Two Queues in Series with a finite, intermediate Waitingroom*

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Abstract

A service unit I, with Poisson input and general service times is in series with a unit II, with negative-exponential service times. The intermediate waitingroom can accomodate at most k persons and a customer cannot leave unit I when the waitingroom is full.

The paper shows that this system of queues can be studied in terms of an imbedded semi-Markov process. Equations for the time dependent distributions are given, but the main emphasis of the paper is on the equilibrium conditions and on asymptotic results.

1. Description of the model.

The system of queues, discussed in this paper, consists of two units. Customers arrive at a first unit (I) according to a homogeneous Poisson process of rate λ. Their service times in unit I are independent, identically distributed random variables with common distribution function $H(\cdot)$. We assume that $H(\cdot)$ has a positive, finite mean α and we will denote the Laplace-Stieltjes transform of $H(\cdot)$ by $h(s)$, $\text{Re } s \geq 0$.

Upon completion of service in unit I, all customers go on to a second unit(II) via a finite waitingroom. We assume that there can be not more than k customers in unit II and in the waitingroom at any time. If upon

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completion of service in unit I a customer finds the waitingroom full, then the unit I <u>blocks</u> until a service in unit II is completed. At that time he is allowed to enter the waitingroom.

We assume that the service times in unit II are independent, identically distributed random variables with a negative-exponential distribution with mean $1/\sigma$. The service times in unit II are also stochastically independent of those in unit I and of the arrival process.

The case $k = 1$, i.e. when no customers can wait between the two units was studied by B. Avi-Itzhak and M. Yadin [1], T. Suzuki [11] and U. N. Prabhu [7]. These authors allow the service times in the second unit to have a general distribution. In the case of a general k , we impose the requirement that these service times are negative-exponentially distributed. so as to preserve the semi-Markov structure of the process.

The techniques used in this paper can easily be adapted to discuss the following alternate models.

Variant 1.

We may assume that there are several servers in unit II and a common waitingroom. Provided all servers have the same negative-exponential service-time distributions, the model will still have an imbedded semi-Markov process.

Variant 2.

If the service time in unit II has an Erlang distribution, the classical method of phases may be employed and the system can again be analyzed in terms of an imbedded semi-Markov process, analogous to that defined below.

Though each of these variants appears to have some practical interest, their detailed discussion leads to expressions which are more complicated still than those given below. We will leave them to the initiative of the more courageous reader.

A practical situation in which our model occurs, is in a vehicle inspection station. The first unit may consist of a clerical worker, who checks such things as registration, license plates etc., whereas the second unit performs an inspection of the technical features of the vehicle. The author thanks Prof. M. Yadin, who communicated this example to him.

2. The imbedded semi-Markov process.

Let the successive completions of service in unit I occur in the instants T_n, $n \geq o$. We assume that $T_o = o$, so that the origin of time is an instant at which a service completion in unit I occurs.

By ξ_n, we denote the number of customers in the system, who have not yet completed service in unit I at the time $T_n + o$. By ζ_n we denote the number of customers in the system who at $T_n + o$ have completed service in unit I, but not yet in unit II. Clearly $1 \leq \zeta_n \leq k + 1$. If at $T_n + o$, we have $\zeta_n \leq k$, then T_n is a departure point from unit I, whereas if $\zeta_n = k + 1$, the customer who finishes service at T_n in unit I finds the waitingroom full. He does not free the server I at time T_n, but only later, when the next departure from unit II occurs. Between T_n and the time a departure from unit II occurs, the server I is inoperative. It is clear that the duration of the interval in which unit I is blocked has a negative-exponential distribution with mean $\frac{1}{\sigma}$.

Between successive epochs T_n and T_{n+1}, the queuelength processes have a simple behavior. The number of customers in unit I behaves like a simple birth process and can only increase, whereas the number of customers in unit II behaves like a simple death process and can only decrease.

It follows from the independence and Poisson assumptions which we have made, that the sequence of triples:

$$(1) \qquad \left\{\xi_n , \zeta_n , T_{n+1} - T_n ; n \geq o\right\}$$

form a <u>semi-Markov sequence</u> with state space $\{0,1,\ldots\} \times \{1,2,\ldots,k+1\}$. A general discussion of semi-Markov sequences may be found in Pyke [8,9]. For their use in the analysis of various queueing models we refer to Neuts [3,4,5,6].

The timedependence and the equilibrium conditions of the model may be studied completely in terms of the semi-Markov sequence (1). It suffices to know the initial distribution of (ξ_o , ζ_o) and the transition probability matrix $Q(x)$, whose entries are defined by:

$$(2) \qquad Q(i,r;\ j,v;\ x) = P\{\xi_{n+1} = j,\ \zeta_{n+1} = v,\ T_{n+1} - T_n \leq x \mid \xi_n = i,\ \zeta_n = r\}$$

for $i \geq o$, $j \geq o$, $1 \leq r$, $v \leq k+1$, $x \geq o$.

We denote by $q(i,r;\ j,v;\ s)$ the Laplace-Stieltjes transform of $Q(i,r;\ j,v;\ x)$.

The explicit expressions for the probabilities (2) are fairly complicated. We list them below.

<u>Case 1</u>: $j < i - 1$ or $v > r + 1$,

$$(3) \qquad Q(i,r;\ j,v;\ x) = o$$

<u>Case 2</u>: $i = o,\ r \leq k,\ 1 < v \leq r + 1,$

(4) $Q(o,r;\ j,v;\ x) =$

$$\int_o^x \int_o^y e^{-(\lambda+\sigma)y} \frac{(\lambda y-\lambda u)^j}{j\,!} \frac{(\sigma y)^{r-v+1}}{(r-v+1)!} \lambda \, du \, d\, H(y-u) ,$$

(5) $q(o,r;\ j,v;\ s) =$

$$\int_0^\infty \int_0^\infty \lambda\, e^{-(s+\lambda+\sigma)(u+t)} \frac{(\lambda t)^j}{j!} \frac{(\sigma u+\sigma t)^{r-v+1}}{(r-v+1)!}\, du\, d\, H(t)\ ,$$

Case 3: $i = o,\ r \leq k,\ v = 1,\ j \geq o$,

(6) $Q(o,r;\ j,1;\ x) =$

$$\sum_{\nu=r}^\infty \int_0^x \int_0^y e^{-(\lambda+\sigma)y} \frac{(\lambda y-\lambda u)^j}{j!} \frac{(\sigma y)^\nu}{\nu!}\, \lambda\, du\, d\, H(y-u)\ ,$$

(7) $q(o,r;\ j,1;\ s) =$

$$\sum_{\nu=r}^\infty \int_0^\infty \int_0^\infty \lambda\, e^{-(s+\lambda+\sigma)(u+t)} \frac{(\lambda t)^j}{j!} \frac{(\sigma u+\sigma t)^\nu}{\nu!}\, du\, d\, H(t),$$

Case 4: $i > o,\ r \leq k,\ 1 < v \leq r+1.\ j \geq i-1$,

(8) $Q(i,r;\ j,v;\ x) =$

$$\int_0^x e^{-(\lambda+\sigma)y} \frac{(\lambda y)^{j-i+1}}{(j-i+1)!} \frac{(\sigma y)^{r-v+1}}{(r-v+1)!}\, d\, h(y)\ ,$$

(9) $q(i,r;\ j,v;\ s) =$

$$\int_0^\infty e^{-(s+\lambda+\sigma)y} \frac{(\lambda y)^{j-i+1}}{(j-i+1)!} \frac{(\sigma y)^{r-v+1}}{(r-v+1)!}\, d\, H(y)\ ,$$

Case 5: $i > o,\ r \leq k,\ v = 1,\ j \geq i-1$,

(10) $Q(i,r;\ j,1;\ x) =$

$$\sum_{\nu=r}^\infty \int_0^x e^{-(\lambda+\sigma)y} \frac{(\lambda y)^{j-i+1}}{(j-i+1)!} \frac{(\sigma y)^\nu}{\nu!}\, d\, H(y)\ ,$$

(11) q(i,r; j,l; s) =

$$\sum_{\nu=r}^{\infty} \int_{0}^{\infty} e^{-(s+\lambda+\sigma)y} \frac{(\lambda y)^{j-i+1}}{j-i+1)!} \frac{(\sigma y)^{\nu}}{\nu!} \, d\, H(y) \, ,$$

Case 6: $r = k+1, \; i = o, \; 1 < v \leq k+1,$

(12) Q(o,k+1; j,v; x) =

$$\int_{0}^{x}\int_{0}^{y}\int_{0}^{u_1} e^{-\lambda u_1} \lambda \, du_1 \; e^{-\sigma u} \sigma \, du \; e^{-\sigma(y-u)} \frac{(\sigma y-\sigma u)^{k-v+1}}{(k-v+1)!}$$

$$e^{-\lambda(y-u_1)} \frac{(\lambda y-\lambda u_1)^{j}}{j!} \, d\, H(y-u_1)$$

$$+ \int_{0}^{x}\int_{0}^{y}\int_{0}^{u} e^{-\lambda u_1} \lambda \, du_1 \; e^{-\sigma u} \sigma \, du \; e^{-\sigma(y-u)} \frac{(\sigma y-\sigma u)^{k-v+1}}{(k-v+1)!}$$

$$e^{-\lambda(y-u_1)} \frac{(\lambda y-\lambda u_1)^{j}}{j!} \, d \; H(y-u)$$

The first integral corresponds to the case, where a customer leaves unit II before a customer arrives in unit I and the second integral corresponds to the case, where a customer arrives in unit I before a departure from unit II occurs.

Upon taking transforms, we obtain after routine calculations that:

(13) q(o,k+1; j,v; s) =

$$\frac{\lambda\sigma}{s+\lambda+\sigma} \left\{ \int_{0}^{\infty}\int_{0}^{\infty} e^{-(\xi+\zeta)(s+\lambda+\sigma)} \frac{(\lambda\xi)^{j}}{j!} \frac{(\sigma\xi+\sigma\zeta)^{k-v+1}}{(k-v+1)!} \, d\zeta \, d\, H(\xi) \right.$$

$$\left. + \int_{0}^{\infty}\int_{0}^{\infty} e^{-(\xi+\zeta)(s+\lambda+\sigma)} \frac{(\lambda\xi+\lambda\zeta)^{j}}{j!} \frac{(\sigma\zeta)^{k-v+1}}{(k-v+1)!} \, d\, \xi \, d\, H(\zeta) \right\}$$

Case 7: $r = k+1,\ i = o,\ v = 1,$

In a manner analogous to case 6, we obtain:

(14) $q(o,k+1;\ j,1;\ s) =$

$$\frac{\lambda\sigma}{s+\lambda+\sigma} \sum_{\nu=k}^{\infty} \Big\{ \int_0^{\infty}\int_0^{\infty} e^{-(\xi+\zeta)(s+\lambda+\sigma)} \frac{(\lambda\xi)^j}{j!} \frac{(\sigma\xi+\sigma\zeta)^{\nu}}{\nu!}\, d\zeta\, d\,H(\xi)$$

$$+ \int_0^{\infty}\int_0^{\infty} e^{-(\xi+\zeta)(s+\lambda+\sigma)} \frac{(\lambda\xi+\lambda\zeta)^j}{j!} \frac{(\sigma\zeta)^{\nu}}{\nu!}\, d\xi\, d\,H(\zeta)\Big\} .$$

Case 8: $r = k+1,\ i > o,\ 1 < v \leq k+1,\ j \geq i-1,$

(15) $Q(i,k+1;\ j,v;\ x) =$

$$\int_0^x\int_0^y \sigma e^{-\sigma u}\, du\, e^{-\lambda y} \frac{(\lambda y)^{j-i+1}}{(j-i+1)!} \frac{(\sigma y-\sigma u)^{k-v+1}}{(k-v+1)!} e^{-\sigma(y-u)} d\,H(y-u)$$

(16) $q(i,k+1;\ j,v;\ s) =$

$$\int_0^{\infty}\int_0^{\infty} e^{-(u+\xi)(s+\lambda+\sigma)} \frac{(\lambda u+\lambda\xi)^{j-i+1}}{(j-i+1)!} \frac{(\sigma\xi)^{k-v+1}}{(k-v+1)!}\, \sigma\, du\, d\,H(\xi) ,$$

Case 9: $r = k+1,\ i > o,\ v = 1,\ j \geq i-1,$

(17) $Q(i,k+1;\ j,1;\ x) =$

$$\sum_{\nu=k}^{\infty} \int_0^x\int_0^y \sigma e^{-\sigma u}\, du\, e^{-\lambda y} \frac{(\lambda y)^{j-i+1}}{(j-i+1)!} \frac{(\sigma y-\sigma u)^{\nu}}{\nu!} e^{-\sigma(y-u)}\, d\,H(y-u) ,$$

(18) $q(i,k+1;\ j,1;\ s) =$

$$\sum_{\nu=k}^{\infty} \int_0^{\infty}\int_0^{\infty} e^{-(u+\xi)(s+\lambda+\sigma)} \frac{(\lambda u+\lambda\xi)^{j-i+1}}{(j-i+1)!} \frac{(\sigma\xi)^{\nu}}{\nu!}\, \sigma\, du\, d\,H(\xi) .$$

3. The Renewal functions of the imbedded semi-Markov process.

We define the powers of the transition matrix Q as follows:

(19) $$Q^{(o)}(i,r;\ j,v;\ x) = \delta_{ij}\ \delta_{rv}\ U_o(x)\ ,$$

where $U_o(\cdot)$ is the distribution function which is degenerate at zero.

(20) $$Q^{(n+1)}(i,r;\ j,v;\ x) = \sum_{\rho=o}^{\infty} \sum_{\nu=1}^{k+1} Q^{(n)}(i,r;\ \rho,\nu;\ x) * Q(\rho,\nu;\ j,v;\ x)\ ,$$

for $n \geq o$.

The renewal functions $M(i,r;\ j,v;\ x)$ of the imbedded semi-Markov process are defined by:

(21) $$M(i,r;\ j,v;\ x) = \sum_{n=o}^{\infty} Q^{(n)}\ (i,r;\ j,v;\ x)$$

$i \geq o,\ j \geq o,\ 1 \leq r,\ v \leq k+1,\ x \geq o$.

The function $M(i,r;\ j,v;\ x)$ expresses the expected number of visits to the state (j,v) during the interval $[o,x]$. The visit to (i,r) at $t = o$ is included in the count.

Analytic expressions for the renewal functions may be obtained in principle as follows. The equations (19) - (21) yield in terms of transforms that:

(22) $$q^{(o)}\ (i,r;\ j,v;\ s) = \delta_{ij}\ \delta_{rv}\ ,$$

and for $n \geq o$.

$$(23)\quad q^{(n+1)}(i,r;\ j,v;\ s) =$$

$$\sum_{\rho=o}^{\infty}\sum_{\nu=1}^{k+1} q^{(n)}(i,r;\ \rho,\nu;\ s)\ q(\rho,\nu;\ j,v;\ s)$$

We define the generating functions:

$$(24)\quad U_v^{(n)}(z,s) = \sum_{j=o}^{\infty} q^{(n)}(i,r;\ j,v;\ s)\ z^j\ ,$$

$$(25)\quad W_v(z,w,s) = \sum_{n=o}^{\infty} U_v^{(n)}(z,s)\ w^n\ ,$$

for $1 \leq v \leq k+1$, $|z| \leq 1$, Re $s > o$, $|w| \leq 1$ or Re $s \geq o$, $|w| < 1$.

Note that these generating functions depend on the initial conditions (i,r).

The series:

$$(26)\quad \sum_{j=o}^{\infty} q(\rho,\nu;\ j,v;\ s)\ z^j$$

sums to one of the following expressions:

(a) $\rho = o$, $v > 1$, $v-1 \leq \nu \leq k$,

$$\int_o^{\infty} \lambda\ e^{-(s+\lambda+\sigma)u}\ du \int_o^{\infty} e^{-(s+\lambda+\sigma-\lambda z)t}\ \frac{(\sigma u+\sigma t)^{\nu-v+1}}{(\nu-v+1)!}\ d\,H(t)\ ,$$

(b) $\rho = o$, $\nu \leq k$, $v = 1$,

$$\sum_{\alpha=\nu}^{\infty} \int_o^{\infty} \lambda\ e^{-(s+\lambda+\sigma)u}\ du \int_o^{\infty} e^{-(s+\lambda+\sigma-\lambda z)t}\ \frac{(\sigma u+\sigma t)^{\alpha}}{\alpha!}\ d\,H(t)\ ,$$

(c) $\rho > o,\ \nu \le k,\ 1 < v \le \nu + 1,$

$$z^{\rho-1} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{\nu-v+1}}{(\nu-v+1)!} \, d\, H(y)\ ,$$

(d) $\rho > o,\ \nu \le k,\ v = 1,$

$$z^{\rho-1} \sum_{\alpha=\nu}^{\infty} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{\alpha}}{\alpha!} \, d\, H(y)\ ,$$

(e) $\rho = o,\ \nu = k+1,\ 1 < v \le k+1,$

$$\frac{\sigma}{s+\lambda+\sigma} \int_0^\infty \int_0^\infty e^{-(\xi+\zeta)(s+\lambda+\sigma)+\lambda\xi z} \frac{(\sigma\xi+\sigma\zeta)^{k-v+1}}{(k-v+1)!} \lambda \, d\, \zeta \, d\, H(\xi)$$

$$+ \frac{\lambda}{s+\lambda+\sigma} \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)\zeta} \frac{(\sigma\zeta)^{k-v+1}}{(k-v+1)!} \, d\, H(\zeta)\ ,$$

(f) $\rho = o,\ \nu = k+1,\ v = 1,$

$$\frac{\sigma}{s+\lambda+\sigma} \int_0^\infty \int_0^\infty e^{-(\xi+\zeta)(s+\lambda+\sigma)+\lambda\xi z} \sum_{\alpha=k}^{\infty} \frac{(\sigma\xi+\sigma\zeta)^{\alpha}}{\alpha!} \lambda \, d\, \zeta \, d\, H(\xi)$$

$$+ \frac{\lambda}{s+\lambda+\sigma} \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)\zeta} \sum_{\alpha=k}^{\infty} \frac{(\sigma\zeta)^{\alpha}}{\alpha!} \, d\, H(\zeta)\ ,$$

(g) $\rho > o,\ \nu = k+1,\ 1 < v \le k+1,$

$$z^{\rho-1} \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_0^\infty e^{-\xi(s+\lambda+\sigma-\lambda z)} \frac{(\sigma\xi)^{k-v+1}}{(k-v+1)!} \, d\, H(\xi)\ ,$$

(h) $\rho > o,\ \nu = k+1,\ v = 1,$

$$z^{\rho-1} \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_0^\infty e^{-\xi(s+\lambda+\sigma-\lambda z)} \sum_{\alpha=k}^{\infty} \frac{(\sigma\xi)^{\alpha}}{\alpha!} \, d\, H(\xi),$$

Substituting in formula (24), we obtain:

(27) $$U_v^{(o)}(z,s) = \delta_{r,v}\, z^i , \quad 1 \le v \le k+1,$$

and for $n \ge o$, $1 \le v \le k+1$:

(28) $$U_v^{(n+1)}(z,s) = \sum_{\rho=o}^{\infty} \sum_{\nu=1}^{k+1} q^{(n)}(i,r;\ \rho,\nu;\ s) \sum_{j=o}^{\infty} q(\rho,\nu;\ j,v;\ s)\, z^j ,$$

which leads to:

(29) $$zU_1^{(n+1)}(z,s) =$$

$$z \sum_{\nu=1}^{k+1} q^{(n)}(i,r;\ o,\nu;\ s) \sum_{j=o}^{\infty} q(o,\nu;\ j,1;\ s)\, z^j$$

$$+ \sum_{\nu=1}^{k+1} [U_\nu^{(n)}(z,s) - q^{(n)}(i,r;\ o,\nu;\ s)] \int_o^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=\nu}^{\infty} \frac{(\sigma y)^\alpha}{\alpha!}\, d\, H(y)$$

$$+ [U_{k+1}^{(n)}(z,s) - q^{(n)}(i,r;\ o,k+1;\ s)] \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_o^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=k}^{\infty} \frac{(\sigma y)^\alpha}{\alpha!}\, d\, H(y) ;$$

and for $1 < v \le k+1$:

(30) $$zU_v^{(n+1)}(z,s) =$$

$$z \sum_{\nu=v-1}^{k+1} q^{(n)}(i,r;\ o,\nu;s) \sum_{j=o}^{\infty} q(o,\nu;\ j,v;\ s)\, z^j$$

$$+ \sum_{\nu=v-1}^{k} [U_\nu^{(n)}(z,s) - q^{(n)}(i,r;\ o,\nu;\ s] \int_o^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{\nu-v+1}}{(\nu-v+1)!}\, d\, H(y)$$

$$+ [U_{k+1}^{(n)}(z,s) - q^{(n)}(i,r;\ o,k+1;\ s)] \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_o^\infty e^{-\xi(s+\lambda+\sigma-\lambda z)} \frac{(\sigma\xi)^{k-v+1}}{(k-v+1)!}\, d\, H(\xi) ,$$

Substituting in formula (24), we obtain:

(27) $$U_v^{(o)}(z,s) = \delta_{r,v}\, z^i , \quad 1 \le v \le k+1,$$

and for $n \ge o$, $1 \le v \le k+1$:

(28) $$U_v^{(n+1)}(z,s) = \sum_{\rho=o}^{\infty} \sum_{\nu=1}^{k+1} q^{(n)}(i,r;\ \rho,\nu;\ s) \sum_{j=o}^{\infty} q(\rho,\nu;\ j,v;\ s)\, z^j ,$$

which leads to:

(29) $$zU_1^{(n+1)}(z,s) =$$

$$z \sum_{\nu=1}^{k+1} q^{(n)}(i,r;\ o,\nu;\ s) \sum_{j=o}^{\infty} q(o,\nu;\ j,1;\ s)\, z^j$$

$$+ \sum_{\nu=1}^{k+1} [U_\nu^{(n)}(z,s) - q^{(n)}(i,r;\ o,\nu;\ s)] \int_o^{\infty} e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=\nu}^{\infty} \frac{(\sigma y)^{\alpha}}{\alpha!}\, d\, H(y)$$

$$+ [U_{k+1}^{(n)}(z,s) - q^{(n)}(i,r;\ o,k+1;\ s)] \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_o^{\infty} e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=k}^{\infty} \frac{(\sigma y)^{\alpha}}{\alpha!}\, d\, H(y) ,$$

and for $1 < v \le k+1$:

(30) $$zU_v^{(n+1)}(z,s) =$$

$$z \sum_{\nu=v-1}^{k+1} q^{(n)}(i,r;\ o,\nu;s) \sum_{j=o}^{\infty} q(o,\nu;\ j,v;\ s)\, z^j$$

$$+ \sum_{\nu=v-1}^{k} [U_\nu^{(n)}(z,s) - q^{(n)}(i,r;\ o,\nu;\ s)] \int_o^{\infty} e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{\nu-v+1}}{(\nu-v+1)!}\, d\, H(y)$$

$$+ [U_{k+1}^{(n)}(z,s) - q^{(n)}(i,r;\ o,k+1;\ s)] \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_o^{\infty} e^{-\xi(s+\lambda+\sigma-\lambda z)} \frac{(\sigma\xi)^{k-v+1}}{(k-v+1)!}\, d\, H(\xi) ,$$

Substituting in formula (24), we obtain:

(27) $$U_v^{(o)}(z,s) = \delta_{r,v}\, z^i , \quad 1 \le v \le k+1,$$

and for $n \ge o$, $1 \le v \le k+1$:

(28) $$U_v^{(n+1)}(z,s) = \sum_{\rho=o}^{\infty}\sum_{\nu=1}^{k+1} q^{(n)}(i,r;\ \rho,\nu;\ s) \sum_{j=o}^{\infty} q(\rho,\nu;\ j,v;\ s)\, z^j ,$$

which leads to:

(29) $$zU_1^{(n+1)}(z,s) =$$

$$z\sum_{\nu=1}^{k+1} q^{(n)}(i,r;\ o,\nu;\ s) \sum_{j=o}^{\infty} q(o,\nu;\ j,1;\ s)\, z^j$$

$$+\sum_{\nu=1}^{k+1} [U_\nu^{(n)}(z,s)-q^{(n)}(i,r;\ o,\nu;\ s)] \int_o^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=\nu}^{\infty} \frac{(\sigma y)^\alpha}{\alpha!}\, d\, H(y)$$

$$+ [U_{k+1}^{(n)}(z,s)-q^{(n)}(i,r;\ o,k+1;\ s)] \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_o^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=k}^{\infty} \frac{(\sigma y)^\alpha}{\alpha!}\, d\, H(y) ,$$

and for $1 < v \le k+1$:

(30) $$zU_v^{(n+1)}(z,s) =$$

$$z\sum_{\nu=v-1}^{k+1} q^{(n)}(i,r;\ o,\nu;s) \sum_{j=o}^{\infty} q(o,\nu;\ j,v;\ s)\, z^j$$

$$+\sum_{\nu=v-1}^{k} [U_\nu^{(n)}(z,s)-q^{(n)}(i,r;\ o,\nu;\ s] \int_o^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{\nu-v+1}}{(\nu-v+1)!}\, d\, H(y)$$

$$+ [U_{k+1}^{(n)}(z,s)-q^{(n)}(i,r;\ o,k+1;\ s)] \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_o^\infty e^{-\xi(s+\lambda+\sigma-\lambda z)} \frac{(\sigma\xi)^{k-v+1}}{(k-v+1)!}\, d\, H(\xi) ,$$

Finally, substituting (29) and (30) in (25), we get:

(31) $z\, W_1(z,w,s) =$

$$\delta_{r,1}\, z^{i+1} + w \sum_{\nu=1}^{k} W_\nu(z,w,s) \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=\nu}^{\infty} \frac{(\sigma y)^\alpha}{\alpha!}\, d\, H(y)$$

$$+ w\, W_{k+1}(z,w,s) \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)\xi} \sum_{\alpha=k}^{\infty} \frac{(\sigma \xi)^\alpha}{\alpha!}\, d\, H(\xi)$$

$$+ w \sum_{\nu=1}^{k} W_\nu(o,w,s) \left\{ z \sum_{j=o}^{\infty} q(o,\nu;\ j,1;s) z^j - \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=\nu}^{\infty} \frac{(\sigma y)^\alpha}{\alpha!}\, d\, H(y) \right\}$$

$$+ w\, W_{k+1}(o,w,s) \left\{ z \sum_{j=o}^{\infty} q(o,k+1;\ j,1;\ s)\, z^j - \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_0^\infty e^{-\xi(s+\lambda+\sigma-\lambda z)} \sum_{\alpha=k}^{\infty} \frac{(\sigma\xi)^\alpha}{\alpha!}\, d\, H(\xi) \right\}$$

and for $1 < v \leq k+1$,

(32) $z\, W_v(z,w,s) = \delta_{r,v}\, z^{i+1}$

$$+ w \sum_{\nu=v-1}^{k} W_\nu(z,w,s) \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{\nu-v+1}}{(\nu-v+1)!}\, d\, H(y)$$

$$+ w\, W_{k+1}(z,w,s) \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)\xi} \frac{(\sigma\xi)^{k-v-1}}{(k-v+1)!}\, d\, H(\xi)$$

$$+ w \sum_{\nu=v-1}^{k} W_\nu\ (o,w,s) \left\{ z \sum_{j=o}^{\infty} q(o,\nu;\ j,v;\ s)\, z^j - \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{\nu-v+1}}{(\nu-v+1)!}\, d\, H(y) \right\}$$

$$+ w\, W_{k+1}(o,w,s) \left\{ z \sum_{j=o}^{\infty} q(o,k+1;\ j,v;\ s) z^j - \frac{\sigma}{s+\lambda+\sigma-\lambda z} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{k-v+1}}{(k-v+1)!} d\, H(y) \right\}$$

In order to simplify the discussion, the following matrices and vectors are defined.

(a) $\underline{W}(z,w,s)$ is a row vector with components $W_v(z,w,s)$, $v = 1,\ldots,k+1$,

(b) I is the unit matrix of order $k+1$,

(c) $\Psi(u)$ and $R(z,s)$ are square matrices of order $k+1$, whose components are defined as follows:

(33) (i) For $i = 1,\ldots,k$, $j=1$,

$$\Psi_{i1}(u) = \int_0^\infty e^{-(u+\sigma)y} \sum_{\alpha=i}^\infty \frac{(\sigma y)^\alpha}{\alpha!} \, d\, H(y) ,$$

$$R_{i1}(z,s) = z\int_0^\infty \lambda e^{-(s+\lambda+\sigma)y} dy \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)t} \sum_{\alpha=i}^\infty \frac{(\sigma y+\sigma t)^\alpha}{\alpha!} \, d\, H(t)$$

$$- \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=i}^\infty \frac{(\sigma y)^\alpha}{\alpha!} \, d\, H(y) ,$$

(ii) $$\Psi_{k+1,1}(u) = \frac{\sigma}{\sigma+u} \int_0^\infty e^{-(u+\sigma)y} \sum_{\alpha=k}^\infty \frac{(\sigma y)^\alpha}{\alpha!} \, d\, H(y) ,$$

$$R_{k+1,1}(z,s) = \frac{\sigma z}{s+\lambda+\sigma} \int_0^\infty \int_0^\infty e^{-(\xi+\zeta)(s+\lambda+\sigma)+\lambda\xi z} \sum_{\alpha=k}^\infty \frac{(\sigma\xi+\sigma\zeta)^\alpha}{\alpha!} \lambda \, d\, \zeta \, d\, H(\xi)$$

$$- \frac{\sigma}{s+\lambda+\sigma} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \sum_{\alpha=k}^\infty \frac{(\sigma y)^\alpha}{\alpha!} \, d\, H(y) ,$$

(iii) For $1 \leq i < j-1$,

$$\Psi_{ij}(u) = R_{ij}(z,s) = 0 ,$$

(iv) For $j-1 \le i < k+1$,

$$\Psi_{ij}(u) = \int_0^\infty e^{-(u+\sigma)y} \frac{(\sigma y)^{i-j+1}}{(i-j+1)!} d\,H(y) ,$$

$$R_{ij}(z,s) = z \int_0^\infty \lambda e^{-(s+\lambda+\sigma)y}\, dy \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)t} \frac{(\sigma y+\sigma t)^{i-j+1}}{(i-j+1)!} d\,H(t)$$

$$- \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{i-j+1}}{(i-j+1)!} d\,H(y) ,$$

(v) For $1 < j \le k+1$,

$$\Psi_{k+1,j}(u) = \frac{\sigma}{u+\sigma} \int_0^\infty e^{-(u+\sigma)y} \frac{(\sigma y)^{k-j+1}}{(k-j+1)!} d\,H(y) ,$$

$$R_{k+1,j}(z,s) = \frac{\sigma z}{s+\lambda+\sigma} \int_0^\infty \int_0^\infty e^{-(\zeta+\xi)(s+\lambda+\sigma)+\lambda\xi z} \frac{(\sigma\xi+\sigma\zeta)^{k-j+1}}{(k-j+1)!} \lambda\, d\,\zeta\, d\,H(\xi)$$

$$- \frac{\sigma}{s+\lambda+\sigma} \int_0^\infty e^{-(s+\lambda+\sigma-\lambda z)y} \frac{(\sigma y)^{k-j+1}}{(k-j+1)!} d\,H(y) ,$$

The equations (31) and (32) may be written as follows:

(34) $\underline{W}(z,w,s)\,[z\,I - w\,\Psi(s+\lambda-\lambda z)] = z^{i+1}\,I + w\,\underline{W}(o,w,s)\,R\,(z,s)$,

Systems of equations, such as (34), occur frequently in the theory of Queues. In the particular problem under discussion, the system may be solved recursively thanks to the almost triangular nature of the matrix $\Psi(\cdot)$. First of all however, it must be shown that the unknown functions $W_v(o,w,s)$ may be determined uniquely. The proof of this follows verbatim proofs given in Neuts [3] and more generally in Çinlar [2]. We will therefore present only a summary of the proof, here.

The coefficient matrix $z\,I - w\,\Psi(s+\lambda-\lambda z)$, has several important properties, which we list below:

Property 1

The matrix $\Psi(w)$, Re $u \geq o$, defined in equations (33) is the matrix of Laplace-Stieltjes transforms of the entries of a $(k+1) \times (k+1)$, irreducible semi-Markov matrix.

Proof:

It suffices to show that each of the entries $\Psi_{ij}(w)$ of the matrix is the Laplace-Stieltjes transform of a mass function on $[o,\infty)$ and that the row-sums of the matrix are the Laplace-Stieltjes transforms of probability distribution functions on $[o,\infty)$.

The first statement follows from (33) by inspection and the second one follows from the fact that:

$$(35) \quad \sum_{j=1}^{k+1} \Psi_{ij}(u) = h(u), \quad 1 \leq i \leq k$$

and

$$(36) \quad \sum_{j=1}^{k+1} \Psi_{ij}(u) = \frac{\sigma}{\sigma+u} h(u), \quad i = k+1,$$

where $h(u)$ is the Laplace-Stieltjes transform of the service time distribution $H(\cdot)$ for unit I.

Property 2

From property 1 and thm. 1 in Neuts [3] it follows that the determinant of the coefficient matrix $z\,I - w\,\Psi(s+\lambda-\lambda z)$ has exactly $\underline{k+1}$ zeros in the unit disk $|z| \leq 1$ for every given pair (w,s) with Re $s > o$, $|w| \leq 1$ or Re $s \geq o$, $|w| < 1$.

Property 3

The unknown vector $\underline{W}(o,w,s)$ may be determined uniquely using property 2. If we solve the equations (34) for $W_j(z,w,s)$, $j = 1,\ldots,k+1$, then we obtain fractions, which have the determinant of $z I - w \Psi(s+\lambda-\lambda z)$ as their common denominator and linear functions in $W_j(o,w,s)$, $j = 1,\ldots,k+1$ as their numerators.

The requirement, that the numerators and the denominator must have the same zeros in the unit circle $|z| \leq 1$, leads to a system of $(k+1)$ independent linear equations for the unknowns $W_j(o,w,s)$, from which the latter may be obtained.

If the matrix $\Psi(u)$ is diagonalizable for all u with $\mathrm{Re}\, u \geq o$, then Neuts [3] has shown that the $k + 1$ zeros of the determinant of $z I - w \Psi(s+\lambda-\lambda z)$ may be found as the roots of the equations

$$(37) \quad z = w\, \eta_\rho(s+\lambda-\lambda z)\ , \quad 1 \leq \rho < k+1$$

which lie in the unit disk! Each such equation then has a unique root in $|z| \leq 1$.

If the matrix $\Psi(u)$ is not diagonalizable, the discussion of the relationship between the eigenvalues of $\Psi(u)$ and the roots is much more complicated, but may be found in Çinlar [2].

Unfortunately this solution for the functions $W_j(z,w,s)$ is purely formal, since it is impossible to supply general expressions for the zeros of the determinant $z I - w \Psi(s+\lambda-\lambda z)$.

Nonetheless, the equations (34) yield many results of a qualitative nature, which we will now discuss in detail.

4. The Equilibrium condition for the Queue.

Since all the first moments of the distributions of the intervals between transitions in the imbedded semi-Markov process are bounded, it suffices to find necessary and sufficient conditions for positive recurrence of the imbedded Markov chain of the semi-Markov process.

If $Q^{(n)}(i,r;\ j,v;\ x)$ is the n-step transition probability between the states (i,r) and (j,v), then the n-step transition probability $P^{(n)}(i,r;\ j,v)$ for the imbedded Markov chain is given by:

$$(38)\quad Q^{(n)}(i,r;\ j,v;\ \infty) = P^{(n)}(i,r;\ j,v) = q^{(n)}(i,r;\ j,v;\ o)$$

Since the imbedded Markov chain is irreducible and aperiodic, the limits

$$(39)\quad \pi_{jv} = \lim_{n\to\infty} P^{(n)}(i,r;\ j,v)$$

exist and are independent of the initial conditions.

If we define

$$(40)\quad \Pi_v(z) = \sum_{j=o}^{\infty} \pi_{jv}\, z^j\ , \quad |z| \leq 1\ , \quad 1 \leq v \leq k+1\ ,$$

then it is well-known that:

$$(41)\quad \Pi_v(z) = \lim_{w\to 1-} (1-w)\, W_v(z,w,o)\ , \quad 1 \leq v \leq k+1\ ,$$

so that equation (35) yields:

$$(42)\quad \underline{\Pi}\,(z)\,[zI - \Psi(\lambda-\lambda z)] = \underline{\Pi}\,(o)\, R\,(z,o)\ ,$$

where $\underline{\Pi}\,(z)$ is the $(k+1)$ - rowvector with components $\Pi_v(z)$.

The discussion of the existence of a nontrivial solution to equation (42) is completely analogous to that given in Neuts [3], particularly thm. 2, given there.

The equilibrium condition may be stated as follows:

Let us denote by $\eta(u)$ the Perron-Frobenius eigenvalue of the matrix $\Psi(u)$ for $u \geq o$, then the system (42) has a unique nontrivial solution, such that:

$$(43) \quad \sum_{v=1}^{k+1} \Pi_v(1) = 1 ,$$

if and only if:

$$(44) \quad \frac{1}{\lambda} > - \eta_1'(o+) ,$$

Moreover the quantity $-\eta_1'(o+)$ is given explicitely by:

$$(45) \quad -\eta_1'(o+) = (\theta_1+\dots+\theta_k)\,\alpha + \theta_{k+1}\left(\alpha + \frac{1}{\sigma}\right) = \alpha + \theta_{k+1}\,\sigma^{-1} ,$$

where the numbers $\theta_1,\dots,\theta_{k+1}$ are the stationary probabilities corresponding to the stochastic matrix $\Psi(o)$. See Neuts [3] p. 206, formula (13).

It follows that the queue is in equilibrium if and only if:

$$(46) \quad \frac{1}{\lambda} > \alpha + \frac{1}{\sigma}\,\theta_{k+1} , \qquad \theta_{k+1} > o .$$

The condition (46) is intuitive. The term α is the expected service time in unit I and the second term corresponds to the interaction between the two units due to blocking. We will calculate θ_{k+1} explicitely in the next section.

5. <u>The quantity θ_{k+1}</u> :

The quantities $\theta_1, \theta_2, \ldots, \theta_{k+1}$ satisfy the following system of linear equations:

$$(47) \quad \text{a.} \quad \theta_1 = \sum_{i=1}^{k} \theta_i \sum_{\alpha=i}^{\infty} p_\alpha + \theta_{k+1} \sum_{\alpha=k}^{\infty} p_\alpha$$

b. for $1 < j \leq k+1$,

$$\theta_j = \sum_{i=j-1}^{k} \theta_i \, p_{i-j+1} + \theta_{k+1} \, p_{k-j+1} ,$$

$$\text{c.} \quad \sum_{j=1}^{k+1} \theta_j = 1 ,$$

where

$$(48) \quad p_j = \int_0^\infty e^{-\sigma u} \frac{(\sigma u)^j}{j!} \, d\, H(u) , \quad j \geq 0 .$$

If we set $\theta_i = \theta_{k+1} A_{k-i+1}$, $i = 1, \ldots, k+1$, we obtain:

$$(49) \quad \text{a.} \quad A_0 = 1 ,$$

$$\text{b.} \quad p_0 \, A_{j+1} = A_j - \sum_{i=1}^{j} p_i \, A_{j-i+1} - p_j ,$$

for $0 \leq j \leq k-1$,

$$\text{c.} \quad \theta_{k+1} = \Big[\sum_{j=0}^{k} A_j \Big]^{-1} ,$$

The relations (49) are very important in discussing the dependence of the equilibrium condition (46) on the size k of the waitingroom. We note that

(49 a. and b.) enable us to calculate the quantities A_j, $j \geq o$, recursively for any j ; the dependence of θ_{k+1} on k appears only in (49 c.)

It is of some interest to study the properties of the generating function of the quantities A_j, $j \geq o$. We define:

$$(50) \quad A(z) = \sum_{j=o}^{\infty} A_j z^j ,$$

where the numbers A_j satisfy (49 a., b.) for all values of $j \geq o$.

A routine calculation leads to:

$$(51) \quad A(z) = \frac{(1-z)h(\sigma-\sigma z)}{h(\sigma-\sigma z) - z} ,$$

inside the largest circle about the origin in which the function on the right is analytic. The denominator in (51) vanishes at most at one point inside the unit circle. This is proved by the classical argument based on Rouché's theorem Takàcs [12].

If $\alpha > \frac{1}{\sigma}$, the denominator has a unique root κ , with $o < \kappa < 1$. In this case the powerseries (50) converges for $|z| < \kappa$.

When $\alpha \leq \frac{1}{\sigma}$, the denominator has no root inside the unit circle, but vanishes for $z = 1$. The radius of convergence of the powerseries is then at least one, but may be larger depending on whether or not $z = 1$ is a removable singularity.

When $\alpha > \frac{1}{\sigma}$, the point $z = \kappa$ is a simple pole of the function $A(z)$. It follows from Tauber's theorem that in this case

$$(52) \quad A_j = O(\kappa^{-j}) ,$$

as $j \to \infty$.

From formula (51), we may also obtain an explicit expression for the numbers A_j, when the service time in unit I is negative-exponential.

Let

$$(53)\quad h(s) = \frac{1}{1+\alpha s},$$

then from (51) we obtain:

$$(54)\quad A(z) = \frac{1}{1-\alpha\sigma z}, \qquad |z| < \frac{1}{\alpha\sigma},$$

and:

$$(55)\quad A_j = (\sigma\alpha)^j, \qquad j \geq 0,$$

$$(56)\quad \theta_{k+1} = \frac{1-\alpha\sigma}{1-\alpha^{k+1}\sigma^{k+1}}, \qquad \alpha\sigma \neq 1,$$

$$\theta_{k+1} = \frac{1}{k+1}, \qquad \alpha\sigma = 1,$$

5. Further Limit Theorems.

Several limit theorems of practical interest may be obtained as special cases of general theorems of Pyke and Schaufele [10] for semi- Markov processes.

These authors proved strong laws of large numbers and central limit theorems for a wide class of functionals defined on a recurrent semi-Markov process. Unfortunately the central limit theorem is of limited applicability due to the very complicated expression for the asymptotic variance. The strong law of large numbers on the other hand yields easily tractable limits, which have considerable qualitative and intuitive appeal.

We will state several instances of such results below and indicate the proof in one of them.

Let the equilibrium condition (46) be satisfied, so that the imbedded semi-Markov process is positive recurrent. We denote the stationary probabilities for the imbedded Markov chain by π_{jv}, $j \geq o$, $1 \leq v \leq k+1$. These probabilities may be determined from formula (42). We further set:

$$(57) \quad \pi_{\cdot v} = \sum_{j=o}^{\infty} \pi_{jv} = \Pi_v(1), \quad 1 \leq v \leq k+1,$$

<u>The Blocked Time</u>

If we denote by $B(t)$ the length of time during $(o,t]$ that the unit I is blocked, then $B(t)$ may be described as follows. Unit I can be blocked only when the imbedded semi-Markov process enters one of the states $(j,k+1)$, $j \geq o$. When this occurs, the time required until the queue becomes unblocked has a negative exponential distribution with mean $1/\sigma$.

We can now associate with each transition in the imbedded semi-Markov process a random variable which is equal to zero if the transition is into a state (j,v), $1 \leq v \leq k$ and which is equal to the duration of the blocking, when the transition is into a state of the type $(j,k+1)$, $j \geq o$.

Asymptotically $B(t)$ is equivalent to the sum of this sequence of random variables defined on the semi-Markov process. It follows from Pyke and Schaufele's law of large numbers that:

$$(58) \quad \lim_{t \to \infty} \frac{B(t)}{t} = \frac{\sigma^{-1} \pi_{\cdot,k+1}}{\sigma+\lambda^{-1}\pi_{\cdot,o}+\sigma^{-1}\pi_{\cdot,k+1}-(\lambda+\sigma)^{-1}\pi_{o,k+1}} \overset{\text{def}}{=} A_I$$

almost surely.

The central limit theorem in [10] asserts that:

$$(59)\qquad \frac{B(t) - A_1 t}{\sqrt{t}}$$

is asymptotically normal with zero mean, but with a complicated asymptotic variance.

In a similar way, if $B_1(t)$ denotes the length of time during $(o,t]$ that a customer in unit I cannot begin service due to blocking, then

$$(60)\qquad \lim_{t\to\infty} \frac{B_1(t)}{t} =$$

$$\frac{\sigma^{-1}\pi_{\bullet,k+1} - (\lambda+\sigma)^{-1}\pi_{o,k+1}}{\alpha+\lambda^{-1}\pi_{\bullet,o} + \sigma^{-1}\pi_{\bullet,k+1} - (\lambda+\sigma)^{-1}\pi_{o,k+1}}, \quad \text{a.s.}$$

If $B_2(t)$ denotes the length of time that unit I is idle during $(o,t]$, then:

$$(61)\qquad \lim_{t\to\infty} \frac{B_2(t)}{t} =$$

$$\frac{\lambda^{-1}\sum_{v=1}^{k+1} \pi_{o,v}}{\alpha+\lambda^{-1}\pi_{\bullet,o} + \sigma^{-1}\pi_{\bullet,k+1} - (\lambda+\sigma)^{-1}\pi_{o,k+1}}, \quad \text{a.s.}$$

If $B_3(t)$ denotes the length of time that unit I is idle, but blocked, then:

$$(62)\qquad \lim_{t\to\infty} \frac{B_3(t)}{t} =$$

$$\frac{\frac{\lambda}{\lambda+\sigma} \; \frac{\pi_{o,k+1}}{\sigma}}{\alpha+\lambda^{-1}\pi_{\bullet,o} + \sigma^{-1}\pi_{\bullet,k+1} - (\lambda+\sigma)^{-1}\pi_{o,k+1}}$$

If $N(t)$ denotes the number of customers served by the system in $(o,t]$, then:

(63) $$\lim_{t\to\infty} \frac{N(t)}{t} =$$

$$[\alpha+\lambda^{-1}\pi_{\cdot,o}+\sigma^{-1}\pi_{\cdot,k+1}-(\lambda+\sigma)^{-1}\pi_{o,k+1}]^{-1}, \quad \text{a.s.}$$

6. The Busy Periods for unit I.

In this section, we will study the probability that in a given number of transitions the queue in unit I does not become empty. In doing so, we will show how the so-called "zero-avoiding transition probabilities" may be obtained in principle. Along with the renewal functions, discussed in section 3, these probabilities may be used to express such items as the distribution of the queuelength in continuous time and the waiting time distributions quite simply. Unfortunately, just as in section 3, we find that only a formalism can be developed and that actual numerical methods must be looked for in other directions.

We will not pursue the subject of transform solutions for the time dependence of this queueing system very far. Only results of some qualitative interest will be derived.

Let ${}_oQ^{(n)}(i,r;\ j,v;\ x)$ denote the probability that, given the initial state (i,r), there are at least n service completions in unit I before one of the states $(o,1),\dots,(o,k+1)$ is reached and that at the end of the n-th service completion the imbedded semi-Markov process is in state (j,v). By ${}_oq^{(n)}(i,r;\ j,v;\ s)$ we denote the Laplace-Stieltjes transform of ${}_oQ^{(n)}(i,r;j,v;s)$.

We define the generating functions:

(64) $${}_oU_v^{(n)}(z,s) = \sum_{j=o}^{\infty} {}_oq^{(n)}(i,r;\ j,v;\ s)z^j \ , \quad n \geq 1 \ ,$$

and:

$$(65)\quad {}_oW_v(z,s) = \sum_{n=1}^{\infty} {}_oU_v^{(n)}(z,s), \quad 1 \le v \le k+1 ,$$

It follows that:

$$(66)\quad {}_oU_v^{(1)}(z,s) = z^{i-1}\, \Psi_{r,v}\,(s+\lambda-\lambda z)\ , \quad i \ge 1 ,$$

and

$$(67)\quad z\,{}_oU_v^{(n+1)}(z,s) = \sum_{\nu=v-1}^{k+1} [{}_oU_\nu^{(n)}(z,s) - {}_oq^{(n)}(i,r;\, o,\nu;s)]\ \Psi_{\nu v}\,(s+\lambda-\lambda z)\ ,$$

for $1 < v \le k+1$, $i \ge 1$, $n \ge 1$,

$$(68)\quad z\ {}_oU_1^{(n+1)}(z,s) = \sum_{\nu=1}^{k+1} [{}_oU_\nu^{(n)}(z,s) - {}_oq^{(n)}(i,r;\, o,\nu;\, s)]\ \Psi_{\nu 1}\,(s+\lambda-\lambda z)\ ,$$

for $i \ge 1$, $n \ge 1$,

$$(69)\quad z\ {}_oW_v(z,s) - \sum_{\nu=v-1}^{k+1} {}_oW_\nu(z,s)\ \Psi_{\nu v}(s+\lambda-\lambda z) =$$

$$z^i\ \Psi_{rv}(z,s) - \sum_{\nu=v-1}^{k+1} {}_oW_\nu(o,s)\ \Psi_{\nu v}\ (s+\lambda-\lambda z)$$

for $1 < v \le k+1$ and:

$$(70)\quad z\ {}_oW_1(z,s) = z^i\ \Psi_{r,k+1}(s+\lambda-\lambda z) + \sum_{\nu=1}^{k+1} {}_oW_\nu(z,s)\ \Psi_{\nu,1}\ (s+\lambda-\lambda z) - \sum_{\nu=1}^{k+1} {}_oW_\nu\ (o,s)\ \Psi_{\nu,1}(s+\lambda-\lambda z)\ ,$$

These equations may be written as:

$$(71) \quad {}_{o}\underline{W}(z,s)\,[z\,I - \Psi(s+\lambda-\lambda z)] =$$

$$z^{i}\,\underline{\Psi}_{r}(s+\lambda-\lambda z) - {}_{o}\underline{W}(o,s)\,\Psi\,(s+\lambda-\lambda z)$$

where ${}_{o}\underline{W}(z,s)$ is the row vector with entries ${}_{o}W_{v}(z,s)$, $1 \leq v \leq k+1$ and $\underline{\Psi}_{r}(s+\lambda-\lambda z)$ is the r-th row of the matrix $\Psi(s+\lambda-\lambda z)$.

When $i = o$, the equations obtained are slightly different from those given above. Instead of the term $z^{i}\,\underline{\Psi}_{r}(s+\lambda-\lambda z)$ the r-th row of the matrix $R(z,s)$ appears.

We restrict our attention further to the case $i \geq 1$.

We will also assume that the eigenvalues $\eta_{\rho}(s)$ of the matrix $\Psi(s)$ are distinct for all points s, Re $s > o$. If this is not the case, then a more delicate analysis due to Çinlar [2] is required.

Diagonalizing the matrix $\Psi(s+\lambda-\lambda z)$, we obtain:

$$(72) \quad \Psi(s+\lambda-\lambda z) = L(s+\lambda-\lambda z)\,H(s+\lambda-\lambda z)\,L^{-1}(s+\lambda-\lambda z)\,;$$

where:

$$(73) \quad L_{ij}(s+\lambda-\lambda z) = \alpha_{ij}(s+\lambda-\lambda z)\,,$$

$$(L^{-1})_{ij}(s+\lambda-\lambda z) = \beta_{ij}(s+\lambda-\lambda z)\,,$$

$$H_{ij}(s+\lambda-\lambda z) = \delta_{ij}\,\eta_{j}(s+\lambda-\lambda z)\,,$$

$1 \leq i,j \leq k+1$,

The equations (71) yield: $(1 \le v \le k+1)$,

(74) $${}_oW_v(z,s) =$$

$$z^i \sum_{\rho=1}^{k+1} \frac{\alpha_{r\rho}(s+\lambda-\lambda z)\, \eta_\rho(s+\lambda-\lambda z)\, \beta_{\rho v}(s+\lambda-\lambda z)}{z-\eta_\rho(s+\lambda-\lambda z)}$$

$$- \sum_{\nu=1}^{k+1} \sum_{\rho=1}^{k+1} {}_oW_\nu(o,s) \frac{\alpha_{\nu\rho}(s+\lambda-\lambda z)\, \eta_\rho(s+\lambda-\lambda z)\, \beta_{\rho\nu}(s+\lambda-\lambda z)}{z-\eta_\rho(s+\lambda-\lambda z)}$$

The $k+1$ roots of the equation:

(75) $$\det [z\, I - \Psi(s+\lambda-\lambda z)] = o, \quad |z| \le 1 ,$$

are then given by the unique roots in the unit circle of the $k+1$ equations

(76) $$z - \eta_\rho(s+\lambda-\lambda z) = o , \quad \rho = 1,\dots, k+1 .$$

We denote these roots by $\gamma_\rho(s)$, $\rho=1,\dots,k+1$.

If we require that the expressions on the right in equations (74) be analytic functions of z in the unit disk, we obtain the following additional conditions:

(77) $$\sum_{\nu=1}^{k+1} {}_oW_\nu(o,s)\, \alpha_{\nu\rho} [s+\lambda-\lambda\gamma_\rho(s)] =$$

$$\alpha_{r\rho}[s+\lambda-\lambda\gamma_\rho(s)]\, \gamma_\rho^i (s), \quad \rho = 1,\dots,k+1 .$$

We define the matrix $T(s)$ by:

(78) $$T_{\nu\rho}(s) = \alpha_{\nu\rho} [s+\lambda-\lambda\gamma_\rho(s)] .$$

Çinlar proved that under our non-singularity assumption the matrix $T(s)$ is non-singular for $\operatorname{Re} s > o$.

Furthermore, let $\Gamma(s)$ be the diagonal matrix with

(79) $\Gamma_{\rho\rho}(s) = \gamma_{\rho}(s), \quad \rho = 1, \ldots, k+1$,

If we write $\widetilde{W}^{(i)}_{r\nu}(s)$ for ${}_{o}W_{\nu}(o,s)$ when the initial state of the semi-Markov process is (i,r) and if we define the matrix

(80) $\widetilde{W}^{(i)}(s) = \{\widetilde{W}^{(i)}_{r\nu}(s)\}\ , \quad i > o$,

then, we obtain from equation (77) that:

(81) $\widetilde{W}^{(i)}(s) = T(s)\, \Gamma^{i}(s)\, T^{-1}(s)$

$$= [\widetilde{W}^{(1)}(s)]^{i}\ , \quad i > o\ .$$

The matrix $\widetilde{W}^{(i)}(s)$ has an important interpretation. Its entry $\widetilde{W}^{(i)}_{r\nu}(s)$ is the Laplace-Stieltjes transform of the probability mass function for the length of a busy period starting in state (i,r) and ending in the state (o,ν). We will prove below that $\widetilde{W}^{(1)^{*}}(s)$ is the transform-matrix of a $(k+1)$ - th order semi-Markov matrix of the only of the queue is in equilibrium.

Formula (81) is important in that it shows that a busy period starting in state (i,r), $i \geq 1$ has a simple relation to a busy period starting in a state with $i = 1$.

If as before, $\eta_1(s+\lambda-\lambda z)$ is the Perron-Frobenius eigenvalue of $\Psi(s+\lambda-\lambda z)$ for $s+\lambda-\lambda z \geq o$, then equation (77) with $\rho = 1$, yields:

$$(82) \quad \sum_{\nu=1}^{k+1} \tilde{W}_{r\nu}^{(1)}(s)\, \alpha_{\nu 1}[s+\lambda-\lambda\gamma_1(s)] = \alpha_{r1}[s+\lambda-\lambda\gamma_1(s)]\, \gamma_1(s) , \quad \text{Re } s \geq o .$$

If and only if the equilibrium condition:

$$(83) \quad \frac{1}{\lambda} \geq \alpha + \frac{1}{\sigma}\, \theta_{k+1} ,$$

holds does $\gamma_1(s)$ tend to one as $s \to o+$. This was shown in Neuts [3]. In this case also we have:

$$(84) \quad \alpha_{r1}(o+) = 1 ,$$

so that:

$$(85) \quad \sum_{\nu=1}^{k+1} \tilde{W}_{r\nu}^{(1)}(o+) = 1 , \quad r = 1,\dots, k+1 ,$$

proving that in the equilibrium queue, the matrix $\tilde{W}^{(1)}(s)$ is the transform of a <u>stochastic</u> semi-Markov matrix.

If the equilibrium condition does not hold, we obtain:

$$(86) \quad \sum_{\nu=1}^{k+1} \tilde{W}_{r\nu}^{(1)}(o)\, \alpha_{\nu 1}[\lambda-\lambda\gamma_1(o)] = \alpha_{r1}[\lambda-\lambda\gamma_1(o)]\, \gamma_1(o) ,$$

with $\gamma_1(o) < 1$ and $\alpha_{r1}[\lambda-\lambda\gamma_1(o)] > o$, $r = 1,\dots, k+1$.

It follows from the Perron-Frobenius theory that $\tilde{W}^{(1)}(o+)$ is a strictly substochastic matrix.

7. Concluding remarks, Acknowledgement.

To find the relations linking the queuelength in continuous time to the renewal functions for the imbedded semi-Markov process is a routine matter and is analogous to arguments presented in Neuts [4,5,6] for other types of queues. For a general service time distribution in unit I such relations are again purely formal and do not lead to further qualitative results. We will not present them here.

When the service time in the unit I is negative exponential, the queuelength in continuous time may be studied as a bivariate birth and death process. This course of investigation was pursued by M. Yadin [13], to whom the author is grateful for regenerating his interest in this model and for communicating his own results prior to publication.

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