{i} b_2^i\ b_1^{k-i}\ \omega_2^i\ \omega_1^{k-i}\right]$$

where

$$b_1 = -\tfrac{1}{2}(\sin\theta_{22} - \cos\theta_{22}) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \sin\theta_{22} \\ -\cos\theta_{22} \end{pmatrix},$$

$$b_2 = -\tfrac{1}{2}(\cos\theta_{22}\ \sin\theta_{22}) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta_{22} \\ \sin\theta_{22} \end{pmatrix},$$

and

$$b_3 = \tfrac{1}{2}\ h_3'\ \underset{\sim}{D}_{\gamma}\ h_3 \quad .$$

Let $\ell = \omega_1/\omega_2$, then the distribution of $\theta_{33}, \theta_{32}, \theta_{22}, \ell_1, \omega_2, \omega_3$ is given by

$$(5.13)\ k\ \omega_3^{\frac{1}{2}(n-4)}\ \omega_2^{n-2}(\omega_3 - \omega_2) \sin\theta_{33} \exp(-b_3\omega_3) \left[\sum_{k=0}^{\infty} \frac{\omega_2^k}{k!} \sum_{i=0}^{k} \binom{k}{i}\right.$$

$$\left. b_2^i\ b_1^{k-i}(\omega_3(1-\ell)\ell^{\frac{1}{2}n+k-i-2} - \omega_2(1-\ell)\ \ell^{\frac{1}{2}n+k-i-1})\right] \quad .$$

Integrate (5.5) with respect to ℓ, then

$$(5.14)\ k\ \omega_2^{n-2}\ \omega_3^{\frac{1}{2}(n-4)}\ (\omega_3 - \omega_2) \sin\theta_{33} \exp(-b_3\ \omega_3)$$

$$\left[\sum_{k=0}^{\infty} \frac{\omega_2^k}{k!} \sum_{i=0}^{k} \binom{k}{i}\ b_2^i\ b_1^{k-i}\ (\omega_3\ \beta(\tfrac{1}{2}n+k-i-1,2) - \omega_2\ \beta(\tfrac{1}{2}n+k-i,2))\right],$$

Again make the transformation $t = \omega_2/\omega_3$, integrate with respect to t and then with respect to ω_3, we can write the distribution of $\theta_{33}, \theta_{32}, \theta_{22}$ in the form

$$(5.15)\quad k \sin\theta_{33} \Big[\sum_{k=0}^{\infty} \frac{\Gamma\{3n/2)+k\}}{k!\ b_3^{(3n/2)+k}} \sum_{i=0}^{k} \binom{k}{i} b_2^{i}\, b_1^{k-i}$$

$$\beta(n+k-1,2)\ \beta(\tfrac{1}{2}n+k-i-1,2)(1- \frac{(n+k)(\frac{1}{2}n+k-i)}{(n+k+2)(\frac{1}{2}n+k-i+2)}) \Big] .$$

For any p, integrate (5.10) with respect to $\frac{1}{2}(p-2)(p-3)$ independent elements of $\underset{\sim}{H}_{p-2}$ by using Lemma (3.2) of Sugiyama [15], we can write the distribution of $\omega_1,\ldots,\omega_p$, $\theta_{ij}(i=p,p-1;j=i,\ldots 2)$ in the form

$$(5.16)\quad k\ \{\pi^{(p-2)^2/2}/\Gamma_{p-2}(\tfrac{1}{2}(p-2))\}(\omega_p-\omega_{p-1})^{\frac{1}{2}(n-p-1)}|\underset{\sim}{W}_2|^{\frac{1}{2}(n-p-1)}\exp(-\tfrac{1}{2}h_p'\underset{\sim}{D}_{\gamma} h_p\omega_p)$$

$$\exp(-\tfrac{1}{2}h_{p-1}'\ \underset{\sim}{D}_{p-1}\ h_{p-1}\ \omega_{p-1})\ \prod_{j=p}^{2} \sin^{j-2}\theta_{pj} \prod_{j=p-1}^{2} \sin^{j-2}\theta_{p-1,j}$$

$$\prod_{i>1}(\omega_i-\omega_j) \Big[\sum_{k=0}^{\infty}\sum_{\kappa} \{C_\kappa(-\tfrac{1}{2}\underset{\sim}{D}_{p-2})\ C_\kappa(\underset{\sim}{W}_2)/k!\ C_\kappa(\underset{\sim}{I}_{p-1})\}\Big] .$$

Now make the transformation $\ell_i' = \omega_i/\omega_{p-1}$, $i=1,\ldots,p-2$, and using James [3], the distribution of ℓ_1', $\ell_2',\ldots,\ell_{p-2}'$, ω_{p-1}, ω_p, $\theta_{ij}(i=p,p-1;j=i,\ldots,2)$ can be written in the form

$$(5.17)\quad k\left\{\pi^{(p-2)^2/2}/\ \Gamma_{p-2}(\tfrac{1}{2}(\tfrac{p-2}{2}))\right\}\ \omega_p^{\frac{1}{2}(n+p-5)}\ \omega_{p-1}^{\frac{1}{2}(np-p-n-1)}(\omega_p-\omega_{p-1})$$

$$|\underset{\sim}{L}'|^{\frac{1}{2}(n-p-1)}|\underset{\sim}{I}-\underset{\sim}{L}'|\prod_{i>j}(\ell_i-\ell_j)\ \exp(-\tfrac{1}{2}h_p'\ \underset{\sim}{D}_\gamma\ h_p\ \omega_p)$$

$$\exp(-\tfrac{1}{2}h_{p-1}'\ \underset{\sim}{D}_{p-1}\ h_{p-1}\ \omega_{p-1})\prod_{j=p}^{2}\sin^{j-2}\theta_{pj}\ \prod_{j=p-1}^{2}\ \sin^{j-2}\ \theta_{p-1,j}$$

$$\left[\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{p-2}\left\{C_\kappa(-\tfrac{1}{2}\underset{\sim}{D}_{p-2})\ C_\kappa(\underset{\sim}{L}')\ C_{(1^j)}(\underset{\sim}{L}')(-1)^j(2j)!\ \omega_{p-1}^{j+k}\ /\right.\right.$$

$$\left.\left.\omega_p^j(j!)^2\ k!\ \chi_{(21^j)}(1)\ C_\kappa(\underset{\sim}{I})\right\}\right].$$

Now by multiplication of two zonal polynomials [4] and integrating (5.18) with respect to $o < \ell_1' \le \ell_2' \le \cdots \le \ell_{p-2}' \le 1$, we get the distribution of ω_{p-1}, ω_p, $\theta_{ij}(i=p,p-1;j=i,\ldots,2)$

$$(5.18)\ k\Gamma_{p-2}(\tfrac{p+1}{2})\ \omega_p^{\frac{1}{2}(n+p-5)}\ \exp(-\tfrac{1}{2}h_p'\ \underset{\sim}{D}_\gamma\ h_p\ \omega_p)\cdot\prod_{j=p}^{2}\ \sin^{j-2}\ \theta_{pj}$$

$$\prod_{j=p-1}^{2}\sin^{j-2}\ \theta_{p-1,j}\ \left[\sum_{r=0}^{\infty}\ \sum_{k=0}^{\infty}\sum_{\kappa}\sum_{j=0}^{p-2}\sum_{\tau}\{(-1)^j(2j)!\ g^{\tau}_{(\kappa,1^j)}\ C_\kappa(-\tfrac{1}{2}\underset{\sim}{D}_{p-2})\right.$$

$$\Gamma_{p-2}\{\tfrac{1}{2}(n-2),\tau\}(-\tfrac{1}{2}\ h_{p-1}'\ \underset{\sim}{D}_{p-1}\ h_{p-1})^r\ C_\tau(\underset{\sim}{I})(\omega_p-\omega_{p-1})\omega_{p-1}^{\frac{1}{2}(np-n-p-1)+k+i+r}/$$

$$\left.\omega_p^j\ (j!)^2\ k!\ r!\ \chi_{(21^j)}(1)\ C_\kappa(\underset{\sim}{I})\ \Gamma_{p-2}(\tfrac{1}{2}(n+p-1),\tau)\}\right]$$

where τ and $g^{\tau}_{(\kappa,1^j)}$ and $\chi_{(21^j)}(1)$ as defined in section 4. Further let $\omega_{p-1}=\ell\omega_p$, integrate ℓ and then ω_p, the distribution of $\theta_{ij}(i=p,p-1;j=i,..,2)$ in the form:

$$(5.19)\quad k\Gamma_{p-2}(\tfrac{1}{2}(p+1)) \prod_{j=p}^{2} \sin^{j-2}\theta_{pj} \cdot \prod_{j=p-1}^{2} \sin^{j-2}\theta_{p-1,j} \Big[\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\kappa}$$

$$\sum_{j=o}^{p-2}\sum_{\tau}\{(-1)^j(2j)!\, g^{\tau}_{(\kappa,1^j)}\; C_\kappa(-\tfrac{1}{2}D_{p-2})\; C_\tau(I)\;(-\tfrac{1}{2}h'_{p-1}\, \underset{\sim}{D}_{p-1}\, h_{p-1})^r$$

$$\beta(\tfrac{1}{2}(np-n-p+1)+k+j+r,2)\Gamma\{\tfrac{1}{2}(np+k+r)\}/(j!)^2k!\; r!\; \chi_{(21^j)}(1)$$

$$C_\kappa(\underset{\sim}{I})\;\Gamma_{p-2}\;\{\tfrac{1}{2}(n+p+1),\tau\}(-\tfrac{1}{2}h'_p\,\underset{\sim}{D}_\gamma\, h_p)^{\frac{1}{2}(np)+k+r}\}\Big] \;.$$

6. The Distribution of the Differences of the Characteristic Roots*.

In this section we find the joint and the marginal distributions of the differences $\theta_i - \theta_j$, $i > j$ when $p = 2,3,4$. First we observe that the distribution of $\ell_1,\ldots,\ell_{p-1}, \theta_p$ is given by Pillai [12] in the form

$$(6.1)\quad c(p,m,n)\theta_p^{mp+(p-1)(1+\frac{p}{2})}\;(1-\theta)^n \prod_{i=1}^{p-1}\{\ell_i^m(1-\ell_i\theta_p)^n(1-\ell_i)\} \prod_{i>j}(\ell_i-\ell_j),$$

where

$$\ell_i = \theta_i/\theta_p \;,\; i = 1,\ldots,p-1 \;.$$

Now consider the transformation $d_i = \theta_p(1-\ell_i)$, $i=1,\ldots p-1$, then $d_1,\ldots,d_{p-1}, \theta_p$ will be distributed as

$$(6.2)\quad c(p,m,n)|\underset{\sim}{D}| \prod_{i<j}(d_i-d_j)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)}{k!}\kappa\;\; C_\delta(\underset{\sim}{D})\, C_\kappa(\underset{\sim}{D})\theta_p^{mp-d}(1-\theta_p)^{np-k}\Big],$$

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* This section was written because of the interest of J. W. Tukey in the study.

where κ, δ are the partitions of k and d respectively and $\underset{\sim}{D} = \mathrm{diag}(d_1,\ldots,d_{p-1})$. Now integrate (6.2) with respect to θ_p, then $d_1,\ldots,d_{p-1}$ are distributed in the form

$$(6.3)\quad c(p,m,n)|\underset{\sim}{D}| \prod_{i<j}(d_i - d_j)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\,\kappa\, C_\delta(\underset{\sim}{D})$$

$$C_\kappa(\underset{\sim}{D})\quad I(d_1,1;mp-d,np-k)\Big],\quad 0 < d_{p-1} \leq \cdots \leq d_1 < 1 .$$

For $p = 2$, (6.3) reduces to

$$(6.4)\quad f(d_1) = c(2,m,n)\Big[\sum_{j=0}^{m}\binom{m}{j}(-1)^j \sum_{i=0}^{n}\binom{n}{i}\, d_1^{m+n+1-(i+j)} I(d_1,1;m+j,n+i)\Big] .$$

For $p = 3$, the joint density of d_1, d_2 can be written in the form

$$(6.5)\quad c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{\tau} g^{\tau}_{(\delta,\kappa)}\sum_{i+j=t}$$

$$h^{T}_{ij}\{(d_1^{i+2}\, d_2^{j+1} - d_1^{i+1}\, d_2^{j+2})\qquad I(d_1,1;\, 3m-d,3n-k)\}\Big] ,$$

where $g^{T}_{\delta,\kappa}$ is as defined in the previous sections and h^{T}_{ij} are such that $C_\tau\begin{pmatrix} d_1 & 0 \\ 0 & d_2\end{pmatrix} = \sum_{i+j=t} h^{T}_{ij}\, d_1^i\, d_2^j$, τ is the partition of t and $t = k+d$.

Integrate (6.5) with respect to d_2, then the density of d_1 is of the form

$$(6.6)\ c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i+j=t} h^T_{ij}\{\frac{d_1^{t+4}}{(j+2)_2}\ I(d_1,1;3m-d,3n-k)\}\Big].$$

Again, integrate (6.5) with respect to d_1, by parts, then the density of d_2 is given by

$$(6.7)\ c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i+j=t} h^T_{ij}\ \frac{1}{(i+2)_2}\{d_2^{t+4} I(d_2,1;3m-d,3n-k)$$

$$+ d_2^{j+1}((i+2)I(d_2,1;3m-d+i+3,3n-k) - (i+3)\ d_2^{j+2}\ I(d_2,1;3m-d+i+2,3n-k))\}\Big].$$

Now let $\delta_{12} = d_1 - d_2 = \theta_2 - \theta_1$, then the distribution δ_{12} and d_1 can be written in the form

$$(6.8)\ c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i+j=t} h^T_{ij}$$

$$\{\sum_{r=0}^{j+1}(-1)^r\binom{j+1}{r}\ \delta_{12}^{r+1}\ d_1^{t+2-r}\ I(d_1,1;3m-d,3n-k)\}\Big], \qquad 0 < \delta_{12} \le d_1 < 1 .$$

Integrating (6.8) with respect to d_1, we get the density of δ_{12} in the form

$$(6.9)\ c(3,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i+j=t} h^T_{ij}$$

$$\{\sum_{r=0}^{j+1}[(-1)^r\binom{j+1}{r}/t+r-3](-\delta_{12}^{t+4} I(\delta_{12},1;3m-d,3n-k)+\delta_1^{r+1} I(\delta_{12},1;3m-d+t$$

$$+3-r,3n-k))\}\Big].$$

For $p = 4$, the joint density of d_1, d_2, d_3 can be written in the form

$$(6.10)\quad c(4,m,n)\Big[\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}\; c(d_2-d_3)$$

$$(d_1^2-(d_2+d_3)d_1+d_2d_3)\; I(d_1,1;a,b)\Big],$$

where

$$a = 4m-d,\; b = 4n-k,\; c = d_1^{i_1+1}\, d_2^{i_2+1}\, d_3^{i_3+1} .$$

Integrating (6.10) with respect to d_1, by parts, and further with respect to d_2, we get the density of d_3 in the form

$$(6.11)\quad c(4,m,n)\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}\; d_3^{i_3+1}$$

$$\Big[-\frac{2d_3^{i_1+i_2+7}}{(i_1+2)_3(i_1+i_2+5)_3}\, I(d_3,1;a,b) + \frac{I(d_3,1;e+3,b)}{(i_2+3)_2(i_1+i_2+7)} - \frac{2d_3 I(d_3,1;e+2,b)}{(i_2+2)(i_2+4)(i_1+i_2+6)}$$

$$+\frac{d_3^2 I(d_3,1;e+1,b)}{(i_2+2)_2(i_1+i_2+5)} - \frac{d_3^{i_2+3}\, I(d_3,1;e_1+2,b)}{(i_2+2)_2(i_1+4)}\Big],$$

where

$$e = i_1 + i_2 + 4 + a\;,\; e_1 = a + i_1 + 2$$

Similarly starting with (6.10) we can obtain the density of d_1 as

$$(6.12)\quad c(4,m,n)\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}$$

$$\frac{2(i_2+2i_3+9)}{(i_3+2)_3(i_2+i_3+5)_3}\; I(d_1,1;a,b)\;,$$

and the density of d_2 as

$$(6.13)\quad c(4,m,n)\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}\; d_2^{i_2+i_3+4}$$

$$\left[\frac{2(i_1-i_3)d_2^{i_1+4}}{(i_1+2)_3(i_3+2)_3}\; I\,(d_2,1;a,b) + \frac{I(d_2,1;e_1+2,b)}{(i_1+4)(i_3+2)_2} - \frac{2d_2\; I(d_2,1;e_1+1,b)}{(i_1+3)(i_3+2)(i_3+4)}\right.$$

$$\left. + \frac{d_2^2\; I(d_2,1;e_1,b)}{(i_1+2)(i_3+3)_2}\right].$$

Now make the transformation

$$(6.14)\quad d_1 = \delta_1+\delta_2+\delta_3\;,\; d_2 = \delta_2+\delta_3\;,\; d_3 = \delta_3,\; \delta_{13} = \theta_3 - \theta_1$$

Using (6.14), then from the joint distribution of δ_1 , d_2 can be obtained in the form:

$$(6.15)\quad f(\delta_1,d_2) = c(4,m,n)\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_\delta}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-1)^k(-n)_\kappa}{k!}\sum_{T} g^T_{(\delta,\kappa)}\sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}$$

$$\left[\sum_{r=0}^{i_1+1}\binom{i_1+1}{r}\delta_1^{r+1}\, d_2^{t+5-r}\left(\frac{\delta_1}{(i_3+2)_2}+\frac{2d_2}{(i_3+2)_3}\right) I(d_2+\delta_1,1;a,b)\right].$$

Further, integrate d_2 over $0 \le d_2 \le 1 - \delta_1$ then the distribution of δ_1 can be written in the form

$$(6.16)\quad c(4,m,n) \sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k (-n)_\kappa}{k!} \sum_{T} g^T_{(\delta,\kappa)} \sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}$$

$$\Big[\frac{\delta_1}{(i_3+2)_2} \int_0^{1-\delta_1} \Big\{ \sum_{r=0}^{i_1+1} \binom{i_1+1}{r} \delta_1^{r+1} d_2^{t+6-r} (d_2+\delta_1)^a (1-d_2-\delta_1)^b$$

$$\Big(\frac{\delta_1}{t+6-r} + \frac{2}{(i_3+4)(t+7-r)}\Big)/t+6-r \Big\}\, dd_2 \Big].$$

Similarly the density of δ_2 can be written in the form

$$(6.17)\quad c(4,m,n) \sum_{d=0}^{\infty} \sum_{\delta} \frac{(-m)_\delta}{d!} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-1)^k (-n)_\kappa}{k!} \sum_{T} g^T_{(\delta,\kappa)} \sum_{i_1+i_2+i_3=t} h^T_{i_1,i_2,i_3}$$

$$\Big[\int_0^{1-\delta_2} \Big\{ \frac{\sum_{r=0}^{i_1+i_2+6} \binom{i_1+i_2+6}{r}}{(i_1+3)_2} - \frac{2\sum_{r=0}^{i_1+i_2+5} \binom{i_1+i_2+5}{r}}{(i_1+2)(i_1+4)} + \frac{\sum_{r=0}^{i_1+i_2+4} \binom{i_1+i_2+4}{r}}{(i_1+2)_2} \Big\}$$

$$\frac{\delta_2^{r}\, \delta_3^{t+8-r}}{t+8-r} (\delta_2+\delta_3)^a (1-\delta_2-\delta_3)^b d\delta_3 + \Big(\frac{\sum_{r=0}^{i_2+2} \binom{i_2+2}{r} - \sum_{r=0}^{i_2+1} \binom{i_2+1}{r}}{i_1+4} \Big)$$

$$q(\delta_2,r,0) \qquad - \Big(\frac{\sum_{r=0}^{i_2+3} \binom{i_2+3}{r} - \sum_{r=0}^{i_2+1} \binom{i_2+1}{r}}{i_1+3} \Big)$$

$$q(\delta_2,r,1) \qquad + \Big(\frac{\sum_{r=0}^{i_2+3} \binom{i_2+3}{r} - \sum_{r=0}^{i_2+2} \binom{i_2+2}{2}}{i_1+2} \Big) q(\delta_2,r,2) \Big],$$

where

$$q(\delta_2,r,j) = \delta_2^2 \int_0^{1-\delta_2} \frac{\delta_3^{i_2+i_3+4-r+j}}{i_2+i_3+4-r+j} (\delta_2+\delta_3)^{e_1+2-j} (1-\delta_2-\delta_3)^b \, d\delta_3 \,, \quad j = 0,\ 1,\ 2 \ .$$

Similarly the distribution of δ_{13} can be written in the form

$$(6.18)\ c(4,m,n)\sum_{d=0}^{\infty}\sum_{\delta}\frac{(-m)_{\delta}}{d!}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_{\kappa}(-1)^{k}}{k!}\sum_{T} g^{T}_{(\delta,\kappa)}\sum_{i_1+i_2+i_3=t} h^{T}_{i_1,i_2,i_3}$$

$$\delta_{13}\left[\left\{A(r)\ \delta_{13}^{r}\ I(\delta_{13},1;a+7+t-r,b) - A(r)\ \delta_{13}^{t+7}\ I(\delta_{13},1;a,b)\right\}/t+7-r\right].$$

where

$$A(r) = \left[\sum_{r=0}^{i_3+1}\binom{i_3+1}{r}(-1)^{r} - \sum_{r=0}^{i_2+i_3+5}\binom{i_2+i_3+5}{r}(-1)^{r}\right]/(i_2+3)_2$$

$$+\left[\sum_{r=0}^{i_2+i_3+4}\binom{i_2+i_3+4}{r}(-1)^{r} - \sum_{r=0}^{i_3+2}\binom{i_3+2}{r}(-1)^{r}\right]/(i_2+2)_2$$

7. <u>On the distribution of the ratios of the Ch. roots</u>. The ratios of the ch. roots are useful in various respects, but one immediate use can be seen from section (1), for tests of hypotheses when δ is not known.

Integral (6.1) with respect to θ_p, then the distribution of $\ell_1,\dots,\ell_{p-1}$ is given by

$$(7.1)\ c(p,m,n)|\underset{\sim}{L}|^{m}|\underset{\sim}{I}-\underset{\sim}{L}|\prod_{i>j}(\ell_i-\ell_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_{\kappa}}{k!}C_{\kappa}(\underset{\sim}{L})\ \beta_k$$

where

$\underset{\sim}{L} = \mathrm{diag}(\ell_1,\dots,\ell_{p-1})$ and $\beta_k = \beta(mp+(p-1)(1+\frac{1}{2}p)+k+1,n+1)$

Consider the transformation $m_i = \ell_i/\ell_{p-1}$, $i = 1,\ldots, p-2$ then the distribution of $m_1,\ldots,m_{p-1}, \ell_{p-1}$ can be obtained in the form

$$(7.2)\ c(p,m,n)\ell_{p-1}^{m(p-1)+\frac{1}{2}\{(p-2)(p+1)\}}(1-\ell_{p-1})|\underset{\sim}{M}|^m|\underset{\sim}{I}-\underset{\sim}{M}| \prod_{i>j}(m_i-m_j)$$
$$\left[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\ell_{p-1}^k\,\beta_k\left\{\sum_{j=0}^{p-2}\frac{(-1)^j(2j)!}{(j!)^2 2^j \chi_{[21^j]}}\ell_{p-1}^j\sum_{i=0}^{k}\sum_{\delta}b_{k,\delta}\sum_{T}g_{(\delta,1^j)}^{T}C_\tau(\underset{\sim}{M})\right\}\right],$$

where

$\underset{\sim}{M} = \operatorname{diag}(m_1,\ldots,m_{p-2})$, $g_{(\delta,1^j)}^{T}$ are the constants that have been defined previously, $b_{k,\delta}$ are the constants defined in Khatri and Pillai [5].

Now integrals (7.2) over $0 < m_1 \le m_2 \le \cdots \le m_{p-2} < 1$ by the use of Lemma (3.3) of Sugiyama [15], we can write the density of ℓ_{p-1} in the form

$$(7.3)\ c(p,m,n)\ \ell_{p-1}^{m(p-1)+\frac{1}{2}(p-2)(p+1)}(1-\ell_{p-1})\left[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\ell_{p-1}^k\,\beta_k\right.$$
$$\left.\left\{\sum_{j=0}^{p-2}\frac{(-1)^j(2j)!}{(j!)^2 2^j \chi_{(21^j)}}\ell_{p-1}^j\left(\sum_{i=0}^{k}\sum_{\delta}b_{\kappa,\delta}\sum_{T}g_{(\delta,1^j)}^{T}\ fC_\tau(\underset{\sim}{I})\right)\right\}\right],$$

where

$$f = \{\Gamma_{p-2}(\tfrac{1}{2}(p-2))/\pi^{(p-2)^2/2}\}(\Gamma_{p-2}(m+\tfrac{1}{2}(p-1),\tau)\ \Gamma_{p-2}(\tfrac{1}{2}(p+1))/\Gamma_{p-2}(m+p,n)).$$

For $p = 2$, (7.1) reduces to

$$(7.4)\ c(2,m,n)\ \ell_1^m(1-\ell_1)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\ell_1^k\ \beta(2m+k+3,n+1).$$

For $p = 3$, integrate ℓ_2 from (7.2), then the distribution of m_1 can be written in the form

$$(7.5)\quad c(3,m,n)m_1^m(1-m_1)\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\,C_\kappa\begin{pmatrix}1 & 0\\ 0 & m_1\end{pmatrix}\beta\,(3m+6+k,n+1)$$

$$\{\beta(m+k+3,2)-m_1\,\beta(m+k+4,2)\}\Big]\,.$$

For $p = 4$, integrate ℓ_3 from (7.2) then the joint density of m_1, m_2 is given by

$$(7.6)\quad c(4,m,n)(m_1m_2)^m(1-m_1)(1-m_2)(m_2-m_1)\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\,C_\kappa(\underset{\sim}{M}_1)\,\beta(c_1,n+1)$$

$$\{\beta(c_2,2)-(m_1+m_2)\beta(c_2+1,2)+m_1m_2\,\beta(c_2+2,2)\}\Big]\,,$$

where

$\underset{\sim}{M}_1 = \text{diag}\,(m_1,m_2,1)$, $c_1 = 4m+k+10$, $c_2 = 3m+k+6$.

Now let $n_1 = m_1/m_2$ and integrate with respect to m_2 then the distribution of n_1 can be obtained in the form

$$(7.7)\quad c(4,m,n)n_1^m(1-n_1)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\,\beta(c_1,n+1)\sum_{i=0}^{k}\sum_{\delta}b_{(\kappa,\delta)}\;c_\delta\begin{pmatrix}1 & 0\\ 0 & n_1\end{pmatrix}\{\beta(c_2,2)\beta(s_1,2)$$

$$-\beta(s_1+1,2)((n_1+1)\beta(c_2+1,2)+n_1\beta(c_2,2))+\beta(s_1+2,2)(n_1\beta(c_2+2,2)$$

$$+n_1(n_1+1)\,\beta(c_2+1,2)\,-\,n_1^2\beta(s_1+3,2))\}\,,$$

where

$$s_1 = 2m+i+3 \quad .$$

We may note that the distribution of ℓ_1 can be found from (7.1) as the distribution of the smallest root as in section (1) and that of m_2 by integrating (7.6) with respect to m_1 .

For $p = 5$, integrate (7.2) with respect to ℓ_4 , the joint density of m_1, m_2, m_3 can be written in the form

$$(7.8)\ c(5,m,n)|\underset{\sim}{M}|^m|\underset{\sim}{I}-\underset{\sim}{M}| \prod_{i>j}(m_i-m_j)\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\beta(c_3,n+1)\sum_{j=0}^{3}\frac{(-1)^j(2j)!}{(j!)^2 2^j \chi_{[21^j]}(1)}$$

$$\beta(s_2,2)\Big(\sum_{i=0}^{k}\sum_{\delta} b_{(\delta,\kappa)}\sum_{T} g^T\, C_\tau(\underset{\sim}{M})\Big)\Big] \ ,$$

where

$$c_3 = 5m+k+15 \quad \text{and} \quad s_2 = 4m+10+j+k \quad .$$

Now consider the transformation $n_i = m_i/m_3$, $i = 1,2$ and integrate with respect to m_3, then the joint density of n_1, n_2 can be written in the form

$$(7.9)\ c(5,m,n)(n_1n_2)^m(1-n_1)(1-n_2)(n_2-n_1)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\beta(c_3,n+1)\sum_{j=0}^{3}\frac{(-1)^j(2j)!}{(j!)^2 2^j \chi_{[21^j]}(1)}$$

$$\beta(s_2,2)\sum_{i=0}^{k}\sum_{\delta} b_{(\delta,\kappa)}\ \sum_{T} g^T_{(\delta,1^j)}\ C_\tau(\underset{\sim}{N_1})\ \{\beta(t_1,2)-(n_1+n_2)\beta(t_1+1,2)$$

$$+ n_1n_2\ \beta(t_1+2,2)\} \ ,$$

where

$$t_1 = 3m+i+j+6 \quad \text{and} \quad \underset{\sim}{N}_1 = \text{diag}\,(1,n_1,n_2) \ .$$

Further, let $x = \frac{n_1}{n_2}$ and integrate with respect to n_2, we get the density of x as

$$(7.10) \quad c(5,m,n)\,x^{m}(1-x)\Big[\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(-n)_\kappa}{k!}\,\beta(c_3,n+1)\sum_{j=0}^{3}\frac{(-1)^j(2j)!}{(j!)^2 2^j \chi_{(21^j)}(1)}$$

$$\beta(s_2,2)\sum_{i=0}^{k}\sum_{\delta} b_{\delta,k}\sum_{T} g^{T}_{(\delta,1^j)}\sum_{r=0}^{i+j}\sum_{\eta} b_\eta\, C_\eta \begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix}$$

$$\{(1-x)\beta(t_1,2)\beta(s_3,2)-(1-x^2)\beta(t_1+1,2)\beta(s_3+1,2)+\chi(1-x)\beta(t_1+2,2)\beta(s_3+2,2)\}\Big] ,$$

where

$s_3 = 2m+r+3$, b_η are constants and η denote the partition of $i + j$.

We may note that the distribution of ℓ_1 and ℓ_4 can be found from (7.1) as the smallest and the largest roots respectively and m_3 can be found from (7.8) as its largest root.

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