# A Decomposition for $L^1$-Bounded Martingales*

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1. Introduction. We exhibit a decomposition for an $L^1$-bounded martingale that allows us to obtain bounds for the distributions of various random variables defined in terms of the martingale. The decomposition, which is of some interest in itself, was devised as a tool to obtain direct proofs of certain inequalities due to D.L. Burkholder [1]. Burkholder's proofs are based on an elegant but indirect and rather difficult technique for establishing maximal inequalities developed by him in an earlier paper [2]. The decomposition permits us to estimate the relevant probabilities directly, and its presentation is self-contained to the extent that nothing beyond the standard lore of martingale theory is required.

2. The Martingale Transform Inequalities. Let $f = (f_1, f_2, \ldots)$ denote a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, where the random variable $f_n$ is measurable with respect to a sub-field $\mathcal{F}_n$, $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$, $n \geq 1$. Let $\varphi = (\varphi_1, \varphi_2, \ldots)$ be the f-increment sequence, so that $f_n = \Sigma_{k=1}^n \varphi_k$, $n \geq 1$. Denote by $||f||_p = \sup_k ||f_k||_p$, where $||f_k||_p$, $1 \leq p \leq \infty$ is the usual $L_p$-norm of the random variable $f_n$. The letter $C$ = constant, not always the same from line to line.

We are interested in a class of quasi-linear mappings from <u>sequences of random variables</u> to <u>random variables</u>. This class, which we call

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''class $\mathcal{B}$'', is described below. For the moment, however, we list its most prominent members

(a) $f^* = \sup_n |f_n|$;

(b) $S_n(f) = (\Sigma_{k=1}^n \varphi_k^2)^{1/2}$;

(c) $S(f) = \lim_{n \to \infty} S_n(f)$.

The following theorem concerning these mappings is due to D.L. Burkholder [1].

Theorem 2.1 (Theorem 8 of [1]) If $f$ is a martingale, then

$$P(S(f) > \lambda) \leq C||f||_1/\lambda$$

and

$$P(f^* > \lambda) \leq \frac{C||S(f)||_1}{\lambda}$$

for all $\lambda \geq 0$.

From the inequalities of Theorem 2.1 and the Marcinkiewicz interpolation theorem, one may deduce the following:

Theorem 2.2 (Theorem 9 of [1]) Let $1 < p < \infty$. There are positive real numbers $C_p$ and $C_p'$ such that if $f$ is a martingale, then

$$C_p||S_n(f)||_p \leq ||f||_p \leq C_p'||S_n(f)||_p$$

for all $n \geq 1$.

The inequality in Theorem 2.2 was proved for special cases by Paley, and Marcinkiewicz and Zygmund. Paley [4] proved it for martingales derived from Walsh-Fourier series. In this context, it plays a fundamental role as a substitute for the conjugate function norm inequalities, available for trigonometric series. The results of Marcinkiewicz and Zygmund have been used to obtain generalizations of Kolmogorov's strong law of large numbers, and more recently, Chow [3] has used Theorem 2.2 to the same end.

Burkholder's proof of Theorem 2.1 is based on the following result.

Theorem 2.3 (Theorem 6 of [1]) *If* $f$ *and* $g$ *are martingales relative to the same increasing sequence of* $\sigma$-*fields and* $S_n(g) \leq S_n(f)$, $n \geq 1$, *then*

$$P(g^* > \lambda) \leq C||f||_1/\lambda$$

*for all* $\lambda > 0$.

In the present apprcach, we circumvent this theorem and prove Theorem 2.1 from the decomposition and the properties of class $\mathcal{B}$ mappings.

3. A Decomposition Theorem and the Class $\mathcal{B}$. Our main result is the following decomposition theorem.

Theorem 3.1. *Let* $f$ *be an* $L^1$-*bounded martingale*. *Corresponding to any* $\lambda > 0$ *the martingale* $f$ *may be decomposed into three martingales* $a$, $b$, $d$, *so that* $f = a+b+d$.

(i) *The martingale* $a = (a_1, a_2, \ldots)$, $a_n = \Sigma_{k=1}^n \alpha_k$ *is* $L^1$-*bounded*, $||a||_1 \leq C\ ||f||_1$ *and the increment sequence* $\alpha = (\alpha_1, \alpha_2, \ldots)$ *is such that* $P(\alpha^* \neq 0) \leq C||f||_1/\lambda$.

(ii) *The martingale* $b = (b_1, b_2, \ldots)$, $b_n = \Sigma_{k=1}^n \beta_k$, *is absolutely convergent*, $||\Sigma_{k=1}^{\infty}|\beta_k|\ ||_1 \leq C\ ||f||_1$.

(iii) *The martingale* $d = (d_1, d_2, \ldots)$, $d_n = \Sigma_{k=1}^n \delta_k$, *is uniformly bounded*, $||d||_\infty \leq C\ \lambda$, $||d||_1 \leq C||f||_1$, *and* $||d||_2^2 \leq C\ \lambda\ ||f||_1$.

The proof of the decomposition theorem is postponed until later. We now show how the decomposition may be used to obtain inequalities for a certain class of random variables.

The mappings $f^*$ and $S(f)$ have a few common features which seem to determine the kind of inequalities that one can prove. We abstract these features and list them under the title ''class $\mathcal{B}$''. This definition seems justified if only to focus on the essential character of subsequent arguments.

Definition. A mapping $T$ from random variable sequences to random variables is said to be of class $\mathcal{B}$ if:

1. $T$ is quasi-linear, i.e. $|T(f+g)| \leq C(|Tf| + |Tg|)$.
2. $P(|Tf| \neq 0) \leq CP(f^* \neq 0)$
3. The mapping $T$ satisfies the following norm inequalities:
   (a) $||Tf||_2 \leq C||f||_2$
   (b) If $f = (f_1, f_2, \ldots)$ where $f_n = \Sigma_{k=1}^n \varphi_n$ and
   $$||\Sigma_{k=1}^n |\varphi_k|\ ||_1 \leq C||f||_1 \text{ then } ||Tf||_1 \leq C||f||_1 .$$

The mappings $f^*$, and $S(f)$ belong to class $\mathcal{B}$ when $f$ is a martingale. Consider $T(f) = f^* = \sup|f_n|$. Requirements 1 and 2 are obviously satisfied. Requirement 3a is Kolmogorov's maximal inequality for $L^2$-bounded martingales, and requirement 3b is obviously satisfied.

Consider $T(f) = S(f) = (\Sigma_{k=1}^\infty \varphi_k^2)^{1/2}$. Requirements 1, 2, and 3a are easily checked. Requirement 3b is satisfied since $\Sigma_{k=1}^\infty \varphi_k^2 \leq (\Sigma_{k=1}^\infty |\varphi_k|)^2$ implies $||S(f)||_1 \leq C||\Sigma_{k=1}^\infty |\varphi_k|\ ||_1 \leq C||f||_1$.

For further examples, let $v = (v_1, v_2, \ldots)$ be a sequence of transforms, i.e. $v_n$ is measurable with respect to $\mathcal{F}_{n-1}$, $n \geq 1$, such that

$v^* \leq C < \infty$. Let $f$ be an $L^1$-bounded martingale and define the transform $g = (g_1, g_2, \ldots)$ by setting $g_n = \Sigma_{k=1}^n v_k \varphi_k$. Here again, $T(f) = g^*$ and $T(f) = S(g)$ are class $\beta$ mappings.

Proposition. Let $f$ be an $L^1$-bounded martingale and $T$ a mapping belonging to class $\beta$. Then

$$P(|Tf| > \lambda) \leq C||f||_1/\lambda.$$

This Proposition follows from Theorem 3.1 and the properties 1-3 of class $\beta$. Write $f = a+b+d$, so that $|Tf| \leq C(|Ta|+|Tb|+|Td|)$. Then

$$P(|Tf| > \lambda) \leq P(|Ta| > \lambda/3C) + P(|Tb| > \lambda/3C) + P(|Td| > \lambda/3C),$$

so that it suffices to show that each term on the right hand side of this inequality is bounded above by $C||f||_1/\lambda$.

First

$$P(|Ta| > \lambda) \leq P(|Ta| \neq 0) \leq CP(a^* \neq 0) \leq C||f||_1/\lambda$$

by property 2 of class $\beta$ and (i) of Theorem 3.1.

Second,

$$P(|Tb| > \lambda) \leq C||f||_1/\lambda$$

by the Chebychev inequality, property 3a of class $\beta$, and (ii), Theorem 3.1.

Third,

$$P(|Td| > \lambda) \leq C||d||_2^2/\lambda^2$$

$$\leq C||f||_1/\lambda$$

by the Chebychev inequality, property 3b of the class $\mathcal{B}$, and (iii) of Theorem 3.1.

In summary, all class $\mathcal{B}$ mappings of $L^1$-bounded martingales are weak type (1,1) (see [5], page 111). In particular,

$$P(S(f) > \lambda) \leq C||f||_1/\lambda$$

so that one half of Theorem 2.1 is proved. For the other inequality, we observe, following Burkholder ([1], Proof of Theorem 2.) that by Khinchin's inequality for Rademacher functions $r_k(t)$, $k \geq 1$,

$$||\Sigma_{k=1}^{\infty} r_k(t_o)\varphi_k||_1 \leq C||S(f)||_1$$

for some $t_o$, $0 \leq t_o \leq 1$. Let $v = (r_1(t_o), r_2(t_o),\ldots)$; then $f$ is the transform of $g = (g_1,g_2,\ldots)$, $g_n = \Sigma_{k=1}^{n} r_k(t_o)\varphi_k$ under $v$, and again by the properties of class $\mathcal{B}$ mappings and Theorem 3.1,

$$P(f^* > \lambda) \leq C||g||_1/\lambda \leq C||S(f)||_1/\lambda.$$

This completes the proof of Theorem 2.1.

5. Proof of Theorem 3.1. We may assume that the martingale $f$ is nonnegative since it is well-known that every $L^1$-bounded martingale $f$ may be written as the sum of two nonnegative martingales, $f = f^+ - f^-$, with $\|f^{\pm}\|_1 \leq \|f\|_1$. We let $I(\cdot)$ denote the indicator function of the set in parentheses.

Define the following two stopping times: First, let

$$r = \inf\{n: f_n > \lambda\}.$$

Now recall that $f_n = \Sigma_{k=1}^n \varphi_k$, and write $\epsilon_n = \varphi_n I(r=n)$. The second stopping time

$$s = \inf\{n: \Sigma_{k=0}^n E(\epsilon_{k+1}||\mathcal{F}_k) > \lambda\}$$

and finally,

$$t = \min(r,s).$$

Then

$$\begin{aligned} P(t < \infty) &\leq P(r < \infty) + P(s < \infty) \\ &\leq [\|f_r\|_1 + \|\varphi_r\|_1]/\lambda \\ &\leq C\|f\|_1/\lambda. \end{aligned}$$

Write $a = f - f^t$; then $a = (a_1, a_2, \ldots)$ where $a_n = \Sigma_{k=1}^n \alpha_k = \Sigma_{k=1}^n \varphi_k I(t < k)$.

Clearly, $||a||_1 \leq 2||f||_1$ and $P(\alpha^* \neq 0) \leq P(t < \infty) \leq C||f||_1/\lambda$. Therefore, the martingale a satisfies requirement (i) of Theorem 3.1.

Now let us examine the martingale $f^t$; the typical term $f_n^t = \Sigma_{k=1}^n \varphi_k\, I(t \geq k)$. Since $I(t \geq k) = I(r \geq k).I(s \geq k)$, we may write $\varphi_k\, I(t \geq k) = (\gamma_k + \epsilon_k)\, I(s \geq k)$, where $\gamma_k = \varphi_k\, I(r > k)$ and $\epsilon_k = \varphi_k I(r=k)$. Notice that

$$E(\gamma_k||\mathcal{F}_{k-1}) = E(\gamma_k - \varphi_k\, I(r \geq k)||\mathcal{F}_{k-1})$$

$$= -E(\epsilon_k||\mathcal{F}_{k-1}).$$

Therefore, we may write the martingale $f_n^t = \Sigma_{k=1}^n(\gamma_k + \epsilon_k)\, I(s \geq k)$, $n \geq 1$ as the sum of two martingales

$$d_n = \Sigma_{k=1}^n(\gamma_k + E(\epsilon_k||\mathcal{F}_{k-1}))\, I(s \geq k),\ n \geq 1$$

and

$$b_n = \Sigma_{k=1}^n(\epsilon_k - E(\epsilon_k||\mathcal{F}_{k-1})\, I(s \geq k),\ n \geq 1.$$

We now show that $b = (b_1, b_2, \ldots)$ and $d = (d_1, d_2, \ldots)$ have the prescribed properties (ii) and (iii) of Theorem 3.1.

The sum $\Sigma_{k=1}^\infty \beta_k$, where $\beta_k = \epsilon_k - E(\epsilon_k||\mathcal{F}_{k-1})\, I(s \geq k)$ is absolutely convergent since

$$\int \sum_{k=1}^{\infty} |\beta_k| \leq 2\int \sum_{k=1}^{\infty} \epsilon_k = 2 \int \varphi_t \; I(t < \infty)$$

$$\leq 2 \int f_t \; I(t < \infty) \leq 2||f||_1$$

Therefore, the martingale $b$ satisfies property (ii).

Now we show that the martingale $d_n = \Sigma_{k=1}^n \delta_k$, $n \geq 1$, where $\delta_k = (\gamma_k + E(e_k||\mathcal{F}_{k-1}))\; I(s \geq k)$ satisfies property (iii).

In fact,

$$|\Sigma_{k=1}^n \gamma_k| = |\Sigma_{k=1}^n \varphi_k \; I(r > k)| \leq \lambda, \; n \geq 1$$

and

$$0 \leq \Sigma_{k=1}^n E(\epsilon_k||\mathcal{F}_{k-1}) \; I(s \geq k) \leq \Sigma_{k=0}^{s-1} E(\epsilon_{k+1}||\mathcal{F}_k) \leq \lambda$$

for all $n \geq 1$. Therefore, $||d||_\infty \leq 2\lambda$. Also, $||\Sigma_{k=1}^\infty \gamma_k||_1 \leq ||f||_1$ and $||\Sigma_{k=1}^\infty E(\epsilon_k||\mathcal{F}_{k-1})||_1 \leq C||f||_1$ so that

$$||\Sigma_{k=1}^n \delta_k||_1 \leq ||\Sigma_{k=1}^\infty (\gamma_k + E(\epsilon_k||\mathcal{F}_{k-1}))\; I(s \geq k)||_1 \leq C||f||_1$$

for all $n \geq 1$. Finally,

$$||\Sigma_{k=1}^n \delta_k||_2^2 \leq \int |\Sigma_{k=1}^\infty \delta_k|^2 \leq 2\lambda \int |\Sigma_{k=1}^\infty \delta_k|$$

$$\leq C\lambda||f||_1$$

so that property (iii) is satisfied. This completes the proof of Theorem 3.1.

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