Non-Central Distributions of Some Multivariate Test Criteria and Associated Powers of Tests

by

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CHAPTER I

POWER COMPARISONS OF TESTS OF TWO MULTIVARIATE HYPOTHESES BASED ON FOUR CRITERIA

1. Introduction and Summary

The two multivariate hypotheses which will be considered in this chapter are the following: (i) the independence between a p-set and a q-set of variates in a (p+q)-variate normal population; and (ii) the equality of p-dimensional mean vectors of ℓ p-variate normal populations having a common covariance matrix. In connection with the test of hypothesis (i), let the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ be ν independent normal (p+q)-variates, ($p \leq q$, $p+q \leq \nu$, $\nu+1=n'$, the sample size) with zero means and covariance matrix

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix} \begin{matrix} p \\ q \end{matrix}$$
$$\qquad\qquad p \qquad q$$

Let $R = \operatorname{diag}(r_i)$, where $r_1^2,\ldots,r_p^2$ are the characteristic roots of the equation

$$(1.1) \qquad |XY'(YY')^{-1}YX'-r^2XX'| = 0$$

and $P = \text{diag}(\rho_i)$, where $\rho_1^2,\ldots,\rho_p^2$ are the characteristic roots of the equation

(1.2) $$|\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}' - \rho^2 \Sigma_{11}| = 0.$$

Then, the distribution of $r_1^2,\ldots,r_p^2$ is given by Constantine (1963), in the following form: (James, 1964)

(1.3) $$|I-P^2|^{\frac{\nu}{2}} {}_2F_1(\tfrac{1}{2}\nu, \tfrac{1}{2}\nu; \tfrac{1}{2}f_2; P^2, R^2)$$

$$\cdot\ C(p,m,n)|R^2|^m|I-R^2|^n \prod_{i>j}(r_i^2 - r_j^2)\prod_{i=1}^{p} dr_i^2$$

$$0 < r_1^2 \le \cdots \le r_p^2 < 1$$

where $f_2 = q$, $m = \frac{1}{2}(q-p-1)$, $n = \frac{1}{2}(\nu-q-p-1)$, (and if n is defined as $\frac{1}{2}(f_1-p-1)$, $\nu = f_1+f_2$),

$$C(p,m,n) = \pi^{\frac{1}{2}p}\prod_{i=1}^{p}\Gamma[\tfrac{1}{2}(2m+2n+p+i+2)]/\{\Gamma[\tfrac{1}{2}(2m+i+1)]\Gamma[\tfrac{1}{2}(2n+i+1)]\Gamma(\tfrac{1}{2}i)\},$$

and the hypergeometric function of matrix argument is defined by (James, 1964)

(1.4) $${}_sF_t(a_1,\ldots,a_s;b_1,\ldots,b_t;S,T) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_s)_\kappa\, C_\kappa(S)C_\kappa(T)}{(b_1)_\kappa\cdots(b_t)_\kappa C_\kappa(I_p)k!},$$

where $a_1,\ldots,a_s$, $b_1,\ldots,b_t$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is given by

$$(a)_\kappa = \prod_{i=1}^{p} \left(a - \tfrac{1}{2}(i-1)\right)_{k_i},$$

where

$$(a)_k = a(a+1) \ldots (a+k-1),$$

partition κ of k is such that

$$\kappa = (k_1, k_2, \ldots, k_p),\ k_1 \geq k_2 \geq \ldots \geq k_p \geq 0,$$

$k_1 + \ldots + k_p = k$, and the zonal polynomials, $C_\kappa(S)$, are expressible in terms of elementary symmetric functions (esf) of the characteristic roots of S, (James, 196-).

Now we define below four criteria for testing the hypothesis (i), which are all functions of the characteristic roots, $r_1^2, \ldots r_p^2$, of equation (1.1):

1) Roy's largest root criterion, r_p^2 (Roy, 1945)

2) $U^{(p)} = \sum_{i=1}^{p} [r_i^2/(1-r_i^2)]$ (Pillai, 1954, 1955, 1960)

3) Pillai's criterion $V^{(p)} = \sum_{i=1}^{p} r_i^2$ (Pillai, 1954, 1955, 1960) and

4) Wilks' criterion, $\Lambda^{(p)} = \prod_{i=1}^{p} (1-r_i^2)$ (Wilks, 1932).

Similarly for testing hypothesis (ii), let X be a $p \times f_2$ matrix variate $(p \leq f_2)$ and Y a $p \times f_1$ matrix variate $(p \leq f_1)$ and the columns be all independently normally distributed with covariance Σ, $E(X)$ [illegible] $, \ldots, \ell_p$ be the characteristic roots of

$$(1.5) \qquad |XX' - \ell(YY' + XX')| = 0,$$

and $\omega_1,\ldots,\omega_p$ those of

$$(1.6) \qquad |MM' - \omega\,\Sigma| = 0,$$

then the joint density function of $\ell_1,\ldots,\ell_p$ is given by Constantine (1963), (James, 1964) in the form

$$(1.7) \qquad e^{-\frac{1}{2}\operatorname{tr}\Omega}\ {}_1F_1(\tfrac{1}{2}\nu;\ \tfrac{1}{2}f_2;\ \tfrac{1}{2}\Omega,\ L)\ C(p,m,n)|L|^m|I-L|^n \prod_{i>j}(\ell_i-\ell_j),$$

where $L = X'(YY' + XX')^{-1}X$, $\Omega = M'\Sigma^{-1}M$, (where the determinants are expressed as the product of the roots), $m = \frac{1}{2}(f_2-p-1)$, $n = \frac{1}{2}(f_1-p-1)$ and $\nu = f_1+f_2$. In the context of test (ii), $f_2=\ell-1$ and $f_1=N-\ell$, N being the pooled sample size of the samples from the ℓ populations. It should be pointed out that the same symbols m and n are used in connection with both tests because of the reason that these definitions of m and n will leave the joint distribution of $r_1^2,\ldots,r_p^2$ the same as that of $\ell_1,\ldots,\ell_p$ under the null hypotheses (i) and (ii) respectively as may be seen from (1.3) and (1.7).

Now the four criteria for testing hypothesis (ii) can be obtained from the criteria 1) - 4) above replacing the characteristic root r_i^2 by $\ell_i(i=1,\ldots,p)$. However, the same symbols $U^{(p)}$, $V^{(p)}$ and $W^{(p)}$ will be used to denote the respective criteria in this case also but ℓ_p instead of r_p^2 will denote the largest root criterion. It may be pointed out that $U^{(p)}$ is Lawley-Hotelling criterion multiplied by a constant.

In this chapter for p=2, the exact distributions of the criteria $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ are derived for testing (i) using (1.3) and for testing (ii) using (1.7). Powers of the tests based on these criteria are extensively tabulated under various alternative hypotheses for p=2 and comparisons made. Comparisons of powers of the tests using these three criteria are also made with those of the largest root whose power studies have been made by Pillai (1965).

It may be pointed out that Pillai (1965) has carried out some power function comparisons for tests of (i) and (ii) using certain approximate methods. For test of (ii) Mikhail (1965) and Gnanadesikan et al (1965) have made such comparisons, employing approximate and Monte Carlo methods respectively.

2. Non-Central C.D.F. of $U^{(2)}$ for Test of (i)

Now putting p=2 in (1.3) we get the joint distribution of r_1^2 and r_2^2 in the form:

$$(2.1)\quad K(1+A_{11}a_1+\sum_{i=2}^{3}\sum_{j=1}^{2}A_{ij}a_{ij}+\sum_{i=4}^{5}\sum_{j=1}^{3}A_{ij}a_{ij}+\sum_{j=1}^{4}A_{6j}a_{6j}+\ldots)a_2^m$$

$$[(1-r_1^2)(1-r_2^2)]^n\ (r_2^2-r_1^2)$$

where

$$K = C(2,m,n)[1-\rho_1^2)(1-\rho_2^2)]^{\nu/2},\ a_1 = r_1^2+r_2^2,\ a_2 = (r_1r_2)^2$$

$$a_{21} = 3a_1^2-4a_2,\ a_{22} = a_2,\ a_{31} = 5a_1^3-12a_1a_2,\ a_{32} = a_1a_2$$

$$a_{41} = 35a_1^4-120a_1^2a_2+48a_2^2,\ a_{42} = a_2a_{21},\ a_{43} = a_2^2$$

$$a_{51} = 63a_1^5-280a_1^3a_2+240a_1a_2^2,\ a_{52} = a_2a_{31},\ a_{53} = a_1a_2^2$$

$$a_{61} = 231a_1^6 - 1260a_1^4a_2 + 1680a_1^2a_2^2 - 320a_2^3,\ a_{62} = a_2a_{41},$$

$$a_{63} = a_2^2a_{21},\ a_{64} = a_2^3,$$

$$A_{11} = \nu^2b_1/2^2f_2,\ A_{21} = \frac{[\nu(\nu+2)]^2}{2^2f_2(f_2+2)}\frac{b_{21}}{4!2},\ A_{22} = \frac{[\nu(\nu-1)]^2}{2^2f_2(f_2-1)}\frac{4b_2}{3!}$$

$$A_{31} = \frac{[\nu(\nu+2)(\nu+4)]^2}{2^3f_2(f_2+2)(f_2+4)}\frac{b_{31}}{5!4},\ A_{32} = \frac{[\nu(\nu+2)(\nu-1)]^2}{2^3f_2(f_2+2)(f_2-1)}\frac{b_1b_2}{5},$$

$$A_{41} = \frac{[\nu(\nu+2)\dots(\nu+6)]^2}{2^4f_2(f_2+2)\dots(f_2+6)}\frac{3b_{41}}{8!8},\ A_{42} = \frac{[\nu(\nu+2)(\nu+4)(\nu-1)]^2}{2^4f_2(f_2+2)(f_2+4)(f_2-1)}\frac{b_{42}}{4!7}$$

$$A_{43} = \frac{[\nu(\nu+2)(\nu-1)(\nu+1)]^2}{2^4f_2(f_2+2)(f_2-1)(f_2+1)}\frac{2b_2^2}{3\cdot5},\ A_{51} = \frac{[\nu(\nu+2)\dots(\nu+8)]^2}{2^5f_2(f_2+2)\dots(f_2+8)}\frac{b_{51}}{8!48}$$

$$A_{52} = \frac{[\nu(\nu+2)\dots(\nu+6)(\nu-1)]^2}{2^5f_2(f_2+2)\dots(f_2+6)(f_2-1)}\frac{b_{52}}{6!3},\ A_{53} = \frac{[\nu(\nu+2)(\nu+4)(\nu-1)(\nu+1)]^2}{2^5f_2(f_2+2)(f_2+4)(f_2-1)(f_2+1)}\frac{b_1b_2^2}{7\cdot5},$$

$$A_{61} = \frac{[\nu(\nu+2)\dots(\nu+10)]^2}{2^6f_2(f_2+2)\dots(f_2+10)}\frac{b_{61}}{8!4!176},\ A_{62} = \frac{[\nu(\nu+2)\dots(\nu+8)(\nu-1)]^2}{2^6f_2(f_2+2)\dots(f_2+8)(f_2-1)}\frac{3b_{62}}{8!44}$$

$$A_{63} = \frac{[\nu(\nu+2)\dots(\nu+6)(\nu-1)(\nu+1)]^2}{2^6f_2(f_2+2)\dots(f_2+6)(f_2-1)(f_2+1)}\frac{10b_{63}}{7!3},$$

$$A_{64} = \frac{[\nu(\nu+2)(\nu+4)(\nu-1)(\nu+1)(\nu+3)]^2}{2^6f_2(f_2+2)(f_2+4)(f_2-1)(f_2+1)(f_2+3)}\frac{4b_2^3}{5\cdot7\cdot9},$$

where $b_1 = \rho_1^2 + \rho_2^2$, $b_2 = (\rho_1\rho_2)^2$ and b_{ij}'s are obtained from corresponding a_{ij}'s by replacing in the latter a_1 and a_2 by b_1 and b_2 respectively.

Now let $\lambda_i = r_i^2/(1-r_i^2)(i=1,2)$, $u = \lambda_1 + \lambda_2$ and $g = \lambda_1\lambda_2$, then the c.d.f. of $U^{(2)}$ can be written in the form:

$$F(U^{(2)}) = K[\sum_{j=0}^{6} \sum_{i=0}^{j} (-1)^{i+j} D_{ij} B_{ij} + \ldots], \tag{2.2}$$

where

$D_{00} = 1 + A_{11} + 3A_{21} + 5A_{41} + 35A_{41} + 63A_{51} + 231A_{61}$,

$D_{01} = A_{11} + 6A_{21} + 15A_{31} + 140A_{41} + 315A_{51} + 1386A_{61}$,

$D_{11} = A_{11} + 2A_{21} + A_{22} + 3A_{31} + A_{32} + 20A_{41} + 3A_{42} + 35A_{51} + 5A_{52} + 126A_{61} + 35A_{62}$,

$D_{02} = 3A_{21} + 15A_{31} + 210A_{41} + 630A_{51} + 3465A_{61}$,

$D_{12} = 6A_{21} + 18A_{31} + A_{32} + 180A_{41} + 6A_{42} + 420A_{51} + 15A_{52} + 1890A_{61} + 140A_{62}$,

$D_{22} = 3A_{21} + 3A_{31} + A_{32} + 18A_{41} + 2A_{42} + A_{43} + 30A_{51} + 3A_{52} + A_{53} + 105A_{61} + 20A_{62} + 3A_{63}$,

$D_{03} = 5A_{31} + 140A_{41} + 630A_{51} + 4620A_{61}$,

$D_{13} = 15A_{31} + 300A_{41} + 3A_{42} + 1050A_{51} + 15A_{52} + 6300A_{61} + 210A_{62}$,

$D_{23} = 15A_{31} + 180A_{41} + 6A_{42} + 450A_{51} + 18A_{52} + A_{53} + 2100A_{61} + 180A_{62} + 6A_{63}$,

$D_{33} = 5A_{31} + 20A_{41} + 3A_{42} + 30A_{51} + 3A_{52} + A_{53} + 100A_{61} + 18A_{62} + 2A_{63} + A_{64}$,

$D_{04} = 35A_{41} + 315A_{51} + 3465A_{61}$, $D_{14} = 140A_{41} + 980A_{51} + 5A_{52} + 8820A_{61} + 140A_{62}$,

$D_{24} = 210A_{41} + 1050A_{51} + 15A_{52} + 7350A_{61} + 300A_{62} + 3A_{63}$,

$D_{34} = 140A_{41} + 420A_{51} + 15A_{52} + 2100A_{61} + 180A_{62} + 6A_{63}$,

$$D_{44} = 35A_{41} + 35A_{51} + 5A_{52} + 105A_{61} + 20A_{62} + 3A_{63},$$

$$D_{05} = 63A_{51} + 1386A_{61},\ D_{15} = 315A_{51} + 5670A_{61} + 35A_{62},$$

$$D_{25} = 630A_{51} + 8820A_{61} + 140A_{62},\ D_{35} = 630A_{51} + 6300A_{61} + 210A_{62},$$

$$D_{45} = 315A_{51} + 1890A_{61} + 140A_{62},\ D_{55} = 63A_{51} + 126A_{61} + 35A_{62},$$

$$D_{06} = D_{66} = 231A_{61},\ D_{16} = D_{56} = 1386A_{61},\ D_{26} = D_{46} = 3465A_{61},\ D_{36} = 4620A_{61},$$

and

$$(2.3)\qquad B_{ij} = \int_0^{U^{(2)}} \int_0^{u^2/4} [g^{m+i}/(1+u+g)^{m+n+3+j}]\,dg\,du.$$

In obtaining (2.2) the following recurrence relation has been used repeatedly:

$$(2.4)\qquad B_{hij} = B_{h-1,i,j-1} - B_{h-1,i,j} - B_{h-1,i+1,j},$$

where

$$(2.5)\qquad B_{hij} = \int_0^{U^{(2)}} \int_0^{u^2/4} [u^h g^{m+i}/(1+u+g)^{m+n+j}]\,dg\,du\ (B_{oij} = B_{ij}).$$

Now B_{ij} in (2.3) can be reduced to the form

$$B_{ij} = \{1/(n+j-i-2)\}\ \{-(1-U^{(2)})^{i+2-j-n}\ B_{[U^{(2)}/U^{(2)}+2]^2}\ (m+i+1, n+j-i-1)$$

$$+ B_{[U^{(2)}/U^{(2)}+2]}\ (2(m+i+1),\ 2(n+j-i-1)-1\},$$

where

$$B_x(r,s) = \int_0^x x_1^{r-1}\,(1-x_1)^{s-1}\,dx_1.$$

3. Non-Central C.D.F. of $V^{(2)}$ for Test of (i)

Now, for obtaining the c.d.f. of $V^{(2)}$ let us rewrite (2.1) in the form

$$(3.1) \qquad K\Big(\sum_{i+2j=k=0}^{6} C_{ij}a_1^i a_2^{j+m} + \ldots\Big)[(1-r_1^2)(1-r_2^2)]^n (r_2^2-r_1^2)$$

where, in the summation, only integral solutions (i,j) of $i+2j=k$ are allowed and where,

$C_{00} = 1$, $C_{10} = A_{11}$, $C_{20} = 3A_{21}$, $C_{01} = A_{22} - 4A_{21}$,

$C_{30} = 5A_{31}$, $C_{11} = A_{32} - 12A_{31}$, $C_{40} = 35A_{41}$, $C_{21} = 3A_{42} - 120A_{41}$,

$C_{02} = A_{43} - 4A_{42} + 48A_{41}$, $C_{50} = 63A_{51}$, $C_{31} = 5A_{52} - 280A_{51}$,

$C_{12} = A_{53} - 12A_{52} + 240A_{51}$, $C_{60} = 231A_{61}$, $C_{41} = 35A_{62} - 1260A_{61}$,

$C_{22} = 3A_{63} - 120A_{62} + 1680A_{61}$ and $C_{03} = A_{64} - 4A_{63} + 48A_{62} - 320A_{61}$.

Now transforming (r_1^2, r_2^2) to (a_1, a_2) we get from (3.1)

$$(3.2) \qquad F_1(V^{(2)}) = P_r(a_1 < V^{(2)})$$

$$= K\int_0^{V^{(2)}}\int_0^{a_1^2/4}\Big(\sum_{i+2j=k=0}^{6} C_{ij}a_1^i a_2^{j+m} + \ldots\Big)(1-a_1+a_2)^n da_1\,da_2.$$

For the integration of a_2 and a_1 on the right side of (3.2), consider the integral

$$(3.3) \qquad f_{ij} = \int_o^{V^{(2)}} \int_o^{a_1^2/4} a_1^i a_2^{j+m} (1-a_1+a_2)^n \, da_2 \, da_1$$

$$= \int_o^{V^{(2)}} f_{1\cdot i,j} \, da_1, \qquad 0 < V^{(2)} \leq 1$$

$$= \int_o^1 f_{1\cdot i,j} \, da_1 + \int_1^{V^{(2)}} f_{2\cdot i,j} \, da_1, \quad 1 \leq V^{(2)} \leq 2$$

where, following Mikhail (1965),

$$(3.4) \qquad \int_o^{V^{(2)}} f_{1\cdot i,j} da_1 = [2^{i+1}/(m+j+1)] \sum_{r=0}^{n} [(-1)^r \binom{n}{r} / \binom{m+r+j+1}{r}] \ x$$

$$B_{V^{(2)}/2}(a+4r+2, \ b-4r-2) \quad 0 < V \leq 1$$

and

$$(3.5) \qquad \int_1^{V^{(2)}} f_{2\cdot i,j} \, da_1 = [2^{i+1}/(n+1)] \sum_{r=0}^{m+j} [(-1)^r \binom{m+j}{r} / \binom{n+r+1}{r}]$$

$$[B_{V^{(2)}/2}(a,b) - B_{1/2}(a,b)], \ 1 \leq V^{(2)} \leq 2,$$

where

$$a = 2m + 2j - 2r+i+1 \quad \text{and} \quad b = 2n + 2r + 3.$$

Further, using (3.4) and (3.5) in (3.3) we can write (3.2) in the form (taking only values of k from 0 to 6)

(3.6) $$F_1(V^{(2)}) = K\Big(\sum_{i+2j=k=0}^{6} C_{ij}f_{ij}\Big).$$

4. Non-Central C.D.F. of $W^{(2)}$ for Test of (i)

For obtaining the c.d.f. of Wilks' criterion, $W^{(2)}$, transforming (r_1^2, r_2^2) in (3.1) to $(a_2, \omega = (1-r_1^2)(1-r_2^2))$, we get the c.d.f. of $W^{(2)}$ in the form:

(4.1) $$F_2(W^{(2)}) = P_r(\omega \leq W^{(2)})$$
$$= K\int_0^{W^{(2)}}\int_0^{(1-\omega^{1/2})^2} \Big[\sum_{i+2j=k=0}^{6} C_{ij}a_2^{j+m}\,\omega^n(1-\omega+a_2)^i+\ldots\Big]da_2\,d\omega$$

remembering that $a_2^{1/2}+\omega^{1/2} \leq 1$. Now let

(4.2) $$g_{ij} = \int_0^{W^{(2)}}\int_0^{(1-\omega^{1/2})^2} a_2^{j+m}\,\omega^n(1-\omega+a_2)^i\,da_2\,d\omega.$$

Then it is easy to show that

(4.3) $$g_{ij} = \sum_{r=0}^{i}\Big[(-1)^r 2^{i-r+1}\binom{i}{r}\Big/\binom{j+m+r+1}{r}(j+m+1)\Big]\,B_{(W^{(2)})^{1/2}}(2n+2,2m+2j+r+i+3).$$

Using in (4.2) the value of g_{ij} on the right side of (4.3), we can write (4.1) in the form (taking only values of k from 0 to 6)

(4.4) $$F_2(W^{(2)}) = K\sum_{i+2j=k=0}^{6} C_{ij}\,g_{ij}\,.$$

5. Non-Central C.D.F.'s of $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ for Test of (ii) .

For obtaining the c.d.f.'s of the three criteria for test of (ii), put $p = 2$ in (1.7) and proceeding as in the preceding sections we arrive at the c.d.f.'s of $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ which would be obtained from (2.2), (3.6) and (4.4) respectively by making the following changes:

$$r_i^2 \longrightarrow \ell_i \quad (i = 1,2,\dots,p)$$

$$K \longrightarrow K' = e^{-\frac{1}{2}(\omega_1+\omega_2)} C(2,m,n)$$

$$A_{ij} \longrightarrow A'_{ij} \quad \text{where } A'_{ij} \text{ is obtained from } A_{ij}$$

by multiplying A_{ij} by 2^i and each linear factor involving ν in the numerator of A_{ij} is raised only to a single power instead of power two in A_{ij} and defining b_1 as $\frac{1}{2}(\omega_1+\omega_2)$ and b_2 as $\omega_1\omega_2/4$. For example,

$$A'_{21} = \frac{2^2 A_{21}[3(\frac{1}{2}(\omega_1+\omega_2))^2 - 4(\omega_1\omega_2/4)]}{\nu(\nu+2)(3b_1^2 - 4b_2)} .$$

Further, $D_{ij} \longrightarrow D'_{ij}$ where D'_{ij} is the same function of A'_{ij}'s as D_{ij} is of A_{ij}'s. Similarly

$$C_{ij} \longrightarrow C'_{ij} ,$$

where the meaning of (') is obvious. In other words, the basic changes are in the A_{ij} coefficients, K coefficient and the different definitions of b_1 and b_2.

6. Power Function Tabulations

For tabulation of powers of test of (i) based on $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$, upper 5 and 1 per cent points of the first two criteria were computed accurate to eight decimal places on IBM 7094 so as to give eight decimals accuracy in probability. Percentage points from Pillai's tables (1960) were fed into the computer as starting values (except for some values for $U^{(2)}$ not available) and the final results computed using (2.2) and (3.6) with $\rho_1 = \rho_2 = 0$ showed that the last (third) decimal provided in Pillai's tables (1960) is quite accurate (which were computed using β_1 and β_2). For $U^{(2)}$, percentage points were computed for values of $m = 0,1,2,5$ and for $n=5(5)30$, 40, 60; for $V^{(2)}$ the same values of m and n except $n = 60$. These percentage points are presented in Table 1. For $W^{(2)}$, lower 5 and 1 per cent points were computed using (4.4) putting $\rho_1 = \rho_2 = 0$ for values of m and n as for $U^{(2)}$. These again are presented in Table 1.

Powers of test of (i) using $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ for various pairs of values of (ρ_1^2, ρ_2^2) were computed for values of m and n as stated above. Similarly, tabulations of powers for test of (ii) using the three criteria for various pairs of values of (ω_1, ω_2) have also been made for the same values of m and n. (These tabulations are available with the Department of Statistics, Purdue University, for each criterion separately). Table 5 gives powers of tests of (i) based on the three criteria and Table 6 those of (ii). Some comparisons are made in the following section.

7. Power Comparisons

Let us consider first the comparison of powers of tests with $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ for testing (i) based on Table 5. In this connection, the following observations may be made:

1) For small deviations from the hypothesis $V^{(2)}$ seems to have more power than $W^{(2)}$; and $W^{(2)}$ more power than $U^{(2)}$.

2) For $\rho_1^2 + \rho_2^2$ = constant, the power of $V^{(2)}$ increases as the two roots tend to be equal but, on the other hand, those of $U^{(2)}$ and $W^{(2)}$ decrease. (For $W^{(2)}$ when $m = 0,1$, $\alpha = .01$ a few exceptions may be noted for small values of n).

3) For larger deviations from the hypothesis, when the values of the roots are far apart, the powers of $U^{(2)}$ and $W^{(2)}$ sometimes seem to exceed that of $V^{(2)}$ for larger values of n. But the power of $V^{(2)}$ always exceeds those of $U^{(2)}$ and $W^{(2)}$ when the values of the two roots are close. Further, the power of $U^{(2)}$ is greater than that of $W^{(2)}$ for larger values of n.

4) On the basis of similar earlier tabulations of the power of the largest root (1965), it is observed that the power of the largest root stays behind those of all the other three tests considered here. The power of the largest root decreases as the roots tend to be equal while their sum is kept a constant as for $U^{(2)}$ and $W^{(2)}$ in 2) above.

5) The power of the test based on any of the four criteria does not seem to have monotonicity property with respect to the sum or the product of the roots.

For test of hypothesis (ii), based on Table 6, we may infer the following observations: 1), 4) and 5) above can be repeated without change in this connection as well. However, 2) and 3) above may be changed to 2') and 3') below:

2') For $\omega_1 + \omega_2$ = constant, the powers of $V^{(2)}$ and $W^{(2)}$ increase as the two roots tend to be equal while that of $U^{(2)}$ decreases.

3') Same as 3) except for the addition of the following phrase to the last sentence there: ''except when the roots tend to be equal''.

Now, some detailed comparison may be made with the table of approximate powers given by Mikhail (1965) for test of hypothesis (ii) based on three criteria.

Table 1. Exact and Approximate Powers for Test of Hypothesis (ii) based on $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ for m=2 and n=34.5

ω_1	ω_2	$U^{(2)}$		$V^{(2)}$		$W^{(2)}$	
		Exact	Approx. (Mikhail)	Exact	Approx. (Mikhail)	Exact	Approx. (Mikhail)
0	1	.07332	.074	.07320	.077	.07329	.080
0	2	.10104	.104	.1004	.106	.1008	.120
1	1	.10099		.1016		.1014	
0	4	.1683		.1651		.1670	
1	3	.1682	.173	.1692	.171	.1688	.234
2	2	.1682		.1703		.1694	

The exact power computations in Table 1 again agree with the findingsgiven above. However, in connection with the approximate powers (Mikhail, 1965), it should be pointed out that the second moment of $U^{(p)}$ obtained by Khatri and Pillai (1967) does not seem to reduce to that given by Mikhail (1965) for p=2, even after taking into account that his definitions of the criteria are in terms of the sums of product matrices and not the sample covariance matrices as stated in his paper.

Further, if we consider larger values of the deviation parameters than those given in Tables 5 and 6, it appears that inferences 3) and 3') above are generally true, except that in very large samples the powers of the three criteria tend to be equal. Tables 2 and 3 help to see this point.

For test (ii), the admissibility of $U^{(p)}$ and the largest root has been established by Ghosh (1964) for large values of the parameters in the alternative hypotheses i.e. against unrestricted alternatives, and by Schwartz (1964) that of $V^{(p)}$ in the same sense. Kiefer and Schwartz (1965) have shown that $V^{(p)}$ test is admissible Bayes, fully invariant, similar and unbiased. They have also shown that $W^{(p)}$ is admissible Bayes, under a restriction, although admissibility could be established without this restricition. Further, the monotonicity property of these tests for both hypotheses were shown earlier by several authors (Roy and Mikhail, 1961; Das Gupta, Anderson and Mudholkar, 1964; Anderson and Das Gupta, 1964).

Table 2. Powers for Test of Hypothesis (ii) Based on $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ for n = 30, α = .05 and larger deviation parameters.

ω_1	ω_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 0				m = 1	
0	5	.311	.307	.309	.243	.237	.240
2.5	2.5	.311	.317	.315	.243	.247	.245
0	8	.493	.485	.419	.392	.381	.388
4	4	.496	.505	.501	.394	.402	.398
0	10	.603	.594	.601	.492	.476	.486
5	5	.606	.616	.614	.495	.504	.500
		m = 2				m = 5	
0	5	.204	.199	.202	.148	.143	.146
2.5	2.5	.204	.207	.206	.148	.150	.149
0	8	.330	.317	.324	.230	.215	.226
4	4	.331	.337	.334	.230	.234	.233
0	10	.418	.398	.410	.292	.265	.286
5	5	.419	.440	.424	.293	.297	.296

Table 3. Powers for Test of Hypothesis (i) Based on $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ for n = 30, α = .05 and larger deviation parameters.

ρ_1^2	ρ_2^2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
			m = 0			m = 1	
0	.1	.452	.445	.450	.370	.359	.366
.05	.05	.432	.440	.436	.351	.357	.355
0	.15	.673	.664	.644	.580	.562	.571
			m = 2			m = 5	
0	.1	.321	.308	.316	.245	.228	.240
.05	.05	.303	.308	.306	.231	.235	.234
0	.15	.516	.513	.510	.405	.377	.397

Table 4. Percentage Points of $U^{(2)}$, $V^{(2)}$ and $\{W^{(2)}\}^{1/2}$

m \ n	5	10	15	20	25	30	40	60
	Upper 5% points of $U^{(2)}$							
0	1.45081159	.68072060	.44291576	.32796652	.26031387	.21577262	.16073648	.10642195
1	2.17706150	1.00707275	.65171350	.48120047	.38126837	.31565518	.23478800	.15521283
2	2.88016039	1.31973369	.85077530	.62687850	.49604875	.41031810	.30485243	.20129555
5	4.94085097	2.22596410	1.42430529	1.04505172	.82470471	.68088086	.50462165	.33234008
	Upper 1% points of $U^{(2)}$							
0	2.27036399	.98500435	.62391020	.45580620	.35887340	.29587369	.21893413	.14398890
1	3.26419595	1.38920230	.87385950	.63612770	.49976308	.41143339	.30389113	.19950087
2	4.22195695	1.77364640	1.10991019	.80576751	.63197355	.51968210	.38328813	.25125214
5	7.01979399	2.88041118	1.78423021	1.28794229	1.00647011	.82554010	.60686740	.39644283
	Upper 5% points of $V^{(2)}$							
0	.69762354	.45101523	.33256947	.26325644	.21780901	.18572623	.14344680	
1	.89551940	.60365328	.45463403	.36444433	.30405346	.26080674	.20302167	
2	1.03905412	.72502980	.55597627	.45062425	.37875613	.32662382	.25608331	
5	1.30525874	.98279343	.78628000	.65477309	.56077390	.49029674	.39172758	
	Upper 1% points of $V^{(2)}$							
0	.85427421	.56849660	.42556795	.33988681	.28285815	.24218925	.18807200	
1	1.05014936	.72511545	.55330245	.44713230	.37507250	.32298160	.25273502	
2	1.18471509	.84605946	.65684970	.53651246	.45333115	.39243197	.30927259	
5	1.42312855	1.09422535	.88549870	.74281501	.63943555	.56118572	.4507040	

Table 4. (Continued)

m \ n	5	10	15	20	25	30	40	60
	Lower 5% points of $\{W^{(2)}\}^{1/2}$							
0	.61461031	.76019898	.82618869	.86374146	.88796331	.90487828	.92694376	.95009983
1	.51560358	.68175821	.76352355	.81195816	.84395456	.86665699	.89672190	.92882881
2	.44595445	.62032981	.71196020	.76811827	.80600099	.83326378	.86986428	.90958279
5	.31942102	.49177228	.59589608	.66495928	.7139824	.75054146	.80138236	.85894502
	Lower 1% points of $\{W^{(2)}\}^{1/2}$							
0	.52173571	.69336981	.77494150	.82234395	.85328541	.87506030	.90366941	.93393755
1	.43102910	.61507750	.71021894	.76786442	.80644707	.83405802	.8709177	.91064370
2	.36909450	.55534823	.65820783	.72274585	.76688690	.79894485	.84235554	.88993480
5	.25994369	.43390394	.54423854	.61922703	.67325211	.71394776	.77109271	.83653364

Table 5. Powers of $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ tests for testing $\rho_1=0$, $\rho_2=0$ against different simple alternative hypotheses, $\alpha = .05$

ρ_1^2	ρ_2^2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 0	n = 5		m = 1	n = 5	
0	$.0^51$	.05000043	.05000044	.05000043	.05000032	.05000034	.05000034
0	.0001	.0500423	.0500447	.0500437	.0500320	.0500352	.0500340
0	.0025	.0510638	.0511230	.0511009	.0508057	.0508841	.0508556
.00125	.00125	.0510621	.0511248	.0511006	.0508045	.0508853	.0508553
0	.01	.0543437	.0545617	.0544873	.0532880	.0535863	.0534836
.005	.005	.0543162	.0545901	.0544813	.0532667	.0536054	.0534787
.005	.01	.0565494	.0569695	.0568029	.0549547	.0554679	.0552772
.001	.05	.0746873	.0753383	.0753126	.0686507	.0697676	.0695585
		m = 0	n = 15		m = 1	n = 15	
0	$.0^51$	.05000118	.05000119	.05000119	.05000089	.05000090	.05000090
0	.0001	.0501195	.0501205	.0501202	.0500895	.0500910	.0500906
0	.0025	.0530274	.0530506	.0530449	.0522666	.0523016	.0522924
.00125	.00125	.0530225	.0530542	.0530438	.0522628	.0523045	.0522916
0	.01	.0626238	.0626665	.0626733	.0594345	.0595234	.0595174
.005	.005	.0625413	.0627207	.0626526	.0593701	.0596929	.0595030
.005	.01	.0692811	.0695676	.0694583	.0643921	.0646952	.0645962
.001	.05	.128431	.127511	.128229	.108644	.107744	.108540
		m = 0	n = 40		m = 1	n = 40	
0	$.0^51$	.05000311	.05000312	.05000312	.0500231	.05000233	.05000230
0	.0001	.0503118	.0503123	.0503122	.0502319	.0502326	.0502324
0	.0025	.0580362	.0580374	.0580399	.0559660	.0559721	.0559734
.00125	.00125	.0580230	.0580451	.0580367	.0559557	.0559788	.0559713
0	.01	.0850749	.0849585	.0850333	.0759550	.0758440	.0759229
.005	.005	.0848456	.085[illegible]549	.0849607	.0757749	.0759398	.0758752
.005	.01	.104947	.1052942	.1051379	.0906296	.090879	.0907838
.001	.05	.293	.292	.2919	.235	.233	.2338

Table 5. (Continued) α = .05

ρ_1^2	ρ_2^2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 2	n = 5		m = 5	n = 5	
0	$.0^51$	.05000027	.05000030	.05000030	.05000021	.05000024	.05000023
0	.0001	.0500271	.0500306	.0500294	.0500210	.0500245	.0500234
0	.0025	.0506829	.0507679	.0507384	.0505270	.0506156	.0505883
.00125	.00125	.0506818	.05076864	.0507381	.0505262	.0506159	.0505880
0	.01	.0527847	.0531136	.0530040	.0521457	.0524958	.0523903
.005	.005	.0527668	.0531263	.0529994	.0521324	.0525007	.0523859
.005	.01	.0541946	.0547384	.0545478	.0532302	.0537871	.0536142
.001	.05	.0657494	.0671257	.0668045	.0620581	.0637059	.0632872
		m = 2	n = 15		m = 5	n = 15	
0	$.0^51$	.05000074	.05000076	.05000075	.05000055	.05000057	.05000057
0	.0001	.0500748	.0500766	.0500761	.0500553	.0500574	.0500568
0	.0025	.0518926	.0519340	.0519230	.0513976	.0514470	.0514341
.00125	.00125	.0518894	.0519364	.0519223	.0513951	.0514483	.0514335
0	.01	.0578622	.0579827	.0579658	.0557810	.0559488	.0559152
.005	.005	.0578082	.0580192	.0579535	.0557421	.0559688	.0559049
.005	.01	.0619790	.0622961	.0622003	.0587867	.0591265	.0590335
.001	.05	.09862	.09794	.09865	.08513	.08511	.08551
		m = 2	n = 40		m = 5	n = 40	
0	$.0^51$	.05000192	.05000193	.05000189	.05000138	.05000139	.05000137
0	.0001	.0501922	.0501930	.0501927	.0501384	.0501391	.0501391
0	.0025	.0549336	.0549436	.0549437	.0535390	.0535552	.0535538
.00125	.00125	.0549250	.0549492	.0549419	.0535329	.0535587	.0535523
0	.01	.0713659	.0712785	.0713484	.0651641	.0651140	.0651804
.005	.005	.0712151	.0713619	.0713104	.0650586	.0651834	.0651526
.005	.01	.083379	.083587	.0835174	.0735602	.0727100	.073695
.001	.05	.203	.200	.2018	.156	.152	.1550

Table 5. (Continued) $\alpha = .01$

ρ_1^2	ρ_2^2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 0		n = 5	m = 1		n = 5
0	$.0^5 1$	.01000010	.01000012	.01000011	.010000073	.010000088	.010000087
0	.0001	.0100106	.0100120	.0100114	.0100078	.0100093	.0100087
0	.0025	.0102689	.0103015	.0102879	.0101977	.0102330	.0102199
.00125	.00125	.0102683	.0103027	.0102880	.0101972	.0102335	.0102199
0	.01	.0111090	.0112324	.0111840	.0108127	.0109511	.0109018
.005	.005	.0110099	.0112514	.0111846	.0108055	.0109597	.0109016
.005	.01	.0116766	.0119138	.0118102	.0112271	.0114635	.0113748
.001	.05	.0166554	.0171111	.0170284	.0148152	.0154366	.0152731
		m = 0		n = 15	m = 1		n = 15
0	$.0^5 1$	.01000032	.01000033	.01000033	.01000023	.01000024	.01000024
0	.0001	.0100327	.0100334	.0100332	.0100239	.0100247	.0100245
0	.0025	.0108374	.0108522	.0108483	.0106101	.0106289	.0106238
.00125	.00125	.0108352	.0108554	.0108485	.0106085	.0106308	.0106239
0	.01	.0135984	.0136288	.0136318	.0126009	.0126532	.0126477
.005	.005	.0135609	.0136808	.0136336	.0125745	.0126847	.0126479
.005	.01	.0155778	.0157753	.0156971	.0140135	.0141861	.0141295
.001	.05	.036212	.035546	.036104	.02844	.02795	.028418
		m = 0		n = 40	m = 1		n = 40
0	$.0^5 1$	.01000087	.01000088	.01000087	.01000063	.01000064	.01000064
0	.0001	.0100883	.0100886	.0100885	.0100642	.0100645	.0100644
0	.0025	.0123339	.0123349	.0123364	.0116840	.0116877	.0116884
.00125	.00125	.0123276	.0123428	.0123369	.0116795	.0116928	.0116886
0	.01	.0210019	.0209179	.0209732	.0177940	.0177292	.0177760
.005	.005	.0208846	.0210436	.0209714	.0177109	.0178146	.0177744
.005	.01	.028009	.028289	.028162	.0226441	.022811	.022747
.001	.05	.119	.119	.1182	.0862	.086	.0848

Table 5. (Continued) $\alpha = .01$

ρ_1^2	ρ_2^2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 2		n = 5	m = 5		n = 5
0	$.0^51$	.010000063	.010000076	.010000075	.010000051	.010000054	.010000053
0	.0001	.0100066	.0100079	.0100075	.0100049	.0100062	.0100058
0	.0025	.0101647	.0101995	.01018782	.0101239	.0101561	.0101471
.00125	.00125	.0101644	.0101998	.01018778	.0101237	.0101562	.0101470
0	.01	.0106756	.0108138	.0107688	.0105066	.0106363	.0106005
.005	.005	.0106702	.0108187	.0107683	.0105030	.0106378	.0105996
.005	.01	.0110197	.0112467	.0111700	.0107640	.0109695	.0109114
.001	.05	.0139659	.0146338	.0144549	.0120931	.013602	.0134326
		m = 2		n = 15	m = 5		n = 15
0	$.0^51$	.01000019	.01000020	.01000020	.01000014	.01000015	.01000015
0	.0001	.0100196	.0100205	.0100203	.0100142	.0100151	.0100148
0	.0025	.0105014	.0105215	.01051619	.0103610	.0103817	.0103766
.00125	.00125	.0105002	.0105228	.01051617	.0103601	.0103822	.0103765
0	.01	.0121255	.0121888	.0121785	.0115154	.0115898	.0115750
.005	.005	.0121048	.0122098	.0121778	.0115021	.0115987	.0115733
.005	.01	.0132701	.0134323	.0133842	.0123199	.0124681	.0124301
.001	.05	.02474	.02446	.02479	.020082	.020085	.020241
		m = 2		n = 40	m = 5		n = 40
0	$.0^51$	.01000052	.01000053	.01000053	.01000037	.01000039	.01000037
0	.0001	.0100525	.0100529	.0100528	.0100370	.0100374	.0100373
0	.0025	.0113699	.0113753	.0113753	.0109576	.0109645	.0109643
.00125	.00125	.0113664	.0113790	.0113753	.0109554	.0109665	.0109641
0	.01	.0162536	.0162097	.0162464	.0142632	.0142304	.0142735
.005	.005	.0161891	.0162717	.0162434	.0142211	.0142710	.0142692
.005	.01	.0200735	.020199	.020159	.016777	.016833	.016847
.001	.05	.0717	.069	.0692	.048	.045	.0469

Table 6. Powers of $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ tests for testing $\omega_1=0$, $\omega_2=0$ against different simple alternative hypotheses, α = .05

ω_1	ω_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 0	n = 5		m = 0	n = 15		m = 0	n = 40	
0	$.0^41$	.05000026	.05000027	.05000027	.05000033	.05000033	.05000033	.05000035	.05000036	.05000036
0	.01	.0502644	.0502795	.0502737	.0503320	.0503350	.0503341	.0503626	.0503631	.0503630
0	.1	.0526669	.0528094	.0527580	.0533574	.0533829	.0533766	.0536702	.0536733	.0536731
.05	.05	.0526663	.0528318	.0527664	.0533571	.0533933	.0533812	.0536700	.0536778	.0536752
0	.5	.0638359	.0643622	.0642368	.0675878	.0676187	.0676444	.0693042	.0692771	.0692987
.10	.40	.0638298	.0647178	.0643725	.0675856	.0677851	.0677187	.0693035	.0693479	.0693325
.25	.25	.0638264	.0649178	.0644488	.0675844	.0678787	.0677606	.0693032	.0693878	.0693515
0	1	.0788732	.0794634	.0795383	.0870840	.0869126	.0871010	.0908734	.0907170	.0908232
0	2	.11219	.11157	.11298	.13112	.12992	.13080	.13994	.13926	.13965
1	1	.11223	.12011	.11636	.13120	.13386	.13264	.13999	.14092	.14048
0	3	.1492	.1457	.1494	.1807	.1777	.1797	.1952	.1938	.1946
.01	1	.0791847	.0798606	.0798887	.0874922	.0873556	.0875261	.0913266	.0911838	.0912835
.1	.1	.0553826	.0557403	.0555952	.0567952	.0568791	.0568490	.0574368	.0574570	.0574494
		m = 1	n = 5		m = 1	n = 15		m = 1	n = 40	
0	$.0^41$	.05000018	.05000018	.05000018	.05000023	.05000023	.05000023	.05000026	.05000027	.05000026
0	.01	.0501780	.0501956	.0501891	.0502356	.0502397	.0502385	.0502636	.0502643	.0502641
0	.1	.0517935	.0519640	.0519030	.0523799	.0524167	.0524069	.0526648	.0526703	.0526694
.05	.05	.0517929	.0519765	.0519078	.0523795	.0524238	.0524100	.0526646	.0526736	.0526709
0	.5	.0592586	.0599825	.0597675	.0624121	.0625063	.0625114	.0639615	.0639448	.0639643
.10	.40	.0592498	.0601847	.0598468	.0624063	.0626230	.0625615	.0639588	.0639985	.0639887
.25	.25	.0592448	.0602985	.0598915	.0624031	.0626887	.0625897	.0639573	.0640287	.0640025
0	1	.0692301	.0703464	.0701482	.0760780	.0760396	.0761852	.0794814	.0693360	.0794356
0	2	.09120	.09206	.09262	.10695	.10596	.10677	.11489	.11416	.11460
1	1	.09107	.09722	.09469	.10689	.10895	.10811	.11487	.11553	.11524
0	3	.1156	.1148	.1172	.1420	.1392	.1413	.1555	.1538	.1548
.01	1	.0694348	.0706083	.0703820	.0763627	.0763498	.0764829	.0798067	.0796718	.0797662
.1	.1	.0536139	.0539948	.0538510	.0548098	.0549049	.0548743	.0553930	.0554137	.0554070

Table 6. (Continued) $\alpha = .05$

ω_1	ω_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 2	n = 5		m = 2	n = 15		m = 2	n = 40	
0	$.0^41$	.05000013	.05000015	.05000015	.05000019	.05000019	.05000019	.05000022	.05000022	.05000022
0	.01	.0501358	.0501529	.0501469	.0501871	.0501915	.0501902	.0502135	.0502144	.0502141
0	.1	.0513668	.0515344	.0514768	.0518870	.0519284	.0519173	.0521565	.0521635	.0521621
.05	.05	.0513663	.0515421	.0514800	.0518866	.0519337	.0519195	.0521563	.0521661	.0521633
0	.5	.0570250	.0577820	.0575496	.0597982	.0599335	.0599209	.0612521	.0612476	.0612623
.10	.40	.0570175	.0579065	.0576017	.0597926	.0600188	.0599574	.0612492	.0612899	.0612812
.25	.25	.0570133	.0579765	.0576310	.0597894	.0600668	.0599780	.0612475	.0613137	.0612918
0	1	.0645229	.0658232	.0655039	.0704959	.0705791	.0706656	.0736673	.0735561	.0736409
0	2	.08089	.08258	.08256	.09450	.09389	.09453	.10186	.10123	.10162
1	1	.08075	.08584	.08394	.09441	.09614	.09552	.10182	.10234	.10210
0	3	.0990	.1001	.1010	.1218	.1197	.1214	.1342	.1327	.1336
.01	1	.0646760	.0660182	.0656803	.0707177	.0708209	.0708981	.0739265	.0738237	.0739045
.1	.1	.0527507	.0531108	.0529832	.0538089	.0539072	.0538771	.0543593	.0543806	.0543742
		m = 5	n = 5		m = 5	n = 15		m = 5	n = 40	
0	$.0^41$	.05000008	.05000009	.05000009	.05000012	.05000012	.05000012	.05000014	.05000014	.05000014
0	.01	.0500807	.0500943	.0500901	.0501203	.0501248	.0501235	.0501442	.0501445	.0501449
0	.1	.0508100	.0509455	.0509038	.0512105	.0512538	.0512423	.0514528	.0514615	.0514597
.05	.05	.0508097	.0509483	.0509051	.0512102	.0512563	.0512434	.0514526	.0514631	.0514604
0	.5	.0541284	.0547795	.0545883	.0562276	.0564067	.0563713	.0575132	.0575293	.0575362
.10	.40	.0541246	.0548243	.0546095	.0562239	.0564479	.0563897	.0575110	.0575546	.0575472
.25	.25	.0541224	.0548495	.0546215	.0562218	.0564711	.0564001	.0575098	.0575688	.0575533
0	1	.0584539	.0596835	.0593478	.0628946	.0631535	.0631409	.0656522	.0655988	.0656668
0	2	.06768	.06983	.06937	.07754	.07763	.07786	.08380	.08306	.08370
1	1	.06760	.07103	.06993	.07747	.07876	.07836	.08376	.08407	.08401
0	3	.0777	.0805	.0800	.0940	.0934	.0941	.1044	.1010	.1041
.01	1	.0585415	.0597942	.0594502	.0630316	.0633025	.0632851	.0658209	.0657725	.0658384
.1	.1	.0516268	.0519074	.0518198	.0524375	.0525315	.0525051	.0529299	.0529511	.0529459

Table 6. (Continued) α = .01

ω_1	ω_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 0 n = 5			m = 0 n = 15			m = 0 n = 40		
0	$.0^{4}1$	.01000006	.01000007	.01000007	.01000008	.01000009	.01000009	.01000009	.01000010	.01000010
0	.01	.0100667	.0100749	.0100714	.0100910	.0100929	.0100923	.0101027	.0101030	.0101029
0	.1	.0106769	.0107561	.0107238	.0109291	.0109454	.0109411	.0110506	.0110528	.0110526
.05	.05	.0106758	.0107670	.0107273	.0109283	.0109514	.0109433	.0110502	.0110555	.0110537
0	.5	.0136128	.0139356	.0138341	.0150726	.0150978	.0151126	.0157928	.0157745	.0157897
.1	.4	.0135957	.0141157	.0138921	.0150603	.0151975	.0151497	.0157867	.0158191	.0158077
.25	.25	.0135860	.0142171	.0139247	.0150534	.0152536	.0151706	.0157833	.0158442	.0158178
0	1	.0178053	.0182498	.0182094	.0212415	.0211392	.0212667	.0229765	.0228619	.0229359
0	2	.02796	.02795	.02860	.03700	.03613	.03678	.04171	.04115	.04148
1	1	.02760	.03293	.03027	.03676	.03894	.03792	.04160	.04241	.04203
0	3	.0406	.0396	.0412	.0574	.0551	.0568	.0664	.0653	.0658
.01	1	.0178911	.0183860	.0183177	.0213736	.0212960	.0214106	.0231333	.0230292	.0230988
.1	.1	.0113721	.0115718	.0114831	.0118973	.0119581	.0119314	.0121530	.0121669	.0121616
		m = 1 n = 5			m = 1 n = 15			m = 1 n = 40		
0	$.0^{4}1$	.01000004	.01000005	.01000004	.01000005	.01000006	.01000006	.01000007	.01000007	.01000007
0	.01	.0100436	.0100515	.0100485	.0100630	.0100651	.0100645	.0100729	.0100734	.0100732
0	.1	.0104410	.0105187	.0104903	.0106406	.0106603	.0106550	.0107436	.0107467	.0107462
.05	.05	.0104404	.0105232	.0104920	.0106401	.0106637	.0106563	.0107433	.0107484	.0107468
0	.5	.0123230	.0126799	.0125634	.0134470	.0135051	.0135044	.0140407	.0140320	.0140430
.1	.4	.0123130	.0127559	.0125928	.0134386	.0135618	.0135260	.0140362	.0140603	.0140543
.25	.25	.0123073	.0127987	.0126094	.0134338	.0135936	.0135382	.0140336	.0140762	.0140607
0	1	.0149471	.0155707	.0154096	.0175263	.0175285	.0175996	.0189229	.0188372	.0188973
0	2	.02113	.02195	.02196	.02770	.02718	.02766	.03141	.03092	.03122
1	1	.02089	.02413	.02282	.02751	.02886	.02833	.03131	.03177	.03157
0	3	.0286	.0291	.0297	.0408	.0391	.0403	.0477	.0465	.0472
.01	1	.0149943	.0156506	.0154772	.0176125	.0176295	.0176936	.0190285	.0189491	.0190060
.1	.1	.0108913	.0110645	.0109988	.0113031	.0113545	.0113378	.0115178	.0115296	.0115258

Table 6. (Continued) α = .01

ω_1	ω_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 2 n = 5			m = 2 n = 15			m = 2 n = 40		
0	$.0^{4}1$	.01000003	.01000004	.01000003	.01000005	.01000005	.01000004	.01000005	.01000005	.01000005
0	.01	.0100327	.0100397	.0100373	.0100493	.0100514	.0100508	.0100583	.0100587	.0100586
0	.1	.0103302	.0103993	.0103763	.0104998	.0105199	.0105145	.0105929	.0105965	.0105958
.05	.05	.0103298	.0104017	.0103773	.0104994	.0105220	.0105153	.0105927	.0105977	.0105963
0	.5	.0117234	.0120543	.0119517	.0126613	.0127343	.0127250	.0131886	.0131875	.0131947
.1	.4	.0117169	.0120956	.0119695	.0126553	.0127706	.0127395	.0131852	.0132075	.0132027
.25	.25	.0117133	.0121188	.0119795	.0126519	.0127910	.0127476	.0131833	.0132187	.0132073
0	1	.0136329	.0142511	.0140810	.0157447	.0158113	.0158428	.0169631	.0169065	.0169509
0	2	.01803	.01905	.01888	.02327	.02303	.02333	.02642	.02605	.02629
1	1	.01787	.02024	.01940	.02312	.02412	.02377	.02634	.02667	.02654
0	3	.0232	.0244	.0244	.0327	.0317	.0325	.0386	.0376	.0382
.01	1	.0136714	.0143073	.0141302	.0158094	.0158863	.0159132	.0170440	.0169919	.0170342
.1	.1	.0106659	.0108144	.0107638	.0110140	.0110619	.0110475	.0112070	.0112180	.0112148
		m = 5 n = 5			m = 5 n = 15			m = 5 n = 40		
0	$.0^{4}1$	.01000002	.01000002	.01000002	.01000003	.01000003	.01000003	.01000004	.01000005	.01000004
0	.01	.0100189	.0100239	.0100225	.0100309	.0100328	.0100323	.0100385	.0100390	.0100387
0	.1	.0101906	.0102400	.0102261	.0103124	.0103305	.0103260	.0103902	.0103939	.0103932
.05	.05	.0101904	.0102407	.0102265	.0103122	.0103313	.0103263	.0103901	.0103946	.0103935
0	.5	.0109800	.0112251	.0111583	.0116334	.0117130	.0116974	.0120597	.0120660	.0120707
.1	.4	.0109776	.0112378	.0111648	.0116306	.0117274	.0117037	.0120580	.0120766	.0120767
.25	.25	.0109762	.0112449	.0111684	.0116291	.0117355	.0117072	.0120571	.0120825	.0120770
0	1	.0120304	.0125135	.0123868	.0134521	.0135799	.0135678	.0144037	.0143692	.0144123
0	2	.01435	.01527	.01506	.01768	.01779	.01786	.02003	.01965	.01997
1	1	.01428	.01576	.01524	.01761	.01822	.01805	.01998	.02010	.02010
0	3	.0170	.0182	.0180	.0228	.0226	.0229	.0270	.026	.0268
.01	1	.0120516	.0125424	.0124138	.0134898	.0136226	.0136087	.0144532	.0144205	.0144631
.1	.1	.0103832	.0104855	.0104566	.0106312	.0106704	.0106602	.0107907	.0107995	.0107978

CHAPTER II

POWER COMPARISONS FOR TESTING $\delta\Sigma_1 = \Sigma_2$

1. Introduction and Summary

Let $X(p \times n_1)$ and $Y(p \times n_2)$ with $p \leq n_i$, $i=1,2$ be two matrix variates that are independently distributed. The columns of X and Y are themselves distributed as independent normal variates with mean zero and covariance Σ_1 and Σ_2 respectively. Therefore, XX' and YY' will be independently distributed as Wishart (n_i, p, Σ_i), $i=1,2$. Let $0 < c_1 < c_2 \ldots < c_p < \infty$ be the characteristic roots of the equation

$$|XX' - cYY'| = 0 \tag{1.1}$$

and $0 < \lambda_1 \ldots \leq \lambda_p < \infty$ be the characteristic roots of

$$|\Sigma_1 - \lambda\, \Sigma_2| = 0 \tag{1.2}$$

Define $F = \mathrm{diag}(c_i)$ and $\Lambda = \mathrm{diag}(\lambda_i)$.

To test the null hypothesis, $H_0{:}\delta\Lambda = I$, $\delta > 0$ given, the following criteria, analogous to those in Chapter I, have been suggested [see also Khatri (1967)]

(i) δc_p or $\delta c_p|(1+\delta c_p)$ (Roy, 1945)

(ii) $U^{(p)} = \sum_{i=1}^{p} [\delta c_i]$ (Pillai, 1954, 1955, 1960)

(iii) Pillai's criterion, $V^{(p)} = \sum_{i=1}^{p} [\delta c_i/(1+\delta c_i)]$ (Pillai, 1954, 1955, 1960) and

(iv) Wilks' criterion, $W^{(p)} = \prod_{i=1}^{p} [1/(1+\delta c_i)]$ (Wilks, 1932).

In this chapter we obtain the powers of the last three criteria, in the case $p=2$.

2. Non-central Distributions of $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$

James (1964) has shown that the joint distribution of $c_1,\ldots,c_p$ can be expressed as

$$(2.1)\quad C(p,m,n)\,|\Lambda|^{-\frac{(2m+p+1)}{2}}\,|F|^{m}\ {}_1F_0(\tfrac{1}{2}\nu;\,-\Lambda^{-1},F)\prod_{i>j}(c_i-c_j)$$

where the hypergeometric function ${}_1F_0$ is as defined in the last chapter and $m = \frac{n_1-p-1}{2}$, $n = \frac{n_2-p-1}{2}$ and $\nu = n_1 + n_2$.

Khatri (1967) obtained the joint density of $(\delta c_1,\ldots,\delta c_p)$ in the form

$$(2.2)\quad C(p,m,n)\,|\delta\Lambda|^{-\frac{(2m+p+1)}{2}}\,|\delta F|^{m}\,|I+\delta F|^{-\frac{\nu}{2}}\ \times$$

$${}_1F_0(\tfrac{1}{2}\nu;\,I-(\delta\Lambda)^{-1},\,(\delta F)(I+\delta F)^{-1})\prod_{i>j}\delta(c_i-c_j)\ .$$

For $p = 2$, (2.2) reduces to

$$(2.3)\quad C(2,m,n)\,|\delta\Lambda|^{-\frac{(2m+3)}{2}}\,[(\delta c_1)(\delta c_2)]^{m}\,[(1+\delta c_1)(1+\delta c_2)]^{-\frac{1}{2}\nu}\ \times$$

$${}_1F_0(\tfrac{1}{2}\nu;\,I-(\delta\Lambda)^{-1},\,(\delta F)(I+\delta F)^{-1})(\delta(c_2-c_1))\ .$$

For getting the c.d.f. of any one of the three criteria $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ defined above, we make the following changes in the corresponding c.d.f. derived in Chapter I: $K \rightarrow K'' = |\delta\Lambda|^{-\frac{(2m+3)}{2}} C(2,m,n)$;

$A_{ij} \rightarrow A_{ij}''$ where A_{ij}'' is obtained from A_{ij} by omitting each linear factor involving f_2 in the denominator, each linear factor involving ν in the numerator is raised only to a single power instead of two in A_{ij} and defining b_1 as $2 - (1/\lambda_1 + 1/\lambda_2)/\delta$ and b_2 as $\{1-(1/\delta\lambda_1)\}\{1-(1/\delta\lambda_2)\}$. For example

$$A_{21}'' = \frac{f_2(f_2+2)A_{21}[3\{2-(1/\lambda_1+1/\lambda_2)/\delta\}^2 - 4\{1-(1/\delta\lambda_1)\}\{1-(1/\delta\lambda_2)\}]}{\nu(\nu+2)(3b_1^2 - 4b_2)} .$$

Further, $D_{ij} \rightarrow D_{ij}''$ where D_{ij}' is the same function of A_{ij}'''s as D_{ij} is of A_{ij}'s. Similarly, $C_{ij} \rightarrow C_{ij}''$ where again C_{ij} goes through the same changes as D_{ij} does. Thus all the changes are in A_{ij}, K and the definitions of b_1 and b_2.

3. Power Function Tabulations

For tabulating the powers of the test for H_0 based on $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$, the following procedure was used: The upper 5 percent points of $U^{(2)}$ and $V^{(2)}$ were calculated with eight decimals accuracy. As initial values percentage points from Pillai's tables (1960) were fed to the computer. For $U^{(2)}$ percentage points were obtained for m = 0,1,2, 5,10 and 15 and n = 5(5)30,40,60; but for $V^{(2)}$ these percentage points were obtained for all the same values of m and n, except n = 60. However, for $W^{(2)}$ lower 5 percent points were computed. All these results for m = 0,1,2,5 and n = 5,15,30 and 40 are tabulated in table 7.

4. Power Comparisons

Let us compare the powers of tests with $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ for testing H_0 based on Table 7. We observe the following:

1) For small deviations from the hypothesis $V^{(2)}$ seems to have more power than $W^{(2)}$, in general; and $W^{(2)}$ more power than $U^{(2)}$.

2) For $\delta(\lambda_1 + \lambda_2)$ = constant, the power of $V^{(2)}$ generally increases as the two roots tend to be equal, on the other hand, that of $U^{(2)}$ generally decreases and $W^{(2)}$ increases in most cases, at least for small deviations.

3) For larger deviations from the hypothesis, when the values of $\delta\lambda_1$ and $\delta\lambda_2$ are far apart, the powers of $U^{(2)}$ sometimes seem to exceed those of $V^{(2)}$ and $W^{(2)}$. But the power of $V^{(2)}$ always exceeds those of $U^{(2)}$ and $W^{(2)}$ when the values of $\delta\lambda_1$ and $\delta\lambda_2$ are equal.

4) The power of the test based on any four criteria does not seem to have the monotonicity property with respect to the sum of roots.

It may be pointed out that Anderson and Das Gupta (1964) have established the monotonicity of the power, with respect to each population characteristic root, of some of these tests, for example, $U^{(2)}$, $W^{(2)}$ and the largest root (which is not considered in this chapter).

Table 7. Powers of $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ tests for testing $\delta\lambda_1 = 1$, $\delta\lambda_2 = 1$ against different simple alternative hypotheses, $\alpha = .05$

$\delta\lambda_1$	$\delta\lambda_2$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 0, n = 5			m = 0, n = 15		
1	1.00001	.0500008	.0500008	.0500008	.0500010	.0500010	.0500010
1	1.001	.050079	.050084	.050082	.050099	.050100	.050100
1	1.1	.058298	.058614	.058538	.060546	.060565	.060580
1.05	1.05	.058237	.058703	.058525	.060431	.060527	.060499
1	1.5	.0976	.0976	.0981	.1122	.1110	.1119
1.25	1.25	.0964	.0991	.0983	.1105	.1110	.1109
1	2	.155	.148	.154	.187	.184	.186
1.333	1.5	.133	.137	.136	.160	.160	.160
1	4	.380	.352		.453	.443	
1	5	.468	.436		.545	.534	
2	4	.495	.496		.581	.580	
3	3	.505	.513		.594	.595	
1	8	.645	.613		.713	.703	
4.5	4.5	.720	.724		.793	.793	
1	11	.745	.717		.799	.792	
6	6	.832	.835		.883	.884	
		m = 0, n = 30			m = 0, n = 40		
1	1.00001	.0500011	.0500011	.0500011	.0500011	.0500011	.0500011
1	1.001	.050106	.050107	.050107	.050108	.050109	.050109
1	1.1	.061352	.061338	.061353	.061574	.061558	.061571
1.05	1.05	.061220	.061249	.061241	.061438	.061455	.061451
1	1.5	.1175	.1168	.1172	.1189	.1184	.1187
1.25	1.25	.1155	.1157	.1157	.1170	.1171	.1170
1	2	.199	.197	.198	.202	.200	.201
1.333	1.5	.169	.169	.169	.171	.172	.171
1	4	.476	.471		.482	.478	
1	5	.568	.563		.574	.570	
2	4	.607	.606		.614	.613	
3	3	.621	.622		.628	.629	
1	8	.732	.728		.737	.734	
4.5	4.5	.813	.814		.818	.819	
1	11	.815	.811		.819	.816	
6	6	.897	.897		.901	.901	

Table 7. (Continued)

$\delta\lambda_1$	$\delta\lambda_2$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 1 n = 5			m = 1 n = 15		
1	1.00001	.0500009	.0500009	.0500009	.0500012	.0500012	.0500012
1	1.001	.050089	.050098	.050094	.050117	.050119	.050119
1	1.1	.059399	.060056	.059888	.062660	.062708	.062740
1.05	1.05	.059319	.060273	.059928	.062530	.062754	.062690
1	1.5	.105	.104	.106	.128	.125	.127
1.25	1.25	.104	.110	.108	.126	.127	.127
1	2	.177	.165	.173	.229	.219	.223
1.333	1.5	.150	.159	.157	.192	.194	.193
1	4	.483	.436		.580	.557	
1	5	.597	.550		.687	.665	
2	4	.623	.629		.727	.725	
3	3	.633	.652		.740	.743	
1	8	.795	.762		.853	.839	
4.5	4.5	.860	.868		.915	.916	
1	11	.884	.861		.919	.910	
6	6	.942	.945		.967	.968	
		m = 1 n = 30			m = 1 n = 40		
1	1.00001	.0500013	.0500013	.0500013	.0500013	.0500013	.0500013
1	1.001	.050128	.050129	.050129	.050132	.050132	.050132
1	1.1	.063915	.063882	.063918	.064269	.064230	.064262
1.05	1.05	.063765	.063836	.063817	.064113	.064156	.064145
1	1.5	.1371	.1353	.1363	.139	.138	.139
1.25	1.25	.1349	.1354	.1352	.137	.137	.137
1	2	.249	.243	.245	.254	.250	.251
1.333	1.5	.208	.208	.209	.212	.216	.213
1	4	.613	.601		.621	.613	
1	5	.716	.706		.724	.716	
2	4	.759	.758		.767	.766	
3	3	.773	.774		.782	.782	
1	8	.871	.864		.875	.871	
4.5	4.5	.930	.930		.934	.934	
1	11	.930	.926		.932	.930	
6	6	.974	.974		.976	.976	

Table 7. (Continued)

$\delta\lambda_1$	$\delta\lambda_2$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
		m = 2 n = 5			m = 2 n = 15		
1	1.00001	.0500009	.0500011	.0500010	.0500013	.0500013	.0500013
1	1.001	.050095	.050107	.050103	.050131	.050134	.050133
1	1.1	.060100	.061036	.060793	.064218	.064302	.064347
1.05	1.05	.060015	.061334	.060878	.064081	.064433	.064333
1	1.5	.118	.110	.112	.140	.136	.139
1.25	1.25	.109	.118	.115	.138	.141	.139
1	2	.195	.180	.185	.263	.248	.251
1.333	1.5	.161	.176	.171	.218	.221	.220
1	4	.577	.533		.677	.647	
1	5	.707	.668		.787	.762	
2	4	.725	.735		.823	.821	
3	3	.736	.758		.836	.838	
1	8	.892	.872		.927	.915	
		m = 2 n = 30			m = 2 n = 40		
1	1.00001	.0500014	.0500014	.0500014	.0500015	.0500015	.0500015
1	1.001	.050145	.050146	.050146	.050149	.050150	.050150
1	1.1	.065911	.065859	.065916	.066400	.066336	.066388
1.05	1.05	.065751	.065867	.065837	.066234	.066304	.066286
1	1.5	.153	.150	.152	.157	.154	.155
1.25	.125	.151	.152	.151	.154	.155	.155
1	2	.291	.281	.282	.299	.290	.291
1.333	1.5	.241	.242	.242	.248	.248	.249
1	4	.714	.697		.724	.703	
1	5	.815	.802		.823	.805	
2	4	.854	.850		.861	.857	
3	3	.865	.867		.873	.872	
1	8	.939	.933		.942	.933	
		m = 5 n = 5			m = 5 n = 15		
1	1.00001	.0500011	.0500012	.0500012	.0500016	.0500016	.0500015
1	1.001	.050105	.050123	.050117	.050156	.050162	.050161
1	1.1	.061244	.062739	.062368	.067333	.067529	.067594
1.05	1.05	.061157	.063167	.062543	.067187	.067873	.067685
1	1.5	.122	.121	.122	.168	.160	.164
1.25	1.25	.119	.133	.128	.166	.171	.169
1	2	.248	.232	.197	.349	.319	.298
1.333	1.5	.185	.210	.199	.278	.285	.281
		m = 5 n = 30			m = 5 n = 40		
1	1.00001	.0500018	.0500018	.0500018	.0500019	.0500019	.0500019
1	1.001	.050180	.050182	.050182	.050187	.050188	.050188
1	1.1	.070259	.070351	.070274	.071161	.071181	.071135
1.05	1.05	.070087	.070338	.070273	.070981	.071107	.071098
1	1.5	.192	.194	.186	.200	.201	.196
1.25	1.25	.189	.190	.191	.197	.199	.198
1	2	.398	.355	.353	.413	.372	.371
1.333	1.5	.323	.329	.323	.337	.334	.337

CHAPTER III

EXACT DISTRIBUTION OF PILLAI'S CRITERION IN THE CENTRAL CASE

1. Introduction and Summary

Some unsolved problems of long standing in multivariate analysis are the exact distribution problems of the test criteria given in earlier chapters like $U^{(p)}$, $V^{(p)}$ and $W^{(p)}$. The exact distributions of these criteria are not available (even under each of the null hypotheses) except in special cases. Of these, the c.d.f. of Pillai's criterion, $V^{(p)}$, has been more difficult to obtain since this has to be established separately for each interval, $i \leq z \leq i+1, i=0,1,\ldots,p-1$ The exact c.d.f. of $V^{(2)}$ i.e. for $p = 2$, has been derived earlier. Nanda (1950) attacked this as a mathematical problem to obtain the c.d.f. of $V^{(p)}$ but succeeded in obtaining the c.d.f. only in the very special case of $m = 0$ (see earlier chapters for definition of m) and that too only for $p = 2$ and 3. In the present chapter and the following, a study of the exact distribution of $V^{(p)}$ in the null case is attempted extending the method of Nanda and explicit expressions for the c.d.f. are given for $p = 3$ and values of $m \leq 3$, and $p = 4$ and $m = 0$ and 1 (given in the next chapter). It may be pointed out that these values of m cover a lot of grounds in terms of tests (i) and (ii), since in (i) $m = \frac{1}{2}(q-p-1)$, where q and p are numbers of variables of the two sets, $p \leq q$, and in (ii) $m = \frac{1}{2}(|\ell-p-1|-1)$, where ℓ is the number of samples and p, the number of variables.

2. Nanda's Method for m = 0

In this section, the method of approach of Nanda will be given briefly in order to describe the extension of the method in the next section. It is obvious from the previous chapters that the joint density of the characteristic roots under each of three null hypotheses can be expressed in the form

$$(2.1)\qquad f(e_1,e_2,\dots,e_p) = C(p,m,n) \prod_{i=1}^{p} \{e_i^m(1-e_i)^n\} \prod_{i>j}(e_i-e_j) \prod_{i=1}^{p}(de_i),$$

$$0 < e_1 \leq e_2 \leq \dots \leq e_p < 1.$$

Now let $g_i = e_i n$, $i=1,2,\dots,p$ and $n \to \infty$, then

$$(2.2)\qquad f(g_1,g_2,\dots,g_p) = C'(p,m,o)e^{-\Sigma g_i} \prod_{i=1}^{p} g_i^m \prod_{i>j}(g_i-g_j) \prod_{i=1}^{p} dg_i$$

where by Σg_i in the exponent we mean $\sum_{i=1}^{p} g_i$, where

$$C'(p,m,o) = \pi^{p/2} / \prod_{i=1}^{p} \Gamma(2m+i+1/2)\Gamma(\tfrac{i}{2}).$$

Consider the c.d.f. of the largest root

$$(2.3)\qquad P(g_p < x) = C'(p,m,o) \int_{0<g_1<g_2<\dots<g_p<x} e^{-\Sigma g_i} \prod_{i=1}^{p} g_i^m \prod_{i>j}(g_i-g_j) \prod_{i=1}^{p} dg_i.$$

The transformation $xy_i = g_i$ yields

$$(2.4) \quad C(p,m,o) \int_{0<g_1<g_2<\ldots<g_p<x} e^{-\Sigma g_i} \prod_{i=1}^{p} g_i^m \prod_{i>j} (g_i-g_j) \prod_{i=1}^{p} dg_i$$

$$= C(p,m,o) \int_{0<y_1<y_2<\ldots<y_p<1} x^{mp+p\frac{(p+1)}{2}} e^{-x\Sigma y_i} \prod_{i=1}^{p} y_i^m \prod_{i>j} (y_i-y_j) \times \prod_{i=1}^{p} dy_i$$

Let us replace y_i by $1-y_i (i=1,2,\ldots,p)$ on the right-hand side of (2.4) and then change m to n on both sides

$$(2.5) \quad C(p,o,n)\, V(x;p-1,p-2,\ldots,1,0;-1) = e^{-px}\, x^{np+p\frac{(p+1)}{2}} M(x,o,n,p),$$

where $M(x,o,n,p)$ denotes the moment generating function of $V^{(p)}$ when $m=0$ and $V(x;p-1,p-2,\ldots,1,0;-1)$ represents the Vandermonde determinant defined by Pillai (1956). Here $V(x;p-1,p-2,\ldots,1,0,;-1)$ actually stands for $V(x;n+p-1,n+p-2,\ldots,n+1,n;-1)$ where

$$V(x;q_p,\ldots,q_1;t) = \begin{vmatrix} \int_o^x y_p^{q_p} e^{ty_p} dy_p & \cdots & \int_o^x y_p^{q_1} e^{ty_p} dy_p \\ \cdots\cdots & & \cdots\cdots \\ \int_o^{y_2} y_1^{q_p} e^{ty_1} dy_1 & \cdots & \int_o^{y_2} y_1^{q_1} e^{ty_1} dy_1 \end{vmatrix} .$$

With a view to illustrating how one obtains this, we will derive $M(x,o,n,3)$, using relation (2.5).

$$M(x,o,n,3) = C(3,o,n)\, e^{3x}\, x^{-(3n+6)}\, V(x;2,1,0;-1) \tag{2.6}$$

Now $V(x;2,1,0;-1)$ can be shown to reduce to the form

$$\begin{aligned} V(x;2,1,0;-1) = {} & 2I(x;2n+3,-2)\, I(x;n,-1) \\ & -2I(x;2n+2,-2)\, I(x;n+1,-1) \\ & -2I_o(x;n+2,-1)\, I(x;2n+2,-2) \\ & +I_o(x;2n+2,-2)\, I(x;n+1,-1) \end{aligned} \tag{2.7}$$

where

$$I(x;q,t) = \int_o^x e^{ty}\, y^q\, dy$$

and

$$I_o(x;q,t) = e^{ty}\, y^q \Big|_o^x .$$

Putting xu for y in the definition of $I(x;q,t)$ above, we get

$$\begin{aligned} V(x;2,1,0;-1) = x^{3n+5} [& 2I(1;2n+3,-2x)\, I(1;n,-x) \\ & -2I(1;2n+2,-2x)\, I(1;n+1,-x) \\ & -\frac{2e^{-x}}{(n+1)}\, I(1;2n+2,-2x) \\ & +\frac{e^{-2x}}{(n+1)}\, I(1;n+1,-x)]. \end{aligned} \tag{2.8}$$

By integrating by parts we can get x^{3n+6} as a common factor on the right-hand side of the above equation. We use the value of $V(x;2,1,0,;-1)$ thus obtained in equation (2.6) to get

$$(2.9) \qquad M(x,0,n,3) = C(3,0,n)\,\Big[\frac{1}{(n+1)(n+2)(2n+3)}\, I(n+2,x) + \frac{2}{(n+1)(n+2)}\, I(2n+4,x) - \frac{4}{(n+1)(2n+3)}\, I(2n+3,x) + \frac{2}{(n+1)(2n+3)}\, I(2n+3,2x)\, I(n+1,x)\Big]$$

where $I(q,\alpha x) = \int_0^1 e^{\alpha x(1-u)}\, u^q du = \int_0^1 e^{\alpha ux}(1-u)^q\, du.$

As the m.g.f. is now known, we can obtain the c.d.f., once we know the contribution of each integral on the right-hand side of (2.9). We get corresponding terms for $I(n+\ell,x)$ and $I(2n+k,2x)$ by integrating the density $(1-u)^{n+\ell}$ of u and $(1-u)^{2n+k}$ of $2u$, respectively. Hence

$$I(n+\ell,\, x) \rightarrow \begin{cases} \dfrac{1-(1-z)^{n+\ell+1}}{n+\ell+1} & 0 \leq z \leq 1 \\[2ex] \dfrac{1}{n+\ell+1} & 1 \leq z \leq 2 \end{cases}$$

and

$$I(2n+k,x) \to \begin{cases} \dfrac{1-(1-z/2)^{2n+k+1}}{2n+k+1} & 0 \leq z \leq 2 \\ \\ \dfrac{1}{2n+k+1} & 2 \leq z \leq 3. \end{cases}$$

However to get the contribution of $I(n+\ell,x)\ I(2n+k,x)$ we consider y_1 and y_2, two random variables, distributed in $(0,1)$ with densities $(1-y_1)^{n+\ell}$ and $(1-y_2)^{2n+k}$. Then

$$I(n+\ell,x)\ I(2n+k,x) \to \iint_{y_1+2y_2 \leq z} (1-y_1)^{n+\ell}(1-y_2)^{2n+k} dy_1\, dy_2$$

where $0 \leq z \leq 3$.

The region of integration is given by OPQ, OARS, OATUC (as is Fig.1) for $0 \leq z \leq 1$, $1 \leq z \leq 2$ and $2 \leq z \leq 3$ respectively.

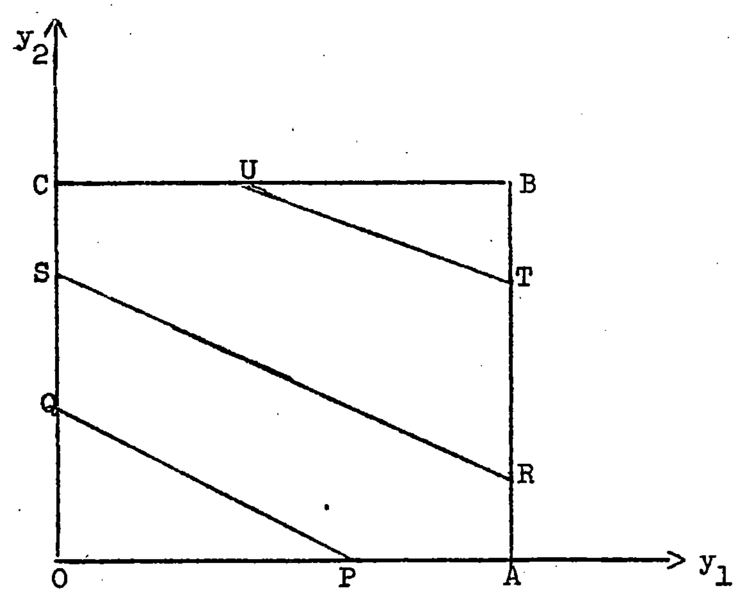

Figure 1. Region of Integration of $I(n+\ell,x)I(2n+k,x)$

Therefore

$$I(n+\ell,x)\ I(2n+k,2x) \rightarrow \begin{cases} \dfrac{1-(1-z/2)^{2n+k+1}}{(n+\ell+1)(2n+k+1)} - sJ(z_1,z_2;2n+k+1,n+\ell+2) & 0 \le z \le 1 \\ \dfrac{1-(1-z/2)^{2n+k+1}}{(n+\ell+1)(2n+k+1)} - sJ(z_1,1;2n+k+1,n+\ell+2) & 1 \le z \le 2, \\ \dfrac{1}{(n+\ell+1)(2n+k+1)} - sJ(0,1;2n+k+1,n+\ell+2) & 2 \le z \le 3 \end{cases}$$

where $s = 2^{n+\ell+1}/(n+\ell+1)z_2^{3n+k+\ell+2}$, $z_1=(2-z)/(3-z)$, $z_2=2/(3-z)$, $J(a,b;c,d) = \int_a^b x^{c-1}(1-x)^{d-1}\,dx$. Hence

$$P(V^{(3)} \le z) = 1-C(3,0,n)\ W(3,0) \tag{2.10}$$

where

$$W(3,0) = \begin{cases} [p_{0,3}(1-z)^{n+3} + \sum_{j=4}^{5} q_{0,j}(1-z/2)^{2n+j} + \omega_{0;3,2}J(z_1,z_2;2n+4,n+3)] & 0 \le z \le 1 \\ [\sum_{j=4}^{5} q_{0,j}(1-z/2)^{2n+j} + \omega_{0;3,2}J(z_1,1;2n+4,n+3)] & 1 \le z \le 2 \\ \omega_{0;3,2}\ J(0,1;2n+4,n+3) & 2 \le z \le 3 \end{cases}$$

$\omega_{m;k,\ell} = 2^{n+\ell}\ z_2^{-(3n+k+\ell+1)}\ h_{m;k,\ell}$ and $p_{m,j}$, $q_{m,j}$ and $h_{m;k,\ell}$ are defined in section 5.

3. Extension of Nanda's Method

To obtain the m.g.f. of $V^{(p)}$ for larger values of m, we proceed as follows: Multiply the right-hand side of (2.2) by the factor

$\prod_{i=1}^{p} (1-g_i)^n$ and replace $C'(p,m,o)$ by $C(p,m,n)$. It is well-known that

$$(3.1) \qquad \prod_{i=1}^{p} (1-g_i)^n = (1-a_1+a_2 \ldots +(-1)^p a_p)^n$$

where $a_j = \sum_{\substack{i_1 < i_2 \ldots < i_j \\ 1 \le i_j \le p}} g_{i_1} g_{i_2} \cdots g_{i_j}$ is the jth elementary symmetric function. Using (3.1) in (2.5) we get

$$(3.2) \; C(p,m,n) \int_{0<g_1<g_2\ldots<g_p<x} e^{-\Sigma g_i} \prod_{i=1}^{p} g_i^n \prod_{i>j} (g_i-g_j)(1-a_1\ldots(-1)^p a_p)^m \prod_{i=1}^{p} dg_i .$$

Let us express $(1-a_1+\ldots(-1)^p a_p)^m$ in the form

$$(3.3) \quad (1-a_1+\ldots+(-1)^p a_p)^m = \sum_{i_0+i_1+\ldots+i_p=m} \left(\frac{m!}{i_0!\ldots i_p!}\right)(-a_1)^{i_1}\ldots((-1)^p a_p)^{i_p}.$$

Thus the integral in (3.2) becomes

$$\int_{0<g_1<g_2<\ldots<g_p<x} e^{-\Sigma g_i} \prod_{i=1}^{p} g_i^n \prod_{i>j} (g_i-g_j) \sum_{i_0+\ldots+i_p=m} \left(\frac{m!}{i_0!\ldots i_p!}\right)(-a_1)^{i_1}\ldots ((-1)^p a_p)^{i_p} \prod_{i=1}^{p} dg_i$$

where each term is a monomial in $a_1,\ldots,a_p$. To express this as a Vandermonde determinant we apply Pillai's lemma (1964) which is briefly stated below:

<u>Lemma</u>: Let $D(q_p, q_{p-1}, \ldots, q_1)$, $(q_j \geq 0,\ j = 1,2,\ldots,p)$, denote the determinant

$$(3.4) \quad D(q_p, q_{p-1}, \ldots, q_1) = \begin{vmatrix} y_p^{q_p} & \cdots & y_p^{q_1} \\ \vdots & & \vdots \\ y_1^{q_p} & \cdots & y_1^{q_1} \end{vmatrix} .$$

If $a_r (r \leq p)$ denotes the rth e.s.f. in p x's, then

(i)

$$(3.5) \quad a_r D(q_p, q_{p-1}, \ldots, q_1) = \Sigma' D(q_p', q_{p-1}', \ldots, q_1'),$$

where $q_p' = q_p + d$, $j=1,2,\ldots,p$, $d=0,1$ and Σ' denotes the sum over all $\binom{p}{r}$ combinations of p q's taken r at a time for which r indices $q_j' = q_j + 1$ such that $d=1$ while for other indices $q_j' = q_j$ such that $d=0$.

(ii)

$$(3.6) \quad a_r a_h D(q_p, q_{p-1}, \ldots, q_1) = \Sigma'' D(q_p'', q_{p-1}'', \ldots, q_1''),$$

where $h \leq p$, $q_j'' = q_j' + d$, $j=1,2,\ldots,p$, $d=0,1$ and Σ'' denotes the summation over the $\binom{p}{r}\binom{p}{h}$ terms obtained by taking h at a time of the p q's in each D in Σ' in (3.5) for which h indices $q_j'' = q_j' + 1$ while for other indices $q_j'' = q_j'$.

(iii) $(a_r)^k (a_h)^\ell D(q_p, q_{p-1}, \ldots, q_1)$, $(k, \ell \geq 0)$ can be expressed as a sum of $\binom{p}{r}^k \binom{p}{h}^\ell$ determinants obtained by performing on $D(q_p, q_{p-1}, \ldots, q_1)$ in any order (i) k times and (ii) ℓ times with $r = h$.

However, if at least two of the indices in any determinant are equal, the corresponding term in the summation vanishes.

Therefore, after using this lemma (3.2) gives

$$(3.7) \qquad M(x,m,n,p) = e^{px}\, x^{-pn-p\frac{(p+1)}{2}} \, \Sigma^{*} V(x;q_p,q_{p-1},\ldots,q_1,\ldots,1)x^{-\alpha}$$

where Σ^{*} denotes summation over all possible determinants obtained by using the above lemma, and $\alpha = \sum_{j=1}^{p} (q_j - j - 1)$.

Our next step is to evaluate the determinants under Σ^{*}. We can apply Pillai's reduction formula (1956) which for our purpose states

$$(3.8) \qquad V(x;q_p,q_{p-1},\ldots,q_1;t) = A^{(p)} + B^{(p)} + q_p\, C^{(p)}$$

where

$$A^{(p)} = -I_0(x;q_p,t)\, V(x;q_{p-1},q_{p-2},\ldots,q_1;t),$$

$$B^{(p)} = 2 \sum_{j=p-1}^{1} (-1)^{p-j-1} I(x;q_p+q_j,2t) V(x;q_p,q_{p-1},\ldots,q_{j+1},q_{j-1},\ldots,q_1;t),$$

$$C^{(p)} = V(x;q_p-1,q_{p-1},\ldots,q_1;t).$$

We will derive the expression for a simple case to show how this is used. When $m=1, p=3$, one of the determinants occuring in (3.7) is $V(x;3,1,0;-1)$ which is evaluated as follows:

$$(3.9) \qquad V(x;3,1,0;-1) = A^{(3)} + B^{(3)} + (n+3)\, C^{(3)}$$

where

$A^{(3)} = -I_0(x;n+3,-1)\ V(x;1,0;-1),$

$B^{(3)} = 2I(x;2n+4,-2)\ I(x;n,-1) - 2I(x;2n+3,-2)\ I(x;n+1,-1),$

$C^{(3)} = V(x;2,1,0;-1).$

But $V(x;3,1,0;-1)$ has a term $x^{-(3n+7)}$ coming with it. As $I_0(x;n+3,-1)\ V(x;1,0;-1)$ gives only x^{3n+5}, therefore we have to integrate $A^{(3)}$ twice, by parts. Similarly $B^{(3)}$ and $C^{(3)}$ have to be integrated to take care of the power of x. In general, this is true for all $V(x;q_3,q_2,q_1,-1)$.

$$(3.10)\quad V(x;3,1,0;-1) = -I_0(x;n+3,-1)\Big[4\frac{I(x;2n+3;-2)}{(n+1)(2n+3)} - \frac{I_0(x;n+1,-1)I(x;n+2,-1)}{(n+1)(n+2)}\Big]$$

$$+ 2\Big[\frac{I(x;2n+4,-2)I(x;n+1,-1)}{n+1} + 4\frac{I_0(x;n+1,-1)I(x;2n+5,-2)}{(n+1)(2n+5)}\Big]$$

$$-2\Big[\frac{I(x;2n+4,-2)I(x;n+1,-1)}{n+2} + \frac{I_0(x;2n+4,-2)I(x;n+2,-1)}{2(n+2)^2}\Big]$$

$$+(n+3)\Big[\frac{I_0(x;2n+3,-2)I(x;n+3,-1)}{(n+1)(n+2)(n+3)(2n+3)}$$

$$-4\frac{I_0(x;n+2,-1)I(x;2n+4;-2)}{(n+1)(n+2)(2n+3)}$$

$$+2\frac{I(x;2n+4,-2)I(x;n+1,-1)}{(n+1)(n+2)(2n+3)}$$

$$+\frac{I_0(x;2n+4,-2)I(x;n+2,-1)}{(n+1)(n+2)^2(2n+3)}$$

$$+4\frac{I_0(x;n+1,-1)I(x;2n+5,-2)}{(n+1)(n+2)(2n+5)}\Big].$$

After simplifying this and going through similar changes, we get

$$(3.11)\quad V(x;3,1,0;-1)x^{-(3n+7)}e^{3x} = I_0\frac{(2n+3,2x)I(n+3,x)}{(n+1)(n+2)(2n+3)}$$

$$- 4\,I_0\frac{(n+3,x)I(2n+3,2x)}{(n+1)(2n+3)}$$

$$+ 6\frac{I(n+1,x)I(2n+4,2x)}{(n+1)(2n+3)}$$

$$+ 3I_0\frac{(2n+4,2x)I(n+2,x)}{(n+1)(n+2)(2n+3)}$$

$$+ 4I_0\frac{(n+1,x)I(2n+5,2x)}{(n+1)(n+2)}$$

$$- 4(n+3)\; I_0\frac{(n+2,x)I(2n+4,2x)}{(n+1)(n+2)(2n+3)}\;.$$

So, in this way, we can evaluate all the determinants and get the m.g.f. from which we can derive the c.d.f.

4. The Exact Distribution of $V^{(3)}$ when $m = 1,2,3$

As the c.d.f. of $V^{(3)}$ when $m = 0$ has been derived in section 2 above, we will derive the c.d.f.'s of $V^{(3)}$ for $m = 1,2,3$. The method described in section 3 will be used with some results from section 2.

Case (i): $m = 1$

$$(4.1)\quad M(x,1,n,3) = e^{3x}\, x^{-(3n+6)}C(3,1,n)[V(x;2,1,0;-1)-x^{-1}V(x;3,1,0;-1)$$
$$+x^{-2}\, V(x;3,2,0;-1)-x^{-3}V(x;3,2,1;-1)]$$

Then

$$(4.2)\quad P(V^{(3)} \leq z) = 1-C(3,1,n)\; W(3,1)$$

where

$$W(3,1)=\begin{cases} \{\sum_{j=3}^{4} p_{1,j}(1-z)^{n+j}+\sum_{j=4}^{7} q_{1,j}(1-z/2)^{2n+j}+\sum_{k=2}^{5}\omega_{1;k,2}J(z_1,z_2;2n+k+1,n+3) \\ \quad +\omega_{1;5,3}J(z_1,z_2;2n+6,n+4)+\omega_{1;3,4}J(z_1,z_2;2n+4,n+5)\} & 0\le z\le 1 \\ \{\sum_{j=4}^{7} q_{1,j}(1-z/2)^{2n+j}+\sum_{k=2}^{5}\omega_{1;k,2}J(z_1,1;2n+k+1,n+3) \\ \quad +\omega_{1;5,3}J(z_1,1;2n+6,n+4)+\omega_{1;3,4}J(z_1,1;2n+4,n+5)\} & 1\le z\le 2 \\ \{\sum_{k=2}^{5}\omega_{1;k,2}J(0,1;2n+k+1,n+3)+\omega_{1;5,3}J(0,1;2n+6,n+4) \\ \quad +\omega_{1;3,4}J(0,1;2n+4,n+5)\} & 2\le z\le 3. \end{cases}$$

Case (ii): $m = 2$

(4.3) $$M(x,2,n,3)=e^{3x}x^{-(3n+6)}C(3,2,n)[V(x;2,1,0;-1)-2x^{-1}V(x;3,1,0;-1)$$
$$+x^{-2}\{V(x;4,1,0;-1)+3V(x;3,2,0;-1)\}$$
$$-2x^{-3}\{V(x;4;2,0;-1)+2V(x;3,2,1;-1)\}$$
$$+x^{-4}\{V(x;4,3,0;-1)+3V(x;4,2,1;-1)\}$$
$$-2x^{-5}V(x;4,3,1;-1)+x^{-6}V(x;4,3,2;-1)]$$

Therefore

(4.4) $$P(V^{(3)}\le z) = 1 - C(3,2,n)\,W(3,2)$$

where

$$W(3,2)=\begin{cases}\{\sum_{j=3}^{5} p_{2j}(1-z)^{n+j}+\sum_{j=4}^{9} q_{2j}(1-z/2)^{2n+j}+\sum_{\ell=2}^{4}\sum_{k=\ell+1}^{7}\omega_{2;k,\ell}J(z_1,z_2;2n+k+1,n+\ell+1)\} \\ \qquad\qquad 0\le z\le 1 \\ \{\text{same as above except } p_{2j}=0\ (j=3,4,5) \text{ and replace } z_2 \text{ by } 1\}\ 1\le z\le 2 \\ \{\text{same as above except } p_{2j}=0,\ j=3,4,5; q_j=0, j=3,\ldots,7 \text{ and replace } z_1 \\ \text{and } z_2 \text{ by } 0 \text{ and } 1 \text{ respectively}\}\quad 2\le z\le 3.\end{cases}$$

<u>Case (iii): m = 3</u>

(4.5) $$M(x,3,n,3)=e^{3x}x^{-(3n+6)}C(3,3,n)[V(x;2,1,0)-3x^{-1}V(x;3,1,0;-1)$$
$$+3x^{-2}\{V(x;4,1,0;-1)+2V(x;3,2,0;-1)\}$$
$$-x^{-3}\{V(x;5,1,0;-1)+8V(x;4,2,0;-1)$$
$$+10V(x;3,2,1;-1)\}$$
$$+3x^{-4}\{V(x;5,2,0;-1)+2V(x;4,3,0;-1)$$
$$+3V(x;4,2,1;-1)\}$$
$$-3x^{-5}\{V(x;5,3,0;-1)+2V(x;5,2,1;-1)$$
$$+3V(x;4,3,1;-1)\}$$
$$+x^{-6}\{V(x;5,4,0;-1)+8V(x;5,3,1,-1)$$
$$+10V(x;4,3,2;-1)\}$$
$$-3x^{-7}\{V(x;5,3,2;-1)+V(x;5,4,1;-1)\}$$
$$+3x^{-8}V(x;5,4,2;-1)-x^{-9}V(x;5,4,3;-1)].$$

(4.6) $$P(V^{(3)} \leq z) = 1-C(3,3,n)\ W(3,3)$$

where

$$W(3,3)=\begin{cases} \{\sum_{j=3}^{6} p_{3j}(1-z)^{n+j}+\sum_{j=4}^{11} q_{3j}(1-z/2)^{2n+j}+\sum_{\ell=2}^{5}\sum_{k=\ell+1}^{9}\omega_{3;k,\ell} \\ \qquad J(z_1,z_2;2n+k+1,n+\ell+1)\} & 0 \leq z \leq 1 \\ \sum_{j=4}^{11} q_{3j}(1-z/2)^{2n+j}+\sum_{\ell=2}^{5}\sum_{k=\ell+1}^{9}\omega_{3;k,\ell}J(z_1,1;2n+k+1,n+\ell+1) & 1 \leq z \leq 2 \\ \sum_{\ell=2}^{5}\sum_{k=\ell+1}^{9}\omega_{3;k,\ell}\ J(0,1;2n+k+1,n+\ell+1) & 2 \leq z \leq 3. \end{cases}$$

On observing (2.10), (4.2), (4.4) and (4.6) closely, we conclude that if for any m, one proceeds systematically, then the c.d.f. will be given by

(4.7) $$P(V^{(3)} \leq z) = 1-C(3,m,n)\ W(3,m)$$

where

$$W(3,m)=\begin{cases} \{\sum_{j=3}^{m+3} p_{mj}(1-z)^{n+j}+\sum_{j=4}^{2m+5} q_{mj}(1-z/2)^{2n+j} \\ \quad +\sum_{\ell=2}^{m+2}\sum_{k=\ell+1}^{2m+3}\omega_{m;k,\ell}\ J(z_1,z_2;2n+k+1,n+\ell+1)\} & 0 \leq z \leq 1 \\ \sum_{j=4}^{2m+5} q_{mj}(1-z/2)^{2n+j}+\sum_{\ell=2}^{m+2}\sum_{k=\ell+1}^{2m+3}\omega_{m;k,\ell}J(z_1,1;2n+k+1,n+\ell+1) & 1 \leq z \leq 2 \\ \sum_{\ell=2}^{m+2}\sum_{k=\ell+1}^{2m+3}\omega_{m;k,\ell}\ J(0,1;2n+k+1,n+\ell+1) & 2 \leq z \leq 3. \end{cases}$$

The constants in (4.7) will have to be obtained as in other cases.

5. Constant Coefficients

The constant coefficients used in sections 3 and 4 are listed below:

$p_{0,3} = 1/(n+1)(n+2)(n+3)(2n+3),$

$p_{1,3} = (2n+1)\, p_{0,3},$

$p_{1,4} = [(4n^3+20n^2+31n + 12)p_{0,3}]/(n+4)(2n+5),$

$p_{2,3} = -[6(n^2+3n+1)p_{0,3}]/(n+3)(2n+5),$

$p_{2,4} = -[(4n^3+12n^2+17n+3)p_{0,3}]/(n+4)(2n+5),$

$p_{2,5} = 30p_{0,3}/(n+4)(2n+5)(2n+7),$

$p_{3,3} = (4n^3+38n^2+85n+33)/r(n) \times (n+3)(n+4),$

$p_{3,4} = -(8n^5+64n^4+216n^3+395n^2+316n+216)/(n+4)(2n+7) \times r(n),$

$p_{3,5} = (8n^5+68n^4+250n^3+439n^2+417n+78)/r(n) \times (n+4)(n+5)(2n+7),$

$p_{3,6} = -630/[r(n)+(n+4)(n+5)(n+6)(2n+7)(2n+9)],$

where

$r(n) = (n+1)(n+2)(n+3)(2n+3)(2n+5).$

$q_{0,4} = -1/(n+1)(n+2)^2,$

$q_{0,5} = 2/(n+1)(n+2)(2n+5),$

$q_{1,4} = [2(n^3+4n^2+3n-3)q_{0,4}]/(n+3)(n+4)(2n+3),$

$q_{1,5} = -2/(n+2)(n+3)(2n+5),$

$q_{1,6} = -2/(n+1)(n+2)(n+3)^2,$

$q_{1,7} = 2/(n+1)(n+2)(n+3)(2n+7),$

$q_{2,4} = -q_{0,4}/(2n+3),$

$q_{2,5} = -4(3n^2+10n+12)/r(n),$

$q_{2,6} = 2(3n^3+29n^2+90n+94)/(n+1)(n+2)(n+3)^2\ (n+4)(2n+5),$

$q_{2,7} = 2n/(n+2)(n+3)(n+4)(2n+7),$

$q_{2,8} = -6/(n+1)(n+2)(n+3)(n+4)^2,$

$q_{2,9} = 4\ p_{0,3}(2n+3)/(n+4)(2n+9),$

$q_{3,4} = q_{2,4},$

$q_{3,5} = -4(6n^2+19n+21)\ p_{0,3}/(2n+5),$

$q_{3,6} = (40n^4+464n^3+1885n^2+3327n+2244)/r(n) \quad (n+3)(n+4)$

$q_{3,7} = +2(10n^5+63n^4-308n^3-3429n-9668n-9120)/[(n+1)(n+2)(n+3)^2(n+4)(n+5)$

$(2n+5)(2n+7)],$

$q_{3,8} = (8n^4+95n^3+339n^2+250n-422)q_{2,8}/3(n+5)(2n+7),$

$q_{3,9} = -2(n^2+3n+8)/(n+2)(n+3)(n+4)(n+5)(2n+9),$

$q_{3,10} = -24/(n+1)..(n+4)(n+5)^2,$

$q_{3,11} = 12/(n+1)...(n+5)(2n+11).$

$h_{m;3,2} = 2/(n+1)(n+2)(2n+3)$, $m = 0,\ldots,3$

$h_{1;4,2} = -6/(n+1)(n+2)(2n+3)$,

$h_{2;4,2} = 2h_{1;4,2}$,

$h_{3;4,2} = 3h_{1;4,2}$,

$h_{1;5,2} = 2/(n+1)(n+2)$,

$h_{2;5,2} = 6(2n^2+10n+13)\, h_{1;5,2}/(2n+3)(2n+5)$,

$h_{3;5,2} = 3(8n^2+44n+63)\, h_{1;5,2}/(2n+3)(2n+5)$,

$h_{2;6,2} = -2h_{1;5,2}(2n+7)/(n+3)$,

$h_{3;6,2} = -2(64n^3+500n^2+1261n+1035)/r(n)$,

$h_{2;7,2} = h_{1;5,2}$,

$h_{3;7,2} = 6(5n+19)/(n+1)(n+2)(n+3)$,

$h_{3;8,2} = -6(2n+9)/(n+1)(n+2)(n+4)$,

$h_{3;9,2} = h_{1;5,2}$, $h_{2;4,3} = -6/(n+2)(n+3) = h_{3;4,3}/2$,

$h_{1;5,3} = -2/(n+2)(n+3)(2n+5)$,

$h_{2;5,3} = 8/(n+2)(n+3)$,

$h_{3;5,3} = 4(16n+43)/(n+2)(n+3)(2n+5)$,

$h_{2;6,3} = -9h_{1;5,3}$,

$h_{3;6,3} = 3(6n^2+26n+25)\, h_{1;5,3}/(n+3)$,

$h_{2;7,3} = -4/(n+2)(n+3),$

$h_{3;7,3} = 3(20n^2+144n+265)\ h_{1;5,3}/(2n+7),$

$h_{3;8,3} = 16(2n+9)/(n+2)(n+3)(n+4),$

$h_{3;9,3} = -6/(n+2)(n+3),$

$h_{1;3,4} = -2/(n+3)(n+4),$

$h_{2;5,4} = -4/(n+4)(2n+5) = h_{3;5,4}/6,$

$h_{2;6,4} = 4/(n+3)(n+4),$

$h_{3;6,4} = 3(7n+20)\ h_{2;6,4}/(2n+5),$

$h_{2;7,4} = 2/(n+3)(n+4)(2n+7),$

$h_{3;7,4} = -2(16n+59)\ h_{2;7,4},$

$h_{3;8,4} = -18h_{2;7,4},$

$h_{3;9,4} = 3h_{2;6,4}/2$

$h_{3;6,5} = -2/(n+3)(n+5),$

$h_{3;7,5} = 12/(n+5)(2n+7),$

$h_{3;8,5} = -6/(n+4)(n+5),$

$h_{3;9,5} = -2/(n+4)(n+5)(2n+9).$

6. Percentage Points of $V^{(3)}$

We give below some tables in order to check the expressions giving the exact c.d.f. For this purpose, approximate percentage points obtained by Pillai (1960), using the two moment quotients, are taken and used to compute on IBM 7094 the probability corresponding to the percentage points. The exact percentage points obtained are given in the last column. The exact percentage points are obtained corresponding to seven decimals accuracy in the probability.

Table 8. Approximate and Exact Percentage Points of $V^{(3)}$

n	Approximate Upper percentage points (Pillai's tables)	Probability	Exact Upper percentage point
		m=1	
		5%	
5	1.288	.95032	1.287223
10	0.892	.95008	0.8918447
15	0.682	.95006	0.6819051
20	0.552	.95012	0.5518451
25	0.464	.95053	0.4634005
		1%	
5	1.459	.99004	1.4585823
10	1.028	.98988	1.0289384
15	0.794	.98991	0.7945599
20	0.647	.98999	0.6470747
25	0.545	.98984	0.545735
		m=2	
		5%	
5	1.476	.94951	1.4771480
10	1.053	.94996	1.053082
15	0.818	.95026	0.8175817
20	0.668	.95001	0.66798817
25	0.565	.95025	0.5647057
		1%	
5	1.639	.98962	1.6423703
10	1.191	.99002	1.1908260
15	0.933	.98996	0.9332362
20	0.767	.98999	0.7670740
25	0.651	.98998	0.6510689
		m=3	
		5%	
10	1.190	.95008	
15	0.937	.95029	
		1%	
5	1.784	.99444	
10	1.326	.98999	
15	1.053	.98996	
20	0.873	.98997	
25	0.745	.99550	

CHAPTER IV

EXACT DISTRIBUTION OF $V^{(4)}$

1. Introduction and Summary

In this chapter details of deriving the c.d.f. of $V^{(p)}$, when $p=4$, will be discussed. From the work of the preceeding chapter, it should be obvious that the derivation for each value of p and even each value of m should be handled separately. The m.g.f. of $V^{(4)}$ involves integrals which did not occur before and, hence, the details of the method of derivation have to be presented. We discuss below, therefore, the steps in obtaining the c.d.f. of $V^{(4)}$ when $m=0$ and 1. The two instances here will illustrate the extent of additional work involved in passing from one value of m to the next higher.

2. Method of Deriving the Exact Distribution of $V^{(4)}$

Results of the last chapter give the m.g.f. of $V^{(4)}$ in the form

$$M(x,m,n,4) = e^{4x}\, x^{-(4n+10)} \sum_{i_0+\dots+i_4=4} \frac{m!}{i_0!\dots i_4!}\, x^{-\alpha} V(x;q_4',\dots,q_1';-1)$$

where $\alpha = \sum_{j=1}^{4} (q_j'-j+1)$ and $V(x;q_4',\dots,q_1';-1)$ is a Vandermonde determinant defined earlier and $q_j'(j=1,\dots,4)$ can take different values from term to term. By applying Pillai's lemma (1956), this determinant is reduced in terms of simple integrals. Finally, the m.g.f. is obtained, after integrating by parts several times, as a sum of terms each of which is a product of integrals of the type $I(n+\ell,x)$, $I(2n+k,2x)$, $I(2n+k,x)\cdot I(n+\ell,x)$ and $I(2n+\ell,2x)\,I(2n+k,2x)$. In order to obtain the c.d.f. of $V^{(4)}$ it is necessary to know the contribution of each of the four types of integral terms occurring in the m.g.f. Of these, the first three can be, immediately, written down

$$(2.2)\qquad I(n+\ell,x) \rightarrow \begin{cases} \dfrac{1-(1-z)^{n+\ell+1}}{n+\ell+1} & 0 \le z \le 1 \\[2ex] \dfrac{1}{n+\ell+1} & 1 \le z \le 4, \end{cases}$$

$$(2.3)\qquad I(2n+k,2x) \rightarrow \begin{cases} \dfrac{1-(1-z/2)^{2n+k+1}}{2n+k+1} & 0 \le z \le 2 \\[2ex] \dfrac{1}{2n+k+1} & 2 \le z \le 4, \end{cases}$$

$$(2.4)\qquad I(n+\ell,x)I(2n+k,2x) \rightarrow \begin{cases} \dfrac{1-(1-z/2)^{2n+k+1}}{(n+\ell+1)(2n+k+1)} - sJ(z_1,z_2;2n+k+1,n+\ell+2) & 0\le z\le 1 \\[2ex] \dfrac{1-(1-z/2)^{2n+k+1}}{(n+\ell+1)(2n+k+1)} - sJ(z_1,1;2n+k+1,n+\ell+2) & 1\le z\le 2 \\[2ex] \dfrac{1}{(n+\ell+1)(2n+k+1)} - sJ(0,1;2n+k+1,n+\ell+2) & 2\le z\le 3 \\[2ex] \dfrac{1}{(n+\ell+1)(2n+k+1)} & 3\le z\le 4, \end{cases}$$

where, as before,

$$z_1 = (2-z)/(3-z),\ z_2 = 2/(3-z),\ s = 2^{n+\ell+1}/z_2^{(3n+k+\ell+2)}(n+\ell+1),$$

$$I(q,\alpha x) = \int_0^1 e^{\alpha ux}(1-u)^q\, du$$

and

$$J(a,b;c,d) = \int_a^b x^{c-1}(1-x)^{d-1}\, dx.$$

However, to obtain the contribution $F(k,\ell)$ of $I(2n+\ell,2x)I(2n+k,2x)$, proceed as follows:

Let y_1 and y_2 be two r.v. distributed in $[0,1]$ with densities $(1-y_1)^{2n+\ell}$ and $(1-y_2)^{2n+k}$ respectively, then

$$F(k,\ell) = \iint_S (1-y_1)^{2n+\ell}(1-y_2)^{2n+k}\, dy_1\, dy_2 \tag{2.5}$$

where $S = \{(y_1,y_2): 0 \le y_i \le 1,\ i=1,2;\ 2y_1 + 2y_2 \le z\}$. It is obvious that the value of $F(k,\ell)$ depends on that of z and let

$$F(k,\ell) = F_i(k,\ell) \qquad z \in S_i \tag{2.6}$$

where $S_i = [2(i-1),2i] \cap S$ and $i=1,2$.

Consider the unit square, OPQR, in Fig. 2 and let AB be the line $y_1 + y_2 = \frac{z}{2}$ $(0 \le z < 2)$ and CD the line $y_1 + y_2 = z/2$ $(2 \le z \le 4)$. Then $F_1(k,\ell)$ and $F_2(k,\ell)$ are obtained by integrating over the areas OAB and OPCDR respectively.

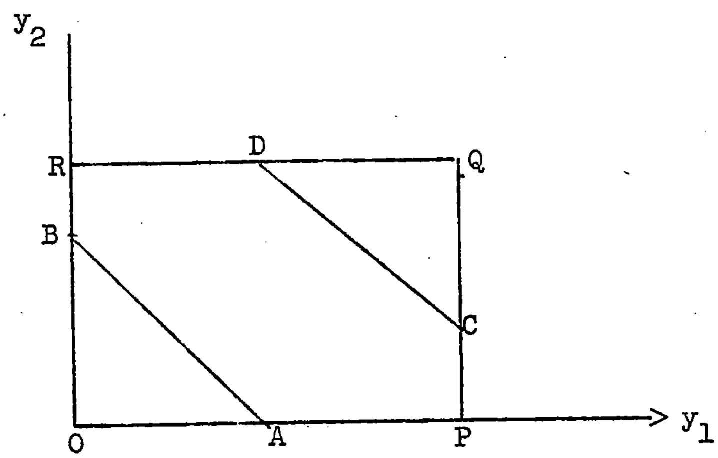

Figure 2. Region of Integration of $F(k,\ell)$

(2.7) $$F_1(k,\ell) = \int_0^{z/2} \int_0^{z/2-y_1} (1-y_1)^{2n+\ell}(1-y_2)^{2n+k}\, dy_1\, dy_2$$

$$= \int_0^{z/2} (1-y_1)^{2n+\ell} \frac{[1-(1-z/2+y_1)^{2n+k+1}]}{2n+k+1}\, dy_1$$

$$= \frac{1-(1-z/2)^{2n+\ell+1}}{(2n+\ell+1)(2n+k+1)} - \frac{J(z_3,z_4;2n+k+2,2n+\ell+1)}{(2n+k+1)z_4^{(4n+k+\ell+2)}}$$

and

(2.8) $$F_2(k,\ell) = \int_{z/2-1}^{1} \int_{z/2-y_1}^{1} (1-y_1)^{2n+\ell}\,(1-y_2)^{2n+k}\, dy_1\, dy_2$$

$$= \frac{1}{(2n+\ell+1)(2n+k+1)} - \frac{J(0,1;2n+k+2,2n+\ell+1)}{(2n+k+1)\, z_4^{(4n+k+\ell+2)}}$$

where $z_3=(2-z)/(4-z)$ and $z_4=2/(4-z)$. Therefore

(2.9) $$I(2n+\ell,2x)\, I(2n+k,2x) \rightarrow \begin{cases} F_1(k,\ell) & 0 \leq z \leq 2 \\ F_2(k,\ell) & 2 \leq z \leq 4 . \end{cases}$$

3. The Exact C.D.F. of $V^{(4)}$ when m = 0

The m.g.f. of $V^{(4)}$ for m=0 can be written in the form

$$(3.1)\qquad M(x,0,n,4) = e^{4x}\, x^{-(4n+10)}\, C(4,0,n)\, V(x;3,2,1,0;-1).$$

By going through the procedure described in the previous section, (3.1) reduces to the form

$$(3.2)\qquad M(x,0,n,4) = C(4,0,n)\Big[\frac{1}{(n+1)(n+2)^2(n+3)(2n+3)(2n+5)}\cdot \{-(4n+9)I(n+2,x)+I(n+1,x)\}$$
$$+\frac{4}{(n+1)(n+2)^2(2n+3)(2n+5)}\, I(2n+4,2x)$$
$$-\frac{4}{(n+1)(n+2)(2n+5)}\, I(2n+5,2x)$$
$$+\frac{2}{(n+1)(n+2)(2n+3)}\, \{2I(2n+6,2x)-I(n+1,x)\cdot I(2n+4,2x)\}$$
$$+\frac{8}{(n+1)(n+2)(2n+3)(2n+5)}\, I(2n+3,2x)\cdot I(2n+5,2x)$$
$$-\frac{2}{(n+1)(n+3)}\, I(n+1,x)\; I(2n+6,2x)$$
$$+\frac{4}{(n+1)(2n+5)}\, \{I(n+1,x)I(2n+5,2x)+I(n+2,x)\; I(2n+5,2x)\}$$
$$-\frac{2}{(n+2)^2}\, I(n+2,x)\; I(2n+4,2x)\Big].$$

Now, the c.d.f. is obtained by using relations (2.2), (2.3), (2.4) and (2.9) in (3.2), which gives

$$(3.3)\quad P[V^{(4)} \leq z]=1-C(4,0,n)\Big[\frac{-(4n+9)}{(n+1)(n+2)^2(n+3)^2(2n+3)(2n+5)}(1-z)^{n+3}$$

$$-\frac{1}{(n+1)(n+2)^2(n+3)(n+4)(2n+3)(2n+5)}(1-z)^{n+4}$$

$$-\frac{4}{(n+1)(n+3)(2n+5)^2}(1-z/2)^{2n+5}$$

$$+\frac{2(n^2+3n+3)}{(n+1)(n+2)^2(n+3)^2(2n+3)}(1-z/2)^{2n+6}$$

$$+\frac{2}{(n+1)(n+2)(n+3)(2n+3)(2n+7)}(1-z/2)^{2n+7}$$

$$+\frac{4}{(n+1)(n+2)^2(2n+3)(2n+5)}z_2^{-(4n+10)}$$

$$J(z_3,z_4;2n+5,2n+6)$$

$$-\frac{2^{n+3}}{(n+1)(n+2)(n+3)}z_2^{-(3n+9)}J(z_1,z_2;2n+7,n+3)$$

$$+\frac{2^{n+4}}{(n+1)(n+2)(2n+5)}z_2^{-(3n+8)}J(z_1,z_2;2n+6,n+3)$$

$$-\frac{2^{n+3}}{(n+1)(n+2)^2(2n+3)}z_2^{-(3n+7)}J(z_1,z_2;2n+5,n+3)$$

$$+\frac{2^{n+5}}{(n+1)(n+3)(2n+5)}z_2^{-(3n+9)}J(z_1,z_2;2n+6,n+4)$$

$$-\frac{2^{n+4}}{(n+2)^2(n+3)}z_2^{-(3n+8)}J(z_1,z_2;2n+5,n+4)\Big]$$

when $0 \leq z \leq 1$.

$$(3.4)\quad P(V^{(4)} \leq z) = 1 - C(4,0,n)\Big[\frac{-4}{(n+1)(n+3)(2n+5)^2}(1-z/2)^{2n+5}$$

$$+ \frac{2(n^2+3n+3)}{(n+1)(n+2)^2(n+3)^2(2n+3)}(1-z/2)^{2n+6}$$

$$+ \frac{2}{(n+1)(n+2)(n+3)(2n+3)(2n+7)}(1-z/2)^{2n+7}$$

$$+ \frac{4}{(n+1)(n+2)^2(2n+3)(2n+5)} z_4^{-(4n+10)} J(z_3,z_4;2n+5,2n+6)$$

$$- \frac{2^{n+3}}{(n+1)(n+2)(n+3)} z_2^{-(3n+9)} J(z_1,1;2n+7,n+3)$$

$$+ \frac{2^{n+4}}{(n+1)(n+2)(2n+5)} z_2^{-(3n+8)} J(z_1,1;2n+6,n+3)$$

$$- \frac{2^{n+3}}{(n+1)(n+2)^2(2n+3)} z_2^{-(3n+7)} J(z_1,1;2n+5,n+3)$$

$$+ \frac{2^{n+5}}{(n+1)(n+3)(2n+5)} z_2^{-(3n+9)} J(z_1,1;2n+6,n+4)$$

$$- \frac{2^{n+4}}{(n+2)^2(n+3)} z_2^{-(3n+8)} J(z_1,1;2n+5,n+4)\Big]$$

when $1 \leq z \leq 2$,

$$(3.5)\quad P[V^{(4)} \leq z] = 1 - C(4,0,n)\Big[\frac{4}{(n+1)(n+2)^2(2n+3)(2n+5)} z_4^{-(4n+10)} J(0,1;2n+5,2n+6)$$

$$- \frac{2^{n+3}}{(n+1)(n+2)(n+3)} z_2^{-(3n+9)} J(0,1;2n+7,n+3)$$

$$+ \frac{2^{n+4}}{(n+1)(n+2)(2n+5)} z_2^{-(3n+8)} J(0,1;2n+6,n+3)$$

$$- \frac{2^{n+3}}{(n+1)(n+2)^2(2n+3)} z_2^{-(3n+7)} J(0,1;2n+5,n+3)$$

$$+ \frac{2^{n+5}}{(n+1)(n+3)(2n+5)} z_2^{-(3n+9)} J(0,1;2n+6,n+4)$$

$$- \frac{2^{n+4}}{(n+2)^2(n+3)} z_2^{-(3n+8)} J(0,1;2n+5,n+4)]$$

when $2 \leq z \leq 3$ and

$$(3.6) \qquad P(V^{(4)} \leq z) = 1 - \frac{4C(4,0,n)}{(n+1)(n+2)^2(2n+3)(2n+5)} z_2^{-(4n+10)} J(0,1;2n+5,2n+6)$$

when $3 \leq z \leq 4$.

On observing (3.4), (3.5) and (3.6), it is seen that they can be obtained from (3.3) by making the following changes:

When $1 \leq z \leq 2$,

(i) Drop all terms involving $(1-z)$

(ii) Change z_2 to 1;

When $2 \leq z \leq 3$,

(i) Drop all terms involving powers of $(1-z)$,

(ii) Drop all terms involving powers of $(1-z/2)$,

(iii) Replace z_1 and z_2 by 0 and 1 respectively,

(iv) Replace z_3 and z_4 by 0 and 1 respectively.

When $3 \leq z \leq 4$,

(i) Drop all terms involving z except those with $J(z_3,z_4;2n+\ell+2,2n+k+2)$,

(ii) Replace z_3 and z_4 by 0 and 1 respectively.

4. The Exact C.D.F. of $V^{(4)}$ when m = 1

In this case, the m.g.f. of $V^{(4)}$ can be written as

$$(4.1)\quad M(x,1,n,4)=C(4,m,n)e^{4x}x^{-(4n+10)}[V(x;3,2,1,0;-1)-x^{-1}V(x;4,2,1,0;-1)$$
$$+x^{-2}V(x;4,3,1,0;-1)-x^{-3}V(x;4,3,2,0;-1)$$
$$+x^{-4}V(x;4,3,2,1;-1)].$$

On simplifying (4.1) and using relations (2.2) to (2.4) and (2.9), the c.d.f. of $V^{(4)}$ can be shown to be

$$(4.2)\quad P[V^{(4)} \le z]=1-C(4,1,n)\ x$$
$$[-\frac{(4n^4+38n^3+136n^2+213n+114)}{(n+4)t(n)}(1-z)^{n+3}$$
$$-\frac{2(8n^6+124n^5+790n^4+2647n^3+4923n^2+4816n+1917)}{(n+4)(2n+7)t(n)}(1-z)^{n+4}$$
$$+\frac{6}{(n+5)(2n+7)t(n)}(1-z)^{n+5}+\frac{2(10n^2+37n+31)(n+4)}{(n+2)(2n+5)s(n)}(1-z/2)^{2n+5}$$
$$+\frac{(16n^5+200n^4+984n^3+2374n^2+2837n+1374)}{(n+2)(n+3)(2n+7)\ s(n)}(1-z/2)^{2n+6}$$
$$-\frac{4(4n^5+56n^4+325n^3+959n^2+1426n+860)}{(n+2)(2n+7)^2s(n)}(1-z/2)^{2n+7}$$
$$+\frac{4(4n^4+40n^3+145n^2+225n+131)}{(n+4)(2n+7)\ s(n)}(1-z/2)^{2n+8}+\frac{6}{(2n+9)s(n)}(1-z/2)^{2n+9}$$
$$+z_2^{-(3n+8)}2^{n+3}\{\frac{2(2n^2+5n+4)}{(n+1)(n+2)^2(2n+3)(2n+5)}J(z_1,z_2;2n+6,n+3)$$
$$-\frac{1}{(n+1)(n+2)(n+3)}z_2^{-1}J(z_1,z_2;2n+7,n+3)$$

$$- \frac{1}{(n+1)(n+2)(n+4)} z_2^{-3} J(z_1,z_2;2n+9,n+3)$$

$$- \frac{2}{(n+2)^2(n+3)} J(z_1,z_2;2n+5,n+4)$$

$$+ \frac{2(2n^2+9n+6)}{(n+1)(n+2)^2(n+3)(2n+5)} z_2^{-1} J(z_1,z_2;2n+6,n+4)$$

$$- \frac{2(2n+7)}{(n+2)(n+3)^2(2n+5)} z_2^{-2} J(z_1,z_2;2n+7,n+4)$$

$$+ \frac{2(2n+3)}{(n+1)(n+2)(n+3)(2n+7)} z_2^{-3} J(z_1,z_2;2n+8,n+4)$$

$$- \frac{2}{(n+2)(n+3)(n+4)} z_2^{-4} J(z_1,z_2;2n+9,n+4)$$

$$+4 \frac{(2n^2+9n+13)}{(n+1)\ldots(n+4)(2n+5)} z_2^{-2} J(z_1,z_2;2n+6,n+5)$$

$$+ \frac{8}{(n+1)(n+3)^2(n+4)} z_2^{-3} J(z_1,z_2;2n+7,n+5)$$

$$+ \frac{4(2n^2+9n+8)}{(n+1)\ldots(n+4)(2n+7)} z_2^{-4} J(z_1,z_2;2n+8,n+5)\}$$

$$+ z_4^{-(4n+12)}\{\frac{1}{(n+1)(n+2)^2(2n+3)(2n+5)} J(z_3,z_4;2n+5,2n+6)$$

$$+ \frac{n}{(n+1)(n+2)(n+3)} z_4^{-1} J(z_3,z_4;2n+5,2n+7)$$

$$+ \frac{1}{(n+1)(n+2)(2n+3)} z_4^{-2} J(z_3,z_4;2n+5,2n+8)$$

$$- \frac{(n^2+2n+2)}{(n+1)(n+2)^2(n+3)(2n+3)} z_4^{-1} J(z_3,z_4;2n+7,2n+5)$$

$$- \frac{(n+5)}{(n+1)(n+2)(n+3)(2n+5)} z_4^{-2} J(z_3,z_4;2n+6,2n+7)$$

$$- \frac{4}{(n+1)(n+2)(n+3)(2n+5)(2n+7)} z_4^{-3} J(z_3,z_4;2n+6,2n+8)$$

$$+ \frac{1}{(n+2)(n+3)(n+4)(2n+5)(2n+7)} z_4^{-4} J(z_3,z_4;2n+9,2n+6)\}]$$

when $0 \leq z \leq 1$,

where $s(n)=(n+1)(n+2)(n+3)(n+4)(2n+3)(2n+5)$ and $t(n)=(n+2)(n+3)s(n)$. For getting the c.d.f. of $V^{(4)}$ when $z \geq 1$, make changes in (4.2) similar to those made in (3.3) for getting (3.4) to (3.6). (See the end of previous section).

5. Percentages Points of $V^{(4)}$

In order to check the expressions giving the exact c.d.f. of $V^{(4)}$, we give below the table of exact and approximate percentage points.

Table 9. Approximate and Exact Percentage Points of $V^{(4)}$

n	Approximate Upper percentage points (Pillai's tables)	Probability	Exact Upper percentage point
		m=0	
		5%	
5	1.411	.95048	1.4097568
10	0.974	.94999	0.9740115
15	0.744	.95008	0.7438563
20	0.602	.95028	0.6016025
25	0.505	.95001	0.5049877
		1%	
5	1.594	.99009	1.5930536
10	1.118	.98994	1.1184651
15	0.861	.98986	0.8619074
20	0.701	.98998	0.7011134
25	0.590	.98982	0.5908809
		m=1	
		5%	
5	1.693	.94983	1.6934337
10	1.203	.95022	1.2025320
15	0.932	.94992	0.9321201
20	0.761	.94934	0.7622401
25	0.643	.94415	

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