Power Comparisons of Tests of Two Multivariate Hypotheses Based on Individual Characteristic Roots

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0. Summary. In this paper, power comparisons are made for tests of each of the following two hypotheses based on individual characteristic roots of a matrix arising in each case: (i) independence between a p-set and a q-set of variates in a (p+q)-variate normal population with $p \leq q$ and (ii) equality of p-dimensional mean vectors of ℓ p-variate normal populations having a common covariance matrix. At first, a few lemmas are given which help to reduce the central distributions of the largest, smallest, second largest, and the second smallest roots in terms of incomplete beta functions or functions of them. Since the central distribution of the largest root has been discussed by Pillai earlier in several papers (1956a, 1960, 1964a, 1965, 1966, 1967) cdf's of the three others in the central case are given. Further, the non-central distributions of the individual roots for $p = 3$ are considered for the two hypotheses and that of the smaller root for $p = 2$; that of the largest root for $p = 2$ has been obtained by Pillai earlier, (Pillai (1966), Pillai and Jayachandran (1967)).

1. Introduction. For test of hypothesis (i) the joint distribution of the characteristic roots $r_1^2,\ldots,r_p^2$ is given by Constantine (1963) in the following form: (James, 1964)

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$$(1.1)\quad |\underset{\sim}{I} - \underset{\sim}{P}^2|^{\frac{1}{2}\nu}\ {}_1F_1(\tfrac{1}{2}\nu, \tfrac{1}{2}\nu; \tfrac{1}{2}f_2; \underset{\sim}{P}^2, \underset{\sim}{R}^2) \cdot f_o(r_1^2,\ldots,r_p^2) \prod_{i=1}^{p} dr_i^2$$

$$0 < r_1^2 \leq \cdots \leq r_p^2 < 1 ,$$

where $f_o(r_1^2,\ldots,r_p^2)$ is the joint distribution of $r_1^2,\ldots,r_p^2$ under the null hypothesis given by

$$(1.2)\quad f_o(r_1^2,\ldots,r_p^2) = C(p,m,n)\ |\underset{\sim}{R}^2|^m |\underset{\sim}{I} - \underset{\sim}{R}^2|^n \prod_{i>j} (r_i^2 - r_j^2) ,$$

$f_2 = q$, $m = \frac{1}{2}(q-p-1)$, $n = \frac{1}{2}(\nu-q-p-1)$, (and if n is defined as $\frac{1}{2}(f_1-p-1)$, $\nu = f_1 + f_2$), $\underset{\sim}{R} = \text{diag.}(r_1,\ldots,r_p)$ $\underset{\sim}{P} = \text{diag.}(\rho_1,\ldots,\rho_p)$, where ρ_i^2's are the population characteristic roots (see Pillai and Jayachandran (1967)),

$$C(p,m,n) = \Pi^{\frac{1}{2}p} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(2m+2n+p+i+2)]/\{\Gamma[\tfrac{1}{2}(2m+i+1)]\Gamma[\tfrac{1}{2}(2n+i+1]\Gamma(\tfrac{1}{2}i)\},$$

and the hypergeometric function of matrix argument, ${}_2F_1$, is defined in James (1964).

Similarly for test of hypothesis (ii) the joint density function of the characteristic roots $\ell_1,\ldots,\ell_p$ is given by Constantine (1963), James (1964), in the form

$$(1.3)\quad e^{-\frac{1}{2}\text{tr}\,\underset{\sim}{\Omega}}\ {}_1F_1(\tfrac{1}{2}\nu; \tfrac{1}{2}f_2; \tfrac{1}{2}\underset{\sim}{\Omega}, \underset{\sim}{L})\ f_o(\ell_1,\ldots,\ell_p) \prod_{i=1}^{p} d\ell_i ,$$

where $f_o(\ell_1,\ldots,\ell_p)$ follows (1.2) with ℓ_i and $\underset{\sim}{L}$ replacing r_i^2 and $\underset{\sim}{R}^2$ respectively, $m = \frac{1}{2}(f_2-p-1)$, $n = \frac{1}{2}(f_1-p-1)$, $\nu = f_1 + f_2$, $\underset{\sim}{L} = \text{diag.}(\ell_1,\ldots,\ell_p)$, $\underset{\sim}{\Omega} = \text{diag.}(\omega_1,\ldots,\omega_p)$, where ω_i's are the non-centrality parameters (See Pillai and Jayachandran (1967)) and in the context of (ii), $f_2 = \ell - 1$ and $f_1 = N - \ell$, N being the pooled sample size of the samples from the ℓ populations.

Pillai has studied (Pillai (1966) and Pillai and Jayachandran (1967)) the power function of the largest roots, r_p^2, for test (i) and ℓ_p for test of (ii) for $p = 2$, and $p = 3$ in the linear case i.e. when there is only one non-zero non-centrality parameter. In this paper the non-central distributions of r_i^2 and ℓ_i $(i=1,\ldots,p)$ are considered for $p = 2$ and 3. In the null case, the cdf of the second largest and second smallest roots are also given for any p, the approach being to reduce by means of certain recursion formulae the cdf's to incomplete beta functions or functions of them. Roy (1945) has attempted to reduce the cdf of the i^{th} root in a similar manner but his results seem to be in error. Hence, some new Lemmas are given in this connection. Further, extensive power function tabulations have been made for various values of the parameters, m=0,1,2,5 and n=5,10,15,20,25,30,40,60. But from these, only selected values are given in the paper to support some of the findings discussed at the end of the paper.

2. Some recursion formulae. Let $V(q_p,n;\ldots;x',x'',q_j,n;\ldots;q_1,n)$ denote the determinant

$$\begin{vmatrix} \int_{x_{p-1}}^{1} x_p^{q_p}(1-x_p)^n dx_p \int_{x_{p-2}}^{1} x_{p-1}^{q_p}(1-x_{p-1})^n dx_{p-1} \cdots \int_{x'}^{x''} x_j^{q_p}(1-x_j)^n dx_j \cdots \int_{o}^{x_3} x_2^{q_p}(1-x_2)^n dx_2 \int_{o}^{x_2} x_1^{q_p}(1-x_1)^n dx_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \int_{x_{p-1}}^{1} x_p^{q_1}(1-x_p)^n dx_p \int_{x_{p-2}}^{1} x_{p-1}^{q_1}(1-x_{p-1})^n dx_{p-1} \cdots \int_{x'}^{x''} x_j^{q_1}(1-x_j)^n dx_j \cdots \int_{o}^{x_3} x_2^{q_1}(1-x_2)^n dx_2 \int_{o}^{x_2} x_1^{q_1}(1-x_1)^n dx_1 \end{vmatrix}$$

(2.1)

Then we state below a lemma due to Pillai (1956b).

Lemma 1. The determinant

(2.2) $V(o,x,q_p,n;\dots;q_1,n) = (q_p+n+1)^{-1} (A_p^{(p)}+B_p^{(p)}+C_p^{(p)})$,

where

(2.3) $A_p^{(p)} = I_o(o,x;q_p,n+1)\ V(o,x,q_{p-1},n;\dots;q_1,n)$,

(2.4) $B_p^{(p)} = 2 \sum_{j=p-1}^{1} (-1)^{p-j-1}\ I(o,x,q_p+q_j,2n+1)V(o,x,q_{p-1}n;\dots,q_{j+1},n;$
$q_{j-1},n;\dots;q_1 n)$

and

(2.5) $C_p^{(p)} = V(o,x,q_p-1,n;\dots;q_1,n)$,

and where

(2.6) $I_o(x',x'',q_p,n+1) = y^{q_p}(1-y)^{n+1}\Big|_{x'}^{x''}$,

and

(2.7) $I(x',x'', a ,b) = \int_{x'}^{x''} y^a(1-y)^b dy$.

Lemma 2. The determinant

(2.8) $V(q_p,n;\dots;x,1,q_1,n) = (q_p+n+1)^{-1} (A_1^{(p)} + B_1^{(p)} + q_p\ C_1^{(p)})$,

where

$$(2.9)\quad A_1^{(p)} = (-1)^p\, I_o(x,1,q_p,n+1)\; V(q_{p-1},n;\ldots;x,1,q_1,n)\ ,$$

$$(2.10)\quad B_1^{(p)} = 2\sum_{j=p-1}^{1} (-1)^{p-j-1}\, I(x,1,q_p+q_j,2n+1)\; V(q_{p-1},n;\ldots;q_{j+1},n;\; q_{j-1},n;\ldots;x,1,q_1,n)$$

and

$$(2.11)\quad C_1^{(p)} = V(q_p-1,n;\ldots;x,1,q_1,n)\ .$$

The lemma can be proved by methods similar to those used by Pillai for proving lemma 1. It may be pointed out that Roy (1945) has given a formula in this context which, however, is incorrect to the extent that it does not show the factor $(-1)^p$ involved in (2.9) above.

Lemma 3. The determinant

$$(2.12)\quad V(q_p,n;o,x,q_{p-1},n;\ldots;q_1,n) = (q_p+n+1)^{-1}\,(A_{p-1}^{(p)} + B_{p-1}^{(p)} + q_p\, C_{p-1}^{(p)})\ ,$$

where

$$(2.13)\quad A_{p-1}^{(p)} = I_o(o,x,q_p,n+1)\sum_{j=p-1}^{1} (-1)^{p-j-1}\, I(x,1,q_j,n)\; V(o,x,q_{p-1},n;\ldots;\; q_{j+1},n;\; q_{j-1},n;\ldots;q_1,n)$$

$$(2.14)\quad B_{p-1}^{(p)} = 2\sum_{j=p-1}^{1} (-1)^{p-j-1}\, I(o,x,q_p+q_j,2n+1)\; V(q_{p-1},n;o,x,q_{p-2},n;\ldots;\; q_{j+1},n;q_{j-1},n;\ldots;q_1,n)$$

and

$$(2.15) \quad C_{p-1}^{(p)} = V(q_p-1,n;o,x,q_{p-1},n;\ldots;q_1,n) \; .$$

The proof is omitted. However, it may be pointed out that the proof follows generally the same lines as in Pillai (1956b) but an additional result has to be used in the proof which is stated in lemma 4 below.

Lemma 4. If $V(q_p'',n'',\ldots;x',x'',q_j'',n'';\ldots;q_1'',n'')^{(i)}$ denotes (2.1) with the indices of the i^{th} column, shown with ' " ', differing from those of the rest in the determinant, then

$$(2.16) \quad V(o,x,q_{p-1}'',n'';\ldots;q_1'',n'')^{(p-1)}+\sum_{i=p-2}^{1}(-1)^{p-i-1}V(q_{p-1}''n'';o,x,q_{p-2}'',n'';\ldots;q_1'',n'')^{(i)}$$

$$=\sum_{j=p-1}^{1}(-1)^{p-j-1}\, I(o,x,q_j'',n'')\; V(q_{p-1},n;o,x,q_{p-2},n;\ldots;q_{j+1},n;q_{j-1},n;\ldots;q_1,n).$$

Now by the use of a result similar to that of lemma 4 and others from Pillai (1956b) the following can be proved:

Lemma 5. The determinant

$$(2.17) \quad V(q_p,n;\ldots;x,1,q_2,n;q_1,n) = (q_p+n+1)^{-1}(A_2^{(p)} + B_2^{(p)} + q_pC_2^{(p)}),$$

where

(2.18) $A_2^{(p)} = (-1)^p I_o(x,1,q_p,n+1) \sum_{j=p-1}^{1} (-1)^{p-j-1} I(o,x,q_j,n)$

$V(q_{p-1},n;\ldots;q_{j+1},n;q_{j-1},n;\ldots;x,1,q_1,n)$,

(2.19) $B_2^{(p)} = 2 \sum_{j=p-1}^{1} (-1)^{p-j-1} I(x,1,q_p+q_j,2n+1)\ V(q_{p-1},n;\ldots;q_{j+1},n;\ q_{j-1},n;$
$\ldots;x,1,q_2,n;q_1,n)$

and

(2.20) $C_2^{(p)} = V(q_p-1,n;\ldots;x,1,q_2,n;q_1,n)$.

3. Cdf's of some individual roots in the central case.

Roy (1945) has given an expression for the cdf of the i^{th} root in the central case $(i=1,\ldots,p)$. However the expression is not correct and even the total probability for the whole range does not equal unity. Since the cdf of the largest root has been dealt with by Pillai in several papers by using lemma 1 (1956a, 1960, 1965), we consider here three other cdf's, namely, those of the smallest, second largest and second smallest.

Since $f_o(r_1^2,\ldots,r_p^2)$ and $f_o(\ell_1,\ldots,\ell_p)$ have the same form, we may label by $(x_1,\ldots,x_p)$ either $(r_1^2,\ldots,r_p^2)$ or $(\ell_1,\ldots,\ell_p)$ and hence

(3.1) $\Pr(x_1 \leq x) = 1 - C(p,m,n)\ V(m+p-1,n;\ldots;x,1,m,n)$,

and to evaluate the right side of (3.1) we may use lemma 2 with $q_j = m+j-1,\ j = 1,\ldots,p$.

Further

$$(3.2)\quad \Pr(x_{p-1} \leq x) = C(p,m,n)\ V(m+p-1,n;o,x,m+p-2,n;\ldots;m,n),$$

and hence the right side of (3.2) can be evaluated using lemma 3 with $q_j = m+j-1$ $(j = 1,2,\ldots,p)$. Again

$$(3.3)\quad \Pr(x_2 \leq x) = 1 - C(p,m,n)\ V(m+p-1,n;\ldots;x,1,m+1,n;m.n)\ ,$$

and, as before, lemma 5 can be used to evaluate the right side of (3.3), noting that $q_j = m+j-1$ $(j = 1,2,\ldots,p)$.

Further we prove below a lemma to show that the cdf of x_i can be derived from that of x_{p-i+1} or <u>vice versa</u>. It has been shown earlier (Nanda (1948), Pillai (1956**a**)) that this result holds for $i = 1$.

<u>Lemma 6</u>. If x_i and x_{p-i+1} are the i^{th} and $(p-i+1)^{th}$ roots $(i = 1,\ldots,p)$, where $x_1,\ldots,x_p$ follow the density $f_o(x_1,\ldots,x_p;m,n)$ of the form in (1.2), then

$$(3.4)\quad \Pr(x_i \leq x;m,n) = 1 - \Pr(x_{p-i+1} \leq 1 - x;n,m)\ ,$$

where on the right side of (3.4) the parameters m and n are interchanged.

<u>Proof</u>: First transform $x_i = 1 - z_i$ $(i = 1,\ldots,p)$, then

$$(3.5)\quad f_1(z_1,\ldots,z_p) = f_o(z_1,\ldots,z_p;n,m)\ , \qquad 1 > z_1 \geq \ldots \geq z_p > o\ .$$

Hence

(3.6) $\Pr(x_i \leq x;m,n) = \Pr(x_{p-i+1} \geq 1-x;n,m) = 1-\Pr(x_{p-i+1} \leq 1-x;n,m)$.

Hence the lemma.

4. Non-central cdf's of individual roots. Denote by $D(q_p,n;\ldots;q_1,n)$ the integrand in (2.1) which is a Vandermonde-type determinant. It is obvious that if $q_j = m+j-1$ $(j=1,\ldots,p)$, then

(4.1) $f_o(x_1,\ldots,x_n;m,n) = C(p,m,n)\, D(q_p,n;\ldots;q_1,n)$.

Now noting that the hypergeometric functions of matrix variates ${}_2F_1$ or ${}_1F_1$ are series whose terms are expressible as functions of elementary symmetric functions (esf's) of the characteristic roots of the matrices involved in the respective series, and that by Pillai's lemma (1964b) the product of a basic Vandermonde-type determinant by powers of esf's can be expressed as a linear compound of Vandermonde-type determinants, it is easy to see that the non-central cdf's of the individual roots in (1.1) or (1.3) can be expressed as a series of determinants of the type (2.1) (see Pillai (1966), Pillai and Jayachandran (1967).

Further reductions would follow by the use of lemmas in section 2. The cdf's of individual roots for $p = 2$ and 3 are considered in detail below.

Largest root. The cdf of the $r_2^{2(2)}$ and $\ell_2^{(2)}$ when $p = 2$, and $r_3^{2(3)}$ and $\ell_3^{(3)}$ in the linear case when $p = 3$ were studied by Pillai (Pillai (1966), Pillai and Jayachandran (1967) and unpublished reports with Department of Statistics, Purdue University). The non-central cdf of $r_3^{2(3)}$ is given below: (using seven terms of (1.1))

$$(4.2) \quad \Pr(r_3^{2(3)} \le x) = K\{-I_o(o,x,m+2,n+1)\Big[(\sum_{i=o}^{6} B_i x^i)\ V(o,x,m+1,n;m,n)$$

$$+ (\sum_{i=2}^{6} C_i x^{i-1})\ V(o,x,m+2,n;m,n) + (\sum_{i=3}^{6} D_i x^{i-2})\cdot$$

$$V(o,x,m+2,n;m+1,n)$$

$$+ \sum_{i=4}^{6} E_i x^{i-2}\ V(o,x,m+3,n;m,n) + \sum_{i=5}^{6} (F_i x^{i-3})$$

$$V(o,x,m+3,n:m+1,n)$$

$$+ \quad G\, x^3\ V(o,x,m+4,n;m,n) + H\, x^2\ V(o,x,m+3,n;$$

$$m+2,n)\Big]$$

$$+ \quad 2I(o,x,m,n) \sum_{i=o}^{6} (B_i^{(1)}\ I(o,x,2m+3+i,2n+1))$$

$$- \quad 2I(o,x,m+1,n) \sum_{i=o}^{6} (B_i^{(2)}\ I(o,x,2m+2+i,2n+1))$$

$$- \quad 2I(o,x,m+2,n) \sum_{i=o}^{4} (B_i^{(3)}\ I(o,x,2m+3+i,2n+1))$$

$$- \quad 2I(o,x,m+3,n) \sum_{i=o}^{2} B_i^{(4)}\ I(o,x,2m+4+i,2n+1)$$

$$- \quad 2G\ I(o,x,m+4,n)\ I(o,x,2m+5,2n+1)\} \ ,$$

where $K = \prod_{i=1}^{3} (1-\rho_i^2)^{\frac{1}{2}\nu} C(3,m,n)$ and the B_i, C_i, D_i, E_i, F_i, G, H and the $B_i^{(j)}$ coefficients are available in an unpublished report (Department of Statistics, Purdue University).

Now the cdf of $\ell_3^{(3)}$ can be obtained from (4.2) by making some simple changes in the coefficients, K and others, given in the preceding paragraph. These changes are descirbed in Pillai and Jayachandran (1967).

Smallest root. For obtaining the non-central cdf's of the smallest root, note first that the recursion formula (2.8) of lemma 2 is similar to that in (2.2) of lemma 1. Moreover, lemma 6 further points out how the central cdf of the smallest root is derivable from that of the largest root and *vice versa*. Hence the non-central cdf's of the smallest root for $p = 2$ and 3 and tests (i) and (ii) may be obtained from the corresponding non-central cdf's of the largest root by making the following changes:

$$(4.3)\quad \begin{cases} -\ I_o(o,x,q_p,n+1) \rightarrow (-1)^p\ I_o(x,1,q_p,n+1) \\ I(o,x,q,r) \rightarrow I(x,1,q,r) \\ V(o,x,q_p,n;\ldots;q_1,n) \rightarrow V(q_p,n;\ldots;x,1,q_1,n)\ . \end{cases}$$

Median root. In obtaining the non-central cdf of the median root for $p = 3$, we use the recursion formula (2.12) of lemma 3 (or alternately (2.17) of lemma 5) to the determinants of the form

(2.1) obtainable in (1.1) or (1.3) as described in the beginning of this section, but note that the coefficients, K and others, involved in (4.2) would remain the same in the process of reduction. Hence for obtaining non-central cdf of $r_2^{2(3)}$, the median root when $p = 3$, the following changes may be made in (4.2):

$$(4.4)\quad \begin{cases} -\ I_o(o,x,m+2,n+1) \rightarrow I_o(o,x,m+2,n+1) \\ V(o,x,q_2,n;q_1,n) \rightarrow \dfrac{I(x,1,q_2,n)I(o,x,q_1,n) - I(x,1,q_1,n)}{I(o,x,q_2,n)} \\ I(o,x,q_j,n)\ I(o,x,q_3+q_j,2n+1) \rightarrow \beta(q_j+1,n+1)I(x,1,q_3+q_j, 2n+1), j = 1,2 . \end{cases}$$

Further, the non-central cdf of $\ell_2^{(3)}$, the median root when $p = 3$, can be obtained from that of $r_2^{(3)}$ by making simple changes (see Pillai and Jayachandran (1967)).

5. Tabulation of percentage points. In order to facilitate the tests of the two hypotheses considered in the paper and others, Pillai (1956a, 1960, 1964, 1965, 1967) has tabulated the upper 5% and 1% points of the largest root for values of p up to 20. Using (3.1) and (3.2) or (3.3), similar percentage points were obtained for the smallest root for $p = 2$ and 3 and for the median root for $p = 3$ and for values of $m = 0(1)5, 7, 10, 15$ and $n = 5(5),30, 40, 60, 100, 130, 160, 200, 300, 500$ and 1000. These are presented in Tables 1 to 6 and are used for the power function tabulations in the following section. The percentage points are believed to be accurate to within a unit of the last decimal quoted.

6. Power function tabulations. Further, powers of test (i) are computed based on the largest root, $r_3^{2(3)}$, for various values $(\rho_1^2, \rho_2^2, \rho_3^2)$ using (4.2), the smallest root, $r_1^{2(3)}$, using (4.3) and the median root, $r_2^{2(3)}$, using (4.4). These are presented in Table 7 for $m = 0,1,2,5$, $n = 5,15$ and 40, $\alpha = .05$ and various values of the vector $(\rho_1^2, \rho_2^2, \rho_3^2)$. Similarly powers of test (ii) are presented for $p = 3$ in Table 8 for the same values of m and n and various values of the vector $(\omega_1, \omega_2, \omega_3)$. In addition, Table 9 gives for $p = 2$ powers based on each of the two roots for test (i) and Table 10 for test (ii). Extensive additional power tabulations are available with the authors, for other values of m and n and also for $\alpha = .01$.

7. Power comparisons. Powers of individual roots for test of hypothesis (i) may first be compared. Cases $p = 2$ and $p = 3$ may be considered separately.

$p = 2$. When $p = 2$, the following observations may be made (Table 9).

1) Although the larger root has generally more power than the smaller root, for small values of n the smaller root has generally greater power for small deviations (except for $m = 0$).

2) For $\rho_1^2 + \rho_2^2$ = constant, while the power of the larger root is greater (and quite so for large deviations) for $\rho_1 = 0$, that of the smaller root is greater generally for small deviations and, almost equal to that of the larger root (and greater for small n), for larger deviations when $\rho_1 = \rho_2$. The power of the larger root decreases as the two roots tend to be equal while the smaller root increases.

3) The individual root possesses monotonicity property of power with respect to individual population root but not with respect to their sum or product.

4) For large values of n, the power of the larger root is greater or at least equal to that of the smaller root.

p = 3. The following observations may be made when $p = 3$: (Table 7)

1') Although the largest root has generally more power than the other two roots, for small values of n the median root has greater power. The power of the smallest root generally stays below those of the others.

2') For $\rho_1^2 + \rho_2^2$ = constant, while the power of the largest root is greater for $\rho_1 = \rho_2 = 0$, that of the median root is greater when $\rho_1 = \rho_2 = \rho_3$. The power of the largest root decreases as the roots tend to be equal while those of the other two increase.

3') is the same as 3) above for $p = 2$.

4') For large n, the power of the largest root is generally greater than those of the others except when the population roots tend to be equal in which case the median root shows larger power.

Now consider the powers of individual roots for test of hypothesis (ii). As before, first let us consider $p = 2$.

p = 2. The findings are somewhat similar to those for test (i), (Table 10).

1'') is the same as 1) and

2'') is the same as 2) with ρ_i^2 changed to ω_i, $i = 1,2$. Further,

3'') and 4'') are the same as 3) and 4) respectively.

p = 3. The observations are somewhat similar to those for test (ii), (Table 8).

1''') Although the largest root has generally more power than the other two roots, for smaller values of n the median root has greater power for small deviations. The power of the smallest root stays below those of the other roots (except in a few cases for small values of n).

2''') For $\omega_1 + \omega_2$ = constant, while the power of the largest root is greater for $\omega_1 = \omega_2 = 0$, that of the median root is greater when $\omega_1 = \omega_2 = \omega_3$ for small deviations. The power of the largest root decreases as the roots tend to be equal while those of the other two increase.

3''') is the same as 3).

4''') For large n, the power of the largest root is greater than those of the others.

It may be pointed out further that the monotonicity property of the power of the individual roots with respect to individual population roots, for tests of hypotheses (i) and (ii) was shown earlier by several authors (Roy and Mikhail, 1961; DasGupta, Anderson and Mudholkar, 1964; Anderson and DasGupta, 1964).

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Table 1

Upper 5% points of the smallest root for p = 2

n \ m	0	1	2	3	4	5	7	10	15
5	.20582	.32254	.40706	.47207	.52392	.56633	.63170	.69943	.76981
10	.12212	.20256	.26731	.32152	.36785	.40804	.47449	.54988	.63645
15	.086781	.14754	.19882	.24352	.28310	.31850	.37940	.45233	.54164
20	.067297	.116008	.15824	.19592	.23001	.26110	.31593	.38399	.47115
25	.054956	.095574	.13140	.16387	.19366	.22119	.27060	.33352	.41679
30	.046439	.081258	.11234	.14083	.16722	.19185	.23662	.29474	.37362
40	.035450	.062524	.087075	.10990	.13135	.15161	.18911	.23909	.30945
60	.024061	.042791	.060057	.076362	.091904	.10680	.13490	.17352	.23027
80	.018211	.032525	.045834	.058504	.070676	.082427	.10484	.13616	.18333
100	.014649	.026232	.037058	.047415	.057414	.067111	.085736	.11204	.15246*
130	.011326	.020331	.028789	.036918	.044802	.052483	.067329	.088513	.12169*
160	.0092318	.016597	.023537	.030227	.036733	.043090	.055428	.073151	.10124*
200	.0074060	.013333	.018932	.024343	.029620	.034789	.044857	.059404	.082697*
300	.0049557	.0089377	.012713	.016375	.019958	.023480	.030374	.040423*	.056740*
500	.0029823	.0053864	.0076726	.009897	.012079	.014229	.018456	.024661*	.034851*
1000	.0014945	.0027022	.0038532	.0049753	.0060786	.0071682	.0084013	.010231*	.012288*

*Values extrapolated

Table 2

Upper 1% points of the smallest root for p = 2

n \ m	0	1	2	3	4	5	7	10	15
5	.29830	.41613	.49670	.55663	.60335	.64093	.69783	.75560	.81448
10	.18145	.26844	.33502	.38921	.43454	.47324	.53613	.60604	.68468
15	.13025	.19781	.25229	.29849	.33863	.37401	.43388	.50419	.58855
20	.10156	.15654	.20219	.24191	.27718	.30891	.36402	.43115	.51544
25	.083222	.12950	.16866	.20330	.23454	.26303	.31340	.37641	.45822
30	.070490	.11042	.14466	.17531	.20324	.22897	.27509	.33391	.41231
40	.053973	.085275	.11259	.13742	.16039	.18183	.22098	.27231	.34332
60	.036748	.058588	.077997	.095939	.11279	.12877	.15853	.19882	.25708
80	.027857	.044621	.059661	.073687	.086973	.099662	.12358	.15653	.20540
100	.022430	.036031	.048304	.059812	.070769	.081287	.10126	.12907	.17124*
130	.017358	.027958	.037575	.046638	.055311	.063675	.079662	.10217	.13701*
160	.014156	.022840	.030745	.038220	.045394	.052334	.065659	.084552	.11416*
200	.011362	.018359	.024748	.030806	.036636	.042291	.053191	.068743	.093379*
300	.0076080	.012317	.016635	.020745	.024715	.028580	.036068	.046854*	.064199*
500	.0045809	.0074282	.010047	.012548	.014971	.017337	.021941	.028624*	.039498*
1000	.0022965	.0037285	.0050489	.0063127	.0075400	.0087409	.0093410	.010554*	.011479*

*Values extrapolated

Table 3

Upper 5% points of the smalles root for p = 3

n \ m	0	1	2	3	4	5	7	10	15
5	.13295	.22860	.30516	.36795	.42038	.46481	.53600	.61323	.69726
10	.079847	.14384	.19956	.24859	.29204	.33079	.39693	.47489	.56809
15	.057048	.10492	.14825	.18770	.22375	.25679	.31522	.38760	.47947
20	.044375	.082571	.11792	.15077	.18135	.20986	.26143	.32745	.41483
25	.036309	.068072	.097899	.12599	.15246	.17743	.22333	.28347	.36558
30	.030724	.057904	.083687	.10820	.13151	.15369	.19492	.24992	.32679
40	.023495	.044585	.064857	.084374	.10316	.12124	.15539	.20208	.26960
60	.015977	.030536	.044728	.058577	.072082	.085243	.11056	.14615	.19972
80	.012104	.023219	.034135	.044862	.055394	.065729	.085805	.11447	.15861
100	.0097422	.018731	.027598	.036350	.044981	.053486	.070108	.094074	.13167*
130	.0075364	.014521	.021856	.028297	.035087	.041805	.055012	.074237	.10492*
160	.0061451	.011856	.017528	.023165	.028761	.034312	.045266	.061309	.087446*
200	.0049313	.0095253	.014099	.018654	.023187	.027693	.036616	.049762*	.071377*
300	.0033011	.0063865	.0094677	.012546	.015619	.018683	.024779	.033830*	.048902*
500	.0019872	.0038495	.0057140	.0075816	.0094504	.011319	.015049*	.020625*	.030004*
1000	.0009959*	.0019312*	.0028696*	.0038116*	.0047560*	.0057021*	.0068507*	.0085560*	.010579*

*Values extrapolated

Table 4

Upper 1% points of the smallest root for p = 3

n \ m	0	1	2	3	4	5	7	10	15
5	.19691	.30010	.37797	.43968	.48179	.53188	.59776	.66772	.74228
10	.12008	.19217	.25176	.30260	.34669	.38539	.45026	.52577	.61284
15	.086341	.14127	.18863	.23053	.26805	.30194	.36085	.43241	.52140
20	.067397	.11167	.15078	.18615	.21844	.24813	.30099	.36739	.45356
25	.055268	.092320	.12552	.15608	.18431	.21059	.25814	.31933	.40128
30	.046838	.078685	.10759	.13438	.15840	.18290	.22595	.28237	.35978
40	.035889	.060740	.083629	.10513	.12548	.14481	.18084	.22926	.29809
60	.024455	.041712	.057855	.073241	.088006	.10227	.12922	.16659	.22193
80	.018546	.031762	.044224	.056194	.067767	.078993	.10053	.13082	.17675
100	.014937	.025644	.035791	.045584	.055095	.064364	.082257	.10769	.14702*
130	.011568	.019896	.027831	.035523	.043127	.050370	.064636	.085115	.11737*
160	.0094309	.016253	.022767	.029100	.035296	.041375	.053233	.070365	.097658*
200	.0075705	.013063	.018322	.023448	.028474	.033418	.043095	.057166*	.079807*
300	.0050701	.0087639	.012313	.015783	.019197	.022567	.029195	.038914*	.054766*
500	.0030532	.0052850	.0074357	.0095436	.011624	.013682	.017748*	.023751*	.033650*
1000	.0015306*	.0026527*	.0037362*	.0048011*	.0058540*	.0068981*	.0075558*	.0087571*	.0097797*

*Values extrapolated

Table 5

Upper 5% points of the median root, for p = 3

n \ m	0	1	2	3	4	5	7	10	15
5	.36344	.46055	.52997	.58280	.62461	.65861	.71073	.76440	.81988
10	.23104	.30763	.36800	.41766	.45956	.49552	.55431	.62013	.69473
15	.16915	.23054	.28124	.32461	.36247	.39596	.45280	.51982	.60058
20	.13338	.18427	.22744	.26528	.29900	.32940	.38230	.44689	.52818
25	.11008	.15343	.19088	.22421	.25436	.28189	.33064	.39169	.47106
30	.093707	.13143	.16442	.19413	.22128	.24631	.29122	.34853	.42495
40	.072219	.10213	.12872	.15303	.17556	.19661	.23506	.28548	.35519
60	.049508	.088674	.089732	.10749	.12420	.14004	.16957	.20951	.26720
80	.037662	.053970	.068867	.082828	.096075	.10873	.13260	.16544	.21464*
100	.030391	.043672	.055873	.067369	.078334	.088863	.10885	.13667	.17926*
130	.023566	.033953	.043547	.052633	.061341	.069743	.085801	.10839	.14368*
160	.019244	.027772	.035676	.043186	.050405	.057392	.070805	.089808	.11986*
200	.015463	.022348	.028748	.034846	.040725	.046429	.057422	.073110*	.098196*
300	.010369	.015016	.019353	.023500	.027514	.031423	.038995	.049907*	.067616*
500	.0062511	.0090667	.011703	.014232	.016687	.019085	.023751	.030527*	.041654*
1000	.0031367	.0045549	.0058861	.0071664	.0084118	.0096311	.011987*	.012996*	.015336*

*Values extrapolated

Table 6

Upper 1% points of the median root, for p = 3

n \ m	0	1	2	3	4	5	7	10	15
5	.45317	.54339	.60601	.65276	.68924	.71859	.76307	.80825	.85433
10	.29593	.37273	.43172	.47940	.51910	.55282	.60730	.66742	.73457
15	.21921	.28277	.33403	.37719	.41440	.44699	.50169	.56529	.64080
20	.17399	.22760	.27211	.31051	.34436	.37459	.42663	.48933	.56709
25	.14420	.19039	.22946	.26374	.29440	.32217	.37083	.43099	.50809
30	.12311	.16361	.19833	.22916	.25704	.28253	.32782	.38490	.45997
40	.095240	.12766	.15596	.18149	.20492	.22664	.26595	.31689	.38642
60	.065548	.088674	.10924	.12812	.14573	.16231	.19294	.23394	.29244
80	.049966	.067922	.084049	.098992	.11305	.12639	.15134	.18533	.23577*
100	.040369	.055039	.068296	.080650	.092335	.10348	.12448	.15343	.19738*
130	.031340	.042848	.053308	.063108	.072424	.081357	.098303	.12193	.15857*
160	.025611	.035078	.043713	.051832	.059576	.067024	.081219	.10115	.13248*
200	.020592	.028247	.035253	.041859	.048179	.054274	.065939	.082444*	.10875*
300	.013821	.018999	.023758	.028264	.032591	.036781	.044844	.056371*	.075031*
500	.0083375	.011481	.014379	.017133	.019787	.022364	.027346	.034527*	.046297*
1000	.0041858	.0057711	.0072372	.0086335	.0099821	.011295	.013352*	.016334*	.020597*

*Values extrapolated

Table 7

Powers of individual roots for $p = 3$ for testing $\rho_1 = 0, \rho_2 = 0, \rho_3 = 0$ against different simple alternative hypotheses, $\alpha = .05$

ρ_1^2	ρ_2^2	ρ_3^2	$r_3^{2(3)}$	$r_2^{2(3)}$	$r_1^{2(3)}$	$r_3^{2(3)}$	$r_2^{2(3)}$	$r_1^{2(3)}$	$r_3^{2(3)}$	$r_2^{2(3)}$	$r_1^{2(3)}$
			m = 0, n = 5			m = 0, n = 15			m = 0, n = 40		
.00125	.00125	.00125	.05093	.05105	.05079	.05265	.05253	.05175	.05708	.05634	.05421
0	0	.012	.05306	.05337	.05250	.05906	.05810	.05538	.07620	.06979	.06205
.004	.004	.004	.05300	.05340	.05257	.05867	.05841	.05578	.07384	.07195	.06446
.000125	.0025	.05	.0649	.0655	.0609	.0992	.0876	.0724	.2177	.1403	.0954
.005	.015	.08	.0803	.0821	.0722	.1550	.1334	.0986	.4092	.2580	.1508
			m = 1, n = 5			m = 1, n = 15			m = 1, n = 40		
.00125	.00125	.00125	.05072	.05088	.05076	.05204	.05210	.05165	.05543	.05520	.05389
0	0	.012	.05238	.05285	.05242	.05695	.05680	.05515	.06998	.06669	.06156
.004	.004	.004	.05234	.05286	.05247	.05667	.05693	.05542	.06826	.06775	.06320
.000125	.0025	.05	.0616	.0632	.0607	.0877	.0822	.0721	.1805	.1292	.0957
.005	.015	.08	.0735	.0771	.0717	.1310	.1199	.0969	.3415	.2305	.1508
			m = 2, n = 5			m = 2, n = 15			m = 2, n = 40		
.00125	.00125	.00125	.05062	.05078	.05072	.05171	.05184	.05154	.05452	.05450	.05357
0	0	.012	.05202	.05254	.05231	.05581	.05598	.05483	.06652	.06465	.06082
.004	.004	.004	.05199	.05254	.05235	.05559	.05605	.05504	.06517	.06527	.06203
.000125	.0025	.05	.0598	.0618	.0603	.0813	.0786	.0711	.1585	.1210	.0940
.005	.015	.08	.0699	.0741	.0708	.1173	.1116	.0943	.2977	.2116	.1469
			m = 5, n = 5			m = 5, n = 15			m = 5, n = 40		
.00125	.00125	.00125	.05047	.05064	.05065	.05125	.05144	.05131	.05322	.05342	.05294
0	0	.012	.05154	.05207	.05208	.05420	.05471	.05417	.06157	.06130	.05916
.004	.004	.004	.05152	.05206	.05210	.05406	.05471	.05429	.06074	.06148	.05983
.000125	.0025	.05	.0574	.0597	.0594	.0723	.0727	.0686	.1263	.1062	.0888
.005	.015	.08	.0651	.0697	.0689	.0982	.0982	.0887	.2299	.1753	.1329

Table 8. Powers of individual roots for p = 3 for testing $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = 0$ against different simple alternative hypotheses $\alpha = .05$

ω_1	ω_2	ω_3	$\ell_3^{(3)}$	$\ell_2^{(3)}$	$\ell_1^{(3)}$	$\ell_3^{(3)}$	$\ell_2^{(3)}$	$\ell_1^{(3)}$	$\ell_3^{(3)}$	$\ell_2^{(3)}$	$\ell_1^{(3)}$
			m = 0, n = 5			m = 1, n = 5			m = 2, n = 5		
.001	.001	.001	.050041	.050046	.050035	.050029	.050035	.050030	.050022	.050028	.050026
0	0	.825	.06218	.06272	.05900	.05845	.05975	.05801	.05648	.05791	.05701
.125	.250	.500	.06249	.06444	.06080	.05871	.06077	.05925	.05671	.05863	.05794
.275	.275	.275	.06168	.06370	.06036	.05816	.06018	.05882	.05629	.05814	.05755
0	0	2	.0825	.0806	.0699	.0723	.0739	.0683	.0670	.0695	.0663
0	1	1	.0803	.0853	.0752	.0709	.0760	.0717	.0659	.0707	.0686
0	0	3	.1025	.0953	.0777	.0858	.0859	.0761	.0771	.0795	.0735
1	1	1	.0970	.1082	.0945	.0829	.0918	.0863	.0755	.0827	.0804
0	0	5	.1491	.1224	.0898	.1178	.1091	.0889	.1013	.0993	.0859
0	0	6	.1754	.134	.0943	.1365	.120	.0939	.1161	.109	.0909
2	2	2	.1754	.171	.1417	.1460	.135	.1232	.1300	.116	.1110
0	0	8	.233	.155	.1017	.180	.139	.1010	.153	.125	.0980
2	2	4	.256	.197	.1542	.217	.154	.1349	.195	.130	.1218
0	0	10	.297	.171	.1035	.233	.152	.1038	.201	.136	.1006
			m = 5, n = 5			m = 0, n = 15			m = 1, n = 15		
.001	.001	.001	.050013	.050018	.050018	.050055	.050053	.050036	.050040	.050042	.050033
0	0	.825	.05382	.05507	.05500	.06690	.06448	.05935	.06223	.06161	.05860
.125	.250	.500	.05386	.05545	.05552	.06710	.06658	.06128	.06245	.06290	.06000
.275	.275	.275	.05374	.05514	.05523	.06593	.06574	.06083	.06161	.06220	.05954
0	0	2	.0598	.0626	.0619	.0967	.0848	.0706	.0836	.0785	.0696
0	1	1	.0594	.0629	.0629	.0924	.0909	.0763	.0807	.0815	.0734
0	0	3	.0655	.0691	.0675	.1272	.1014	.0786	.1054	.0928	.0778
1	1	1	.0653	.0700	.0705	.1154	.1181	.0968	.0978	.1011	.0896
0	0	5	.0801	.0822	.0777	.1998	.1314	.0909	.1585	.1202	.0912
0	0	6	.0903	.0883	.0819	.241	.145	.0956	.1895	.133	.0965
2	2	2	.108	.0986	.0919	.215	.193	.1467	.1787	.156	.1301
0	0	8	.121	.0989	.0880	.329	.167	.1023	.259	.155	.1043
2	2	4	.164	.1062	.1029	.309	.224	.1593	.261	.180	.1423
0	0	10	.168	.115	.0900	.420	.184	.1061	.336	.171	.1083

Table 8 (Cont'd.)

ω_1	ω_2	ω_3	$\ell_3^{(3)}$	$\ell_2^{(3)}$	$\ell_1^{(3)}$	$\ell_3^{(3)}$	$\ell_2^{(3)}$	$\ell_1^{(3)}$	$\ell_3^{(3)}$	$\ell_2^{(3)}$	$\ell_1^{(3)}$
				m = 2, n = 15			m = 5, n = 15			m = 0, n = 40	
.001	.001	.001	.050032	.050035	.050029	.050021	.050024	.050022	.050063	.050055	.050037
0	0	.825	.05968	.05975	.05773	.05605	.05670	.05589	.06951	.06529	.05951
.125	.250	.500	.05989	.06068	.05882	.05624	.05722	.05653	.06964	.06758	.06149
.275	.275	.275	.05923	.06008	.05839	.05584	.05680	.05619	.06824	.06669	.06104
0	0	2	.0763	.0742	.0679	.0661	.0667	.0640	.1048	.0867	.0709
0	1	1	.0742	.0759	.0707	.0650	.0673	.0653	.0992	.0935	.0768
0	0	3	.0932	.0866	.0758	.0760	.0756	.0705	.1413	.1041	.0790
1	1	1	.0880	.0915	.0842	.0740	.0771	.0746	.1256	.1228	.0978
0	0	5	.1348	.1111	.0891	.1011	.0936	.0822	.2288	.1354	.0914
0	0	6	.1596	.123	.0945	.117	.102	.0872	.278	.149	.0961
2	2	2	.1577	.135	.1187	.127	.104	.0995	.238	.204	.1488
0	0	8	.217	.143	.1025	.158	.117	.0945	.381	.172	.1030
2	2	4	.233	.155	.1303	.190	.116	.1101	.339	.236	.1615
0	0	10	.285	.158	.1062	.214	.168	.0974	.485	.190	.1072
				m = 1, n = 40			m = 2, n = 40			m = 5, n = 40	
.001	.001	.001	.050047	.050045	.050034	.050039	.050038	.050030	.050026	.050027	.050024
0	0	.825	.06452	.06254	.05887	.06174	.06074	.05809	.05771	.05773	.05639
.125	.250	.500	.06466	.06398	.06034	.06190	.06180	.05925	.05789	.05835	.05712
.275	.275	.275	.06365	.06323	.05987	.06109	.06114	.05881	.05737	.05787	.05675
0	0	2	.0906	.0808	.0701	.0826	.0767	.0687	.0710	.0694	.06514
0	1	1	.0866	.0843	.0742	.0796	.0788	.0718	.0694	.0702	.0668
0	0	3	.1179	.0962	.0785	.1043	.0904	.0768	.0846	.0798	.0722
1	1	1	.1068	.1060	.0911	.0962	.0964	.0861	.0805	.0818	.0770
0	0	5	.1849	.1255	.0922	.159	.1173	.0906	.1190	.1007	.0848
0	0	6	.224	.139	.0977	.191	.130	.0962	.140	.111	.0901
2	2	2	.199	.167	.1333	.176	.146	.1226	.142	.114	.1042
0	0	8	.310	.162	.1058	.264	.152	.1047	.193	.129	.0981
2	2	4	.289	.194	.1457	.258	.168	.1345	.211	.129	.1151
0	0	10	.402	.180	.1104	.346	.169	.1091	.259	.142	.1018

Table 9. Powers of individual roots for $p = 2$ for testing

$\rho_1 = 0$, $\rho_2 = 0$ against different simple alternate hypotheses, $\alpha = .05$

ρ_1^2	ρ_2^2	$r_2^{2(2)}$	$r_1^{2(2)}$	$r_2^{2(2)}$	$r_1^{2(2)}$	$r_2^{2(2)}$	$r_1^{2(2)}$	$r_2^{2(2)}$	$r_1^{2(2)}$
		$m = 0$, $n = 5$		$m = 1$, $n = 5$		$m = 2$, $n = 5$		$m = 5$, $n = 5$	
0	.001	.050395	.050357	.050290	.050303	.050241	.050271	.050180	.050225
0	.005	.051999	.051785	.051465	.051522	.051217	.051362	.050910	.051129
0	.02	.05836	.05716	.05612	.05615	.05508	.05553	.05379	.05460
.01	.01	.05814	.05753	.05596	.05633	.05495	.05564	.05370	.05465
.005	.03	.06494	.06310	.06095	.06115	.05908	.05998	.05676	.05828
.01	.015	.06027	.05951	.05753	.05799	.05625	.05711	.05466	.05586
.002	.075	.08756	.07817	.07754	.07480	.07278	.07255	.06684	.06906
0	.1	.1026	.0858	.0886	.0822	.0819	.0796	.0735	.0753
.05	.05	.0961	.0957	.0838	.0873	.0780	.0829	.0708	.0768
0	.15	.1402	.1031	.1168	.0991	.1053	.0958	.0907	.0899
		$m = 0$, $n = 15$		$m = 1$, $n = 15$		$m = 2$, $n = 15$		$m = 5$, $n = 15$	
0	.001	.051126	.050858	.050815	.050716	.050666	.050628	.050473	.050495
0	.005	.055777	.054277	.054179	.053588	.053410	.053153	.052419	.052492
0	.02	.07532	.06685	.06832	.06441	.06491	.06278	.06048	.06021
.01	.01	.07398	.06929	.06737	.06565	.06416	.06359	.06001	.06058
.005	.03	.09615	.08261	.08349	.07714	.07724	.07385	.06910	.06884
.01	.015	.08055	.07460	.07214	.06990	.06805	.06726	.06273	.06342
.002	.075	.1772	.1133	.1439	.1067	.1266	.1016	.1033	.0927
0	.1	.2332	.1244	.1874	.1198	.1628	.1150	.1286	.1054
.05	.05	.1986	.1833	.1599	.1542	.1399	.1391	.1129	.1179
0	.15	.3673	.1510	.2976	.1595	.2573	.1453	.1977	.1346
		$m = 0$, $n = 40$		$m = 1$, $n = 40$		$m = 2$, $n = 40$		$m = 5$, $n = 40$	
0	.001	.052984	.052102	.052152	.051737	.051746	.051507	.051211	.051152
0	.005	.065803	.060359	.061389	.058671	.059219	.057567	.056352	.055820
0	.02	.1258	.0889	.1050	.0840	.0944	.0804	.0801	.07396
.01	.01	.1180	.1039	.0994	.0921	.0900	.0857	.0774	.0765
.005	.03	.1908	.1346	.1538	.1196	.1342	.1107	.1071	.0966
.01	.015	.1381	.1197	.1142	.1042	.1019	.0960	.0855	.0840
.002	.075	.4495	.1784	.3672	.1711	.3168	.1634	.2373	.1467
0	.1	.594	.177	.503	.184	.441	.181	.336	.167
.05	.05	.486	.451	.395	.368	.340	.321	.254	.251
0	.15	.823	.207	.747	.264	.687	.274	.565	.238

Table 10. Powers of individual roots for $p = 2$ for testing $\omega_1 = 0$, $\omega_2 = 0$ against different simple alternative hypotheses, $\alpha = .05$

ω_1	ω_2	$\ell_2^{(2)}$	$\ell_1^{(2)}$	$\ell_2^{(2)}$	$\ell_1^{(2)}$	$\ell_2^{(2)}$	$\ell_1^{(2)}$	$\ell_2^{(2)}$	$\ell_1^{(2)}$
		m = 0, n = 5		m = 1, n = 5		m = 2, n = 5		m = 5, n = 5	
0	.001	.050025	.050022	.050016	.050017	.050012	.050014	.050007	.050009
0	.2	.05503	.05442	.05327	.05336	.05244	.05271	.05140	.05173
.1	.1	.05498	.05460	.05324	.05344	.05242	.05275	.05139	.05174
0	1	.07723	.07123	.06775	.06658	.06298	.06351	.05731	.05870
.5	.5	.07607	.07560	.06690	.06850	.06256	.06461	.05714	.05907
0	3	.1452	.1070	.1111	.0973	.0947	.0896	.0743	.0763
1.5	1.5	.1357	.1435	.1055	.1147	.0909	.0999	.0728	.0798
1	2	.1368	.1394	.1061	.1128	.0914	.0988	.0730	.0794
.5	4	.1973	.1580	.1455	.1325	.1198	.1169	.0878	.0926
0	5	.226	.134	.164	.124	.133	.1135	.0944	.0936
2.5	2.5	.203	.227	.150	.171	.123	.143	.0902	.1041
0	8	.358	.162	.257	.155	.202	.144	.130	.1181
4	4	.310	.364	.223	.269	.178	.218	.119	.146
0	10	.446	.176	.324	.172	.253	.161	.158	.133
5	5	.383	.458	.276	.340	.218	.273	.141	.178
		m = 0, n = 15		m = 1, n = 15		m = 2, n = 15		m = 5, n = 15	
0	.001	.050031	.050024	.050021	.050019	.050017	.050016	.050010	.050011
0	.2	.05639	.05473	.05438	.05376	.05339	.05314	.05208	.05215
.1	.1	.05632	.05493	.05433	.05385	.05336	.05319	.05207	.05218
0	1	.08541	.07266	.07414	.06851	.06855	.06565	.06119	.06088
.5	.5	.08348	.07765	.07291	.07092	.06769	.06711	.06078	.06144
0	3	.1780	.1103	.1381	.1024	.1174	.0957	.0894	.0829
1.5	1.5	.1625	.1518	.1275	.1242	.1096	.1096	.0856	.0885
1	2	.1642	.1471	.1287	.1218	.1105	.1080	.0860	.0879
.5	4	.2481	.1659	.1886	.1430	.1565	.1284	.1123	.1043
0	5	.289	.138	.219	.131	.180	.123	.125	.104
2.5	2.5	.252	.243	.192	.190	.159	.162	.115	.121
0	8	.463	.166	.358	.163	.292	.156	.193	.134
4	4	.393	.391	.299	.303	.244	.253	.166	.179
0	10	.571	.179	.453	.180	.374	.174	.245	.152
5	5	.485	.491	.374	.383	.306	.320	.203	.222
		m = 0, n = 40		m = 1, n = 40		m = 2, n = 40		m = 5, n = 40	
0	.001	.050034	.050025	.050024	.050020	.050019	.050017	.050012	.050012
0	.2	.05705	.05485	.05497	.05394	.05393	.05335	.05255	.05240
.1	.1	.05696	.05507	.05491	.05404	.05389	.05341	.05253	.05243
0	1	.08943	.07324	.07773	.06939	.07184	.06669	.06390	.06214
.5	.5	.08709	.07851	.07615	.07203	.07068	.06836	.06329	.06284
0	3	.1942	.1116	.1532	.1047	.1313	.0986	.1006	.0867
1.5	1.5	.1756	.1553	.1396	.1287	.1207	.1144	.0948	.0937
1	2	.1777	.1504	.1411	.1259	.1219	.1126	.0954	.0930
.5	4	.2726	.1691	.2125	.1476	.1790	.1340	.1306	.1111
0	5	.319	.140	.249	.134	.209	.127	.149	.110
2.5	2.5	.276	.249	.215	.199	.181	.171	.133	.131
0	8	.509	.168	.410	.167	.346	.161	.241	.142
4	4	.432	.402	.339	.319	.284	.271	.199	.199
0	10	.623	.181	.516	.184	.442	.179	.311	.161
5	5	.530	.505	.425	.403	.358	.343	.250	.249

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