On the Distribution of the Maximum of a Semi-Markov Process

by

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INTRODUCTION

If $\{X(t): t \geq 0\}$ is a separable stochastic process, the problem of computing the distribution of $Z(t) = \sup[X(s): 0 \leq s \leq t]$ is of great interest particularly in level crossing (detection) problems and in queuing theory.

Spitzer [5] used combinatorial methods to find the distribution of $Z(t)$ in the case of a discrete time random walk. In [1] Baxter used operator theoretic techniques to give a characterization of the distribution of $Z(t)$ and many other functionals on a discrete time Markov process. In the case of continuous time processes with stationary independent increments Baxter and Donsker [2] obtained the double Laplace transform of the distribution of $Z(t)$. Using a generalization of the classical ballot theorem, Takács [6], has computed the distribution of $Z(t)$ for many interesting cases involving processes with interchangeable increments.

However, there are many cases in which one must deal with continuous time Markov processes and semi-Markov processes. The purpose of this paper is to extend the results of Baxter [1] by characterizing the distribution of $Z(t)$ for a wide class of semi-Markov processes.

Define $m_{ij}(s)$ to be the Laplace transform of the function $M_{ij}(t) = P[Z(t)=j \mid X_0 = i]$ and let $m(s) = (m_{ij}(s))$. The main result of this paper is in the form of a recurrence relation for m(s)

$$m(s) = g(s) + (q(s)\ m(s))^{\sigma}$$

where q(s) and g(s) are matrices whose elements are Laplace transforms of distributions which occur in the definition of the semi-Markov process and σ is an operator on matrices. Moreover, m(s) is the unique solution of the above equation under a condition on the matrices q(s) which guarentees that the process makes a finite number of transitions in any finite interval of time.

The main result appears in Section III. In Sections IV and V, continuous and discrete time Markov chains are considered as special cases of semi-Markov processes, and specific results are determined for them. For processes with stationary independent increments, Spitzer's identity is derived. In Section VI, some other functionals are characterized by methods similar to those in Section III. The paper concludes with a class of examples in which the recurrence relation for m(s) can be solved to give $m_{ij}(s)$.

Our indebtedness to the methods of Baxter [1] will be apparent.

I PRELIMINARIES

First it is necessary to discuss linear operations defined on a space, $\mathcal{L}$, of bounded sequences, $\{s_i\}$ $i \in I$, where I may be an arbitrary subset of the integers. The exact nature of $\mathcal{L}$ will depend on the state space of the semi-Markov process in question. For us, the important properties of $\mathcal{L}$ are that it is a Banach space under the supremum norm and that any bounded linear operator, A, on $\mathcal{L}$ into $\mathcal{L}$ is of the form

(1.1) $$A\ \{s_i\} = \{\sum_j a_{ij}\ s_j\}$$

where $\sum_j |a_{ij}|$ is uniformly bounded in i. Clearly we may identify A with the matrix (a_{ij}), and the norm of $A = ||A|| = \sup_i (\sum_j |a_{ij}|)$.

DEFINITION 1.1 For any operator A of the form (1.1) we define an operator A^σ by

$$A^\sigma = (a^\sigma_{ij}) \text{ where } a^\sigma_{ij} = \begin{cases} a_{ij} & \text{if } j > i \\ \sum_{k \leq i} a_{i,k} & \text{if } j = i \\ 0 & \text{if } j < i \end{cases}$$

We also define $A^\tau = A - A^\sigma$

Let I be the identity operator, then the following properties hold.

(i) $I^\sigma = I$ (ii) $I^\tau = 0$

(iii) $(A^\sigma)^\sigma = A^\sigma$ (iv) $(A^\tau)^\tau = A^\tau$

(v) $(A^\sigma B^\sigma)^\sigma = A^\sigma B^\sigma$ (vi) $(A^\tau B^\tau)^\tau = A^\tau B^\tau$

(vii) $||A^\sigma|| \le ||A||$ (viii) $||A^\tau|| \le 2||A||$

(1.2) (ix) $(\alpha A + \beta B)^\sigma = \alpha A^\sigma + \beta B^\sigma$

(x) If $A_0 + A_1 + \cdots$ is a series of bounded operators of the form (1.1) whose partial sums form a Cauchy sequence in the operator norm, then $T = A_0 + A_1 + \cdots$ is a bounded linear operator of the form (1.1). Moreover, $A_0^\sigma + A_1^\sigma + \cdots$ and $A_0^\tau + A_1^\tau + \cdots$ converge in the operator norm, also $T^\sigma = A_0^\sigma + A_1^\sigma + \cdots$ and $T^\tau = A_0^\tau + A_1^\tau + \cdots$.

(xi) If $A = A_1 + A_2$ where $A_1 = A_1^\sigma$ and $A_2 = A_2^\tau$, then $A_1 = A^\sigma$ and $A_2 = A^\tau$.

We prove only (x). Since the space of bounded linear operators on $\mathfrak{L}$ into $\mathfrak{L}$ is a Banach space, T is a bounded linear operator on $\mathfrak{L}$ and, therefore, must be of the form (1.1). Let $T_n = A_0 + A_1 + \cdots + A_n$. Since $|| T_n^\sigma - T^\sigma || \le || T_n - T ||$ and $|| T^\tau - T_n^\tau || \le 2 \, || T - T_n ||$, the second statement in (x) follows.

Note that properties (i) - (xi) say that any bounded linear operator A on $\mathfrak{L}$ into $\mathfrak{L}$ can be split uniquely into the sum of two operators A^σ and A^τ each of which is an element in a proper subspace of the Banach algebra of bounded linear operators on $\mathfrak{L}$ into $\mathfrak{L}$.

II DEFINITION OF A SEMI-MARKOV PROCESS

Following Yackel [7], we take $\{X_t, t \geq 0\}$ to be a separable process with a countable state space I. Let

$$Y_t = \begin{cases} t \text{ if } X_s = X_t \text{ for all } 0 \leq s \leq t \\ t - \sup[s: 0 \leq s \leq t;\ X_s \neq X_t] \text{ otherwise} \end{cases}$$

If the two dimensional process $\{(X_t, Y_t): t \geq 0\}$ is a strong Markov process with stationary Borel measurable transition probabilities, then we say that $\{X_t, t \geq 0\}$ is a semi-Markov process (S.M.P.).

DEFINITION 2.1. Let

$$w_t = \begin{cases} \inf[s: s \geq t; X_s \neq X_t] \text{ if } X_u = X_t \text{ for } 0 \leq u \leq t \\ \inf[s: s \geq t,\ X_s \neq X_t] - \sup[s: s \leq t,\ X_s \neq X_t] \\ \qquad \text{otherwise} \end{cases}$$

$$F_i(t) = P\,[w_0 \leq t \mid (X_0, Y_0) = (i, 0)]$$

For convenience we shall denote w_0 by w and $P\,[S \mid (X_t, Y_t) = (i, 0)]$ by $P_{i,0}[S]$ where S is a Borel subset of the state space of $\{(X_t, Y_t): t \geq 0\}$.

For this paper we shall require that $F_i(t) \to 0$ as $t \to 0+$ for all $i \in I$. In this case, once the process enters a state it stays there for a positive length of time with probability one. That is, the process is a step process.

DEFINITION 2.2. If $\{X_t: t \geq 0\}$ is a S.M.P. for which $F_i(t) \to 0$ as $t \to 0+$ for all i in I, then we call $\{X_t, t \geq 0\}$ a <u>semi-Markov step process</u> (S.M.S.P.)

DEFINITION 2.3. Let

$$Q_{ij}(t) = P_{i,0}[w \leq t \text{ and } X_w = j] \text{ if } i \neq j$$

$$Q_{ii}(t) = 0$$

$$Z(t) = \sup[X_s: 0 \leq s \leq t]$$

$$M_{ij}(t) = P_{i,0}[Z(t) = j].$$

In this paper we take the point of view that the $Q_{ij}(t)$ are the known and that the distributions of certain functionals on $\{X_t, t \geq 0\}$ are to be solved in terms of them. This is an acceptable point of view even for a continuous parameter Markov chain since the $Q_{ij}(t)$ may be easily calculated from the transition probabilities $p_{ij}(t)$. In fact, in [3] p. 246 it is shown that

$$Q_{ij}(t) = p_{ij}(1 - e^{-c_i t})$$

where $$c_i = \lim_{t \to 0+} (1 - p_{ii}(t))/t$$

and $$c_i p_{ij} = \lim_{t \to 0+} p_{ij}(t)/t.$$

III SEMI-MARKOV PROCESSES

If for each t, A(t) and B(t) are matrices, then let

$$A(t) * B(t) = \left(\sum_k \int_0^t B_{kj}(t-s)\, d\, A_{ik}(s)\right)$$

when this makes sense. In the context of this paper $A_{ij}(t)$ will be a non-decreasing function, and $B_{ij}(t)$ will be a Borel function. The above integrals are to be understood as Lebesgue-Stieltjes integrals with respect to the measures induced by the $A_{ik}(s)$'s.

The next theorem is one of the main results of this paper. There is given an equation involving the known functions $Q_{ij}(t)$ which is satisfied by the distribution $M_{ij}(t)$ of Z(t). This is the generalization to semi-Markov processes of Baxter's results in [1] for discrete Markov processes.

<u>THEOREM 3.1.</u> For a S.M.S.P., let $M(t) = (M_{ij}(t))$ and $Q(t) = (Q_{ij}(t))$. Then for all $t > 0$

(3.1) $M(t) = (\delta_{ij}(1-F_i(t))) + (Q(t) * M(t))^{\sigma}$

and $M_{ij}(t) \to \delta_{ij}$ as $t \to 0+$.

<u>Proof</u> We consider three cases.

<u>Case 1 $j < i$</u> Clearly $M_{ij}(t) = 0$

<u>Case 2 $j > i$</u> If the process starts at i and has maximum $j > i$ over the interval $[0, t]$, then there must have been a transition in $[0, t]$. Since we are dealing with a step process we can suppose that the first jump is to $k = X_w$, where obviously $k \leq j$.

Partitioning on the first jump, we have

$$M_{ij}(t) = P_{i,0}[Z_t = j] = \sum_{k \leq j} P_{i,0}[Z_t = j;\ X_w = k].$$

Let E_k denote expectations taken over $[X_w = k]$. Then

$$P_{i,0}[Z_t = j;\ X_w = k] = E_k(P_{i,0}[Z_t = j \mid (X_w,\ Y_w) = (k,\ 0)])$$

By the strong Markov property and the stationarity of $(X_t,\ Y_t)$, the process regenerates itself at jump times. Thus for $k \leq j$,

$$P_{i,0}[Z_t = j \mid (X_w,\ Y_w) = (k,\ 0)] = P_{k,0}[Z_{t-w} = j] = M_{kj}(t-w),$$

and

$$P_{i,0}[Z_t = j;\ X_w = k] = E_k(M_{kj}(t-w)) = \int_0^t M_{kj}(t-s)\ d\ Q_{ik}(s)$$

by a transformation theorem p. 342 [4].

Finally,

$$M_{ij}(t) = \sum_{k \leq j} \int_0^t M_{kj}(t-s)\ d\ Q_{ik}(s) = \sum_{k \in I} \int_0^t M_{kj}(t-s)\ d\ Q_{ik}(s).$$

<u>Case 3 $i = j$</u> If $Z_t = i$ and $X_0 = i$, there are two possibilities (i) $w > t$. $P_{i,0}[w > t \text{ and } Z_t = i] = P_{i,0}[w > t] = 1-F_i(t)$

(ii) $w \leq t$. Since $Z_t = i$, $X_w < i$, and $Z_{t-w} \leq i$.

So by an argument similar to that given before

$$p_{i,0}[w \le t \text{ and } Z_t = i] = \sum_{j \le i} \sum_{k < i} \int_0^t M_{kj}(t-s) \quad dQ_{ik}(s)$$

Since $Q_{ii}(t) = 0$, and $M_{kj}(t) = 0$ for $k > i$, we may write

$$p_{i,0}[w \le t \text{ and } Z_t = i] = \sum_{j \le i} \sum_{k \in I} \int_0^t M_{kj}(t-s) \quad dQ_{ik}(s)$$

It follows that

$$M_{ii}(t) = 1 - F_i(t) + \sum_{j \le i} \sum_{k \in I} \int_0^t M_{kj}(t-s) \quad Q_{ik}(s).$$

By checking the definition of the σ operator we see that

$$(M_{ij}(t)) = (\delta_{ij}(1-F_i(t))) + (Q(t) * M(t))^{\sigma}$$

To show that $M_{ij}(t) \to \delta_{ij}$ as $t \to 0+$ we observe that

$$0 \le M_{ij}(t) \le p_{i,0}[w \le t] = F_i(t) \to 0 \text{ as } t \to 0+ \ (i \ne j)$$

and

$$1 \ge M_{ii}(t) \ge p_{i,0}[w > t] = 1 - F_i(t) \to 1 \text{ as } t \to 0.$$

This completes the proof.

The convolutions appearing in Theorem 1 suggest that it might be convenient to work with Laplace transforms.

Define

$$m_{ij}(s) = \int_{[0,\infty)} e^{-st} M_{ij}(t)\, dt$$

$$(3.2) \quad g_i(s) = \int_{[0,\infty)} e^{-st}(1 - F_i(t))\, dt$$

$$q_{ij}(s) = \int_{[0,\infty)} e^{-st}\, d\, Q_{ij}(t)$$

where the integrals above are to be understood as Lebesque-

Stieltjes integrals. Let $q(s) = (q_{ij}(s))$, $g(s) = (\delta_{ij}\, g_i(s))$, and $m(s) = (m_{ij}(s))$. We may now write Theorem 3.1 in the following convenient form.

COROLLARY 3.1. For a S.M.S.P.

(3.3) $m(s) = g(s) + (q(s)\, m(s))^{\sigma}$

Notice that we are now using a simple matrix product when we write $q(s)\, m(s)$.

With no restrictions on $M_{ij}(t)$, $m_{ij}(s)$ defines $M_{ij}(t)$ only almost surely. If, however, $M_{ij}(t)$ is continuous then we know that there is only one continuous function whose transform is $m_{ij}(s)$. In this case one may apply the standard inversion formula to recover $M_{ij}(t)$ from $m_{ij}(s)$. Below we give a condition on the $Q_{ij}(t)$'s which assures that $M_{ij}(t)$ will be continuous.

THEOREM 3.2. If $Q'_{ij}(t) = \varphi_{ij}(t)$ is continuous for all i and j then $M_{ij}(t)$ is continuous for all i and j.

Proof Fix i and j. Let

$$f_k(t) = \int_0^t M_{kj}(t-s)\, dQ_{ik}(s) = \int_0^t M_{kj}(t-s)\, \varphi_{ik}(s)\, ds.$$

$$|f_k(t) - f_k(t+h)| =$$

$$= \left|\int_0^{t+h} M_{kj}(t+h-s)\, \varphi_{ik}(s) ds - \int_0^t M_{kj}(t-s)\, \varphi_{ik}(s)\, ds\right|$$

$$\le \left|\int_0^h M_{kj}(t+h-s)\, \varphi_{ik}(s)\, ds\right| + \left|\int_0^t M_{kj}(t-s)\, (\varphi_{ik}(s+h) - \varphi_{ik}(s)) ds\right|$$

The first term goes to zero as $h \to 0$. The second term is less than

$\int_0^t |M_{kj}(t-s)|\ |\varphi_{ik}(s+h) - \varphi_{ik}(s)|\ ds \le \int_0^t |\varphi_{ik}(s+h) - \varphi_{ik}(s)|\ ds$ $\to 0$ as $h \to 0$. Thus $f_k(t)$ is continuous.

If $i > j$, then $M_{ij}(t) = 0$.

If $j > i$, then

$$M_{ij}(t) = \sum_{k \in I} f_k(t)$$

Since $|f_k(t)| \le \int_0^t \varphi_{ik}(s)\ dt = P[w \le t;\ X_w = k] \le P[X_w = k]$ and

$\sum_k P[X_w = k] = 1$, we have that $\sum_k f_k(t)$ converges uniformly to $M_{ij}(t)$ and that $M_{ij}(t)$ is continuous.

If $j = i$, then

$$M_{ii}(t) = (1 - F_i(t)) + \sum_{j \le i} \sum_k \int_0^t M_{kj}(t-s)\ \varphi_{ik}(s)\ ds.$$

Since $Q_{ik}(t)$ is continuous, and $\sum_k Q_{ik}(t)$ converges uniformly to $F_i(t)$, $1 - F_i(t)$ is continuous.

$$\sum_{j \le i} \sum_k \int_0^t M_{kj}(t-s)\ \varphi_{ik}(s)\ ds = \sum_k \int_0^t \sum_{j \le i} M_{kj}(t-s)\ \varphi_{ik}(s)\ ds.$$

By an argument similar to that which showed $f_k(t)$ to be continuous we can show $\int_0^t \sum_{j \le i} M_{kj}(t-s)\ \varphi_{ik}(s)\ ds$ to be continuous. The continuity of $\sum_k \int_0^t \sum_{j \le i} M_{kj}(t-s)\ \varphi_{ik}(s)$ then follows by uniform convergence, and we have that $M_{ii}(t)$ is continuous. This ends the proof.

Although we have shown that m(s) satisfies equation (3.3) in Corollary 3.1, we have no quarantee that m(s) is the only family of matrices satisfying equation (3.3). The aim of the next theorem is to give conditions under which equation (3.3) uniquely determines m(s). Note that m(s) is a bounded operator on $\mathcal{L}$ into $\mathcal{L}$ for every $s > 0$.

THEOREM 3.3. If for some $s_0 > 0$, $\| q(s_0) \| < 1$, then $m(s) = g(s) + (q(s)\, m(s))^{\sigma}$ has a unique bounded solution for $s \geq s_0$.

Proof Let m(s) be a bounded solution of (3.3). Iterating equation (3.3) n times one obtains

$$m(s) = m_0(s) + m_1(s) + \cdots + m_n(s) + L_n(s)$$

where

$$m_0(s) = q(s), \quad m_{n+1}(s) = (q(s)\, m_n(s))^{\sigma}$$

and $$L_n(s) = (q(s)(\cdots (q(s)\, m(s)\underbrace{)^{\sigma} \cdots)^{\sigma}}_{n+1 \text{ times}}$$

So, by properties (1.2) $\|m_n\| \leq \|q(s)\|^n \, \|g(s)\|$

and $\|L_n(s)\| \leq \|q(s)\|^{n+1} \, \| m(s)\|$.

Since $\|g(s)\| < \infty$ and $\|q(s)\| \leq \|q(s_0)\| < 1$ for $s \geq s_0$, the series $\sum_{n=0}^{\infty} m_n(s)$ converges in the strong operator sense, and $\|L_n\| \to 0$ as $n \to \infty$. Thus,

$$m(s) = \sum_{n=0}^{\infty} m_n(s)$$

is the unique bounded solution of equation (3.3).

COROLLARY 3.3. If for a S.M.S.P., $\sum_k Q'_{ik}(t) \leq B$ for all i, then m(s) in (3.2) is the unique bounded solution of equation (3.3).

Proof If $\sum_k Q'_{ik}(t) \leq B$, then $||q(s)|| \leq B/s$. So for $s > B$, $||q(s)|| < 1$ and Theorem 3.3 applies.

In the next theorem we prove an analog of Spitzer's identity for semi-Markov processes under assumptions which are very strong and are satisfied only in special cases. Yet, the method of proof suggests an approach to solving the general case in (3.3). This will be discussed after the next theorem is stated and proved.

THEOREM 3.4. For a S.M.S.P. suppose that $F_i(t) = F(t)$ for all i and that $[q^k(s)]^\sigma\, q(s) = q(s)\, [q^k(s)]^\sigma$ for all $k \geq 0$. If $||q(s_0)|| < 1$ for some s_0, then for $s \geq s_0$ let

$$L = \log(I - q(s)) = -\sum_1^\infty \frac{q^k(s)}{k}$$

and $\eta(s) = \int_0^\infty e^{-st}(1 - F(t))\, dt.$

Then for $s \geq s_0$,

$$m(s) = \eta(s)\, \exp\left(\sum_1^\infty \frac{(q^k(s))^\sigma}{k}\right)$$

Proof The condition $[q^k(s)]^\sigma\, q(s) = q(s)\, [q^k(s)]^\sigma$ guarantees that $\exp(L) = \exp(L^\sigma)\, \exp(L^\tau)$. By Theorem 3.3 we know that for $s \geq s_0$, m(s) is the unique bounded solution of

$$m(s) = g(s) + (q(s)\, m(s))^\sigma$$

which in this case has the form

$$\eta(s)\ I = [(I - q(s))\ m(s)]^{\sigma} = [\exp(L)\ m(s)]^{\sigma}.$$

By verifying that $[\exp(L^{\tau})]^{\sigma} = I$, it is easily seen that for $s \geq s_0$ $\eta(s)\ \exp(-L^{\sigma})$ is a bounded, and hence the unique, bounded solution of the above equation. Thus

$$m(s) = \eta(s)\ \exp\left(\sum_{k=1}^{\infty} \frac{(q^k(s))^{\sigma}}{k}\right)$$

COROLLARY 3.4. Let the S.M.S.P. have the integers for its state space and be spacially homogeneous (i. e. $Q_{ik}(t) = Q(k-i,\ t)$. If the $Q_{ik}(t)$'s are continuous and if $||q(s_0)|| < 1$ for some s_0, then for $s \geq s_0$

$$m(s) = \eta(s)\ \exp\left(\sum_{1}^{\infty} (q^k(s)/k)^{\sigma}\right)$$

Proof $F_i(t) = \sum_{k\in I} Q_{ik}(t) = \sum_{k\in I} Q(k-i,\ t)$ which is independent of i.

We need only show that $q(s)(q^k(s))^{\sigma} = (q^k(s))^{\sigma}q(s)$, and we may apply Theorem 3.4 to get the result. We shall show this by proving that

$$\left(Q_k(t)\right)^{\sigma} * Q(t) = Q(t) * \left(Q_k(t)\right)^{\sigma}$$

where, $Q_1(t) = (Q_{ij}(t))$, and $Q_{n+1}(t) = Q(t) * Q_n(t)$.

Denote the elements of $(Q_k(t))^\sigma$ by $Q_k^\sigma\,(j-i,\ t)$.

$$Q(t)\ *\ \left(Q_k(t)\right)^\sigma = \left(\sum_{k\leq j} \int_0^t Q_k^\sigma(j-k,\ t-s)\ dQ(k-i,\ s)\right)$$

By making the change of variable $Z = j + i - k$ we obtain

$$\left(\sum_{Z\geq i} \int_0^t Q_k^\sigma(Z-i,\ t-s)\ dQ(j-Z,\ s)\right).$$

Integration by parts gives

$$\left(\sum_{Z\geq i} \int_0^t Q(j-Z,\ t-s)\ dQ_k^\sigma(Z-i,\ s)\right) = \left(Q_k(t)\right)^\sigma * Q(t).$$

The construction in Theorem 3.4 is based on a Wiener-Hopf factorization. That is, we write (uniquely)

$$\text{(3.5)} \qquad I - q(s) = \exp(L^\tau)\ \exp(L^\sigma)$$

where $(\exp(L^\tau))^\sigma = I$ and $(\exp(L^\sigma))^\sigma = \exp(L^\sigma)$. That this factorization is the unique one of the type $I - q(s) = \exp(A)\ \exp(B)$ where $A^\tau = A$ and $B^\sigma = B$ may be seen by taking logarithms in (3.5) and using property (xi) of (1.2).

The solution of Equation (3.3) given in Theorem 3.4 is then a multiple of the inverse of the right factor matrix on the right hand side of (3.5). If $g(s)$ is not a multiple of the identity the situation becomes more complicated. In fact, we then want to find matrices $A(s)$ and $B(s)$ satisfying the conditions

(1) $I - q(s) = \left(I + B(s)\right)\left(I + A(s)\right)$.

(2) $\left(I + A(s)\right)$ has a bounded right inverse, $\left(I + A(s)\right)^{-1}$.

(3) $\left(A(s)\right)^{\sigma} = A(s)$.

(4) $B(s)$ is subdiagonal, and $\sum_{k=-\infty}^{i} B_{ik}\, g_k = 0$.

Then the (unique) bounded solution of (3.3) is $m(s) = (I + A(s))^{-1}\, g(s)$. Of course if $g(s)$ is a constant multiple of I, condition (4) becomes $(B(s))^{\tau} = B(s)$, and (1) is the familiar Wiener-Hopf factorization.

A simple example will serve to illustrate the above method and its difficulties. Suppose that we have a two state semi-Markov process in which $Q_{ij}(t) = \int_0^t Q'_{ij}(s)\, ds$. Then

$$q(s) = \begin{pmatrix} 0 & q_{01}(s) \\ q_{10}(s) & 0 \end{pmatrix}$$

and

$$g(s) = \begin{pmatrix} (1 - q_{01}(s))/s & 0 \\ 0 & (1 - q_{10}(s))/s \end{pmatrix}$$

Of course, one can easily write down $m(s)$ from probabilistic considerations. Namely,

$$m(s) = \begin{pmatrix} (1 - q_{01}(s))/s & q_{01}(s)/s \\ 0 & 1/s \end{pmatrix}$$

However, it is instructive to carry out the factorization described in (1) - (4). We have

$$I - q(s) = \left(I + B(s)\right)\left(I + A(s)\right)$$

where

$$B(s) = \begin{pmatrix} 0 & 0 \\ -q_{10} & q_{10}(1-q_{01})/(1-q_{10}) \end{pmatrix}$$

and

$$A(s) = \begin{pmatrix} 0 & -q_{01}(s) \\ 0 & -q_{10}(s) \end{pmatrix}$$

Also,

$$\left(I + A(s)\right)^{-1} = \begin{pmatrix} 1 & q_{01}(s)/(1-q_{10}(s)) \\ 0 & 1/(1-q_{10}(s)) \end{pmatrix}$$

One can easily check that properties (1) - (4) hold and that

$$m(s) = \left(I + A(s)\right)^{-1} g(s) = \begin{pmatrix} (1-q_{01}(s))/s & q_{01}(s)/s \\ 0 & 1/s \end{pmatrix}$$

IV CONTINUOUS PARAMETER MARKOV CHAINS

We now consider a continuous parameter Markov chain with stationary transition probabilities, $p_{ij}(t)$, as a special case of a S.M.P.

If a S.M.P. is in state i at time t, the probability that there will be a change of state by time t + s depends, in general, on the length of time that the process has been in state i at time t. If, however, the probability of a change of state by time t + s is independent of the length of time the process has been in state i at time t, then we have the special case of a Markov process. In short, the semi-Markov process remembers the past (up to the time of the last jump) and the Markov process does not.

If our S.M.P. process is a stationary Markov process, we have a continuous parameter Markov chain (M.C.) with stationary transition probabilities. In this case (See [3] p. 246)

$$F_i(t) = 1 - e^{-c_i t} \text{ and } Q_{ij}(t) = p_{ij}(1 - e^{-c_i t})$$

where

$$c_i \geq 0, \ p_{ij} \geq 0, \ p_{ii} = 0, \text{ and } \sum_{j \in I} p_{ij} = 1.$$

Our assumption that we are dealing with a step process amounts to saying $c_i < \infty$ for all i, which means that there are no instantaneous states.

If a M.C. has a standard transition matrix (i.e. $p_{ij}(t) \to \delta_{ij}$ as $t \to 0+$) and no instantaneous states then it has an infinitesimal generator $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} p'_{ij}(0) = Q'_{ij}(0) = p_{ij}c_i & \text{if } i \neq j \\ -c_i & \text{if } j = i \end{cases}$$

We now state for M.C.'s the analogs of the theorems of Section III. In spite of its simplicity, the following theorem appears not to be in the literature.

<u>THEOREM 4.1.</u> For a stationary M.C. with a standard transition matrix and no instantaneous states,

(4.1) $$M'(t) = \left(A\ M(t)\right)^{\sigma}$$

where $$M'(t) = \left(M'_{ij}(t)\right)$$

<u>Proof</u> Again we consider three cases.

<u>Case 1</u> If $j < i$, then $M'_{ij}(t) = 0$.

<u>Case 2</u> If $j > i$, then by Theorem 3.1.

$$M_{ij}(t) = \sum_k \int_0^t M_{kj}(t-s)\ p_{ik}\ c_i\ e^{-c_i s}\ ds$$

Letting $y = t - s$ we have

$$M_{ij}(t) = e^{-c_i t}\Big(\sum_k \int_0^t M_{kj}(y)\ p_{ik}\ c_i\ e^{c_i y}\ dy\Big).$$

By the usual theorem allowing termwise differentiation,

$$M'_{ij}(t) = - c_i e^{-c_i t}(\sum_k \int_0^t M_{kj}(y)\ p_{ik} c_i\ e^{c_i y}\ dy)$$

$$+ e^{-c_i t}(\sum_k M_{kj}(t)\ p_{ik}\ c_i\ e^{c_i t})$$

$$= - c_i\ M_{ij}(t) + \sum_k c_i\ p_{ik}\ M_{kj}(t)$$

$$= \sum_k a_{ik}\ M_{kj}.$$

<u>Case 3</u> $i = j$

$$M_{ii}(t) = e^{-c_i t} + \sum_{j \leq i} \sum_k \int_0^t M_{kj}(t-s)\ p_{ik}\ c_i\ e^{-c_i s}\ ds$$

$$= e^{-c_i t} + \sum_k \int_0^t \sum_{j \leq i} M_{kj}(t-s)\ c_i\ p_{ik}\ e^{-c_i s}\ ds$$

In a manner similar to the above we get

$$M'_{ii}(t) = -c_i\ M_{ii}(t) + \sum_k c_i\ p_{ik} \sum_{j \leq i} M_{kj}(t)$$

$$= - c_i\ M_{ii}(t) + \sum_{j \leq i} \sum_k c_i\ p_{ik}\ M_{kj}(t).$$

Since $M_{ij}(t) = 0$ for $j < i$, we may write

$$M'_{ii}(t) = \sum_{j \leq i} \sum_k a_{ik}\ M_{kj}(t).$$

It follows that $M'(t) = \left(A\ M(t)\right)^{\sigma}$

<u>COROLLARY 4.1.</u> For a stationary M.C. with standard transition matrix and no instantaneous states

(4.2) $$sm(s) = I + \left(A\ m(s)\right)^{\sigma}$$

Proof

$$M_{ij}(t) = \int_0^t M'_{ij}(r)\,dr + \delta_{ij}$$

Let Lf(t) denote the Laplace transform of a function f(t). Then

$$L\,M_{ij}(t) = L\int_0^t M'_{ij}(r)\,dr + \delta_{ij}/s$$

$$= (1/s)\,L\,M'_{ij}(t) + \delta_{ij}/s.$$

Using (4.1) we have $\left(L\,M'_{ij}(t)\right) = \left(A\,m(s)\right)^{\sigma}$.

So $s\,m(s) = \left(A\,m(s)\right)^{\sigma} + I$.

THEOREM 4.3. For a stationary M.C. with a standard transition matrix and no instantaneous states, $\|A\| = b < \infty$ implies that for $s > b$ Equation (4.2) has a unique bounded solution m(s).

Proof For $s > b$, $\|A\|/s < 1$ and we may apply the proof of Theorem 3.3 to show that Equation (4.2) has a unique bounded solution.

THEOREM 4.4. Consider a stationary M.C. with a standard transition matrix and no instantaneous states. If $(A^k)^{\sigma} A = A(A^k)^{\sigma}$ for all $k \geq 1$, and if $\|A\| = b < \infty$, then for $s > b$

$$m(s) = (1/s)\exp\left(\sum_1^{\infty} (A^k)^{\sigma}/k\,s^k\right)$$

Proof Rewriting equation 4.2 as

$$(1/s)\ I = \left((I - A/s)\ m(s)\right)^{\sigma},$$

it is easily seen by a method similar to that used in Theorem 3.4 that $(1/s)\ \exp\left(\sum_{1}^{\infty} (A^k)^{\sigma}/k\ s^k\right)$ is the unique bounded solution of (4.2).

The corollary below follows from Theorem 4.4 in same manner as Corollary 3.4 follows from Theorem 3.4.

COROLLARY 4.4. In the case of a stationary M.C. with a standard transition matrix and no instantaneous states which is homogeneous in space, we have for $s > 2c$

$$m(s) = (1/s)\ \exp\left(\sum_{1}^{\infty} (A^k)^{\sigma}/k\ s^k\right)$$

Proof We need only note that $\| A \| = 2\ | a_{ii} | = 2\ c < \infty$.

In the case described in the preceding corollary we can go a step farther and deal with Fourier series instead of matrices. We may think of the process involved as being a random sum, $X_1 + \ldots + X_{N(t)}$, of independent identically distributed random variables having distribution $P[X = k] = p_k \quad k = 0 \pm 1, \pm 2, \ldots$ (Note $p_0 = 0$) where $N(t)$ is an independent Poisson variable with parameter $c < \infty$.

Since $m_{ij}(s) = m_{j-i}(s)$ we may define

$$m(s,\theta) = \sum_{n} m_n(s)\ e^{in\theta}$$

$$p(\theta) = \sum_{n} p_n\ e^{in\theta} - 1. \tag{4.3}$$

In analogy with the results in the matrix case we may prove

THEOREM 4.5. For a process of the type in Corollary 4.4, $m(s,\theta)$ in (4.3) is the unique solution for $s > 2c$ of

$$s\, m(s,\theta) = 1 + (cp(\theta)\, m(s,\theta))^{\sigma}$$

where $p(\theta)\, m(s,\theta)$ denotes pointwise multiplication and σ is an operator which transforms the Fourier series $\Sigma\, b_n e^{in\theta}$ into $\Sigma\, b_n^{\sigma} e^{in\theta}$ with

$$b_n^{\sigma} = b_n \quad \text{for } n > 0,$$

$$b_0^{\sigma} = \sum_{n \le 0} b_n,$$

and

$$b_n^{\sigma} = 0 \quad \text{for } n < 0.$$

We may apply the Wiener-Hopf factorization technique to obtain

THEOREM 4.6. In the case of a Markov chain of the type in Corollary 4.4, we have for $s > 2c$,

$$m(s,\theta) = (1/s)\exp\left([\log(1 - cp(\theta)/s)]^{\sigma}\right)$$

V DISCRETE PARAMETER MARKOV CHAINS

In the case of a discrete parameter M.C. with one step transition matrix (p_{ij}), our basic distributions become

$$F_i(t) = 1 - (p_{ii})^n \qquad n \leq t < n+1 \qquad n = 0, 1, 2, \ldots$$

$$Q_{ii}(t) = 0$$

and if $i \neq j$,

$$Q_{ij}(t) = \begin{cases} 0 & 0 \leq t < 1 \\ p_{ij}(p_{ii})^{n-1} & n \leq t < n+1 \qquad n = 1, 2, \ldots \end{cases}$$

<u>THEOREM 5.1.</u> If $P = (p_{ij})$ is the transition matrix of a M.C. then

$$M(0) = I$$

$$M(n+1) = [P\ M(n)]^{\sigma} \qquad n = 0, 1, 2 \ldots$$

<u>Proof</u> Clearly $M(0) = I$. Equation (3.1) becomes,

$$M(n+1) = (\delta_{ij}(p_{ii})^{n+1}) + (\sum_{k\neq i} \sum_{r=0}^{n} (p_{ii})^r p_{ik} M_{kj}(n-r))^{\sigma}$$

$$= (\delta_{ij}\ p_{ii}(p_{ii})^n) + (\sum_{k\neq i} p_{ik} M_{kj}(n))^{\sigma}$$

$$+ (p_{ii} \sum_{k\neq i} \sum_{s=0}^{n-1} (p_{ii})^s\ p_{ik} M_{kj}(n-1-s))^{\sigma}$$

$$= (\sum_{k \neq i} p_{ik} M_{kj}(n))^{\sigma} + (p_{ii} M_{ij}(n))^{\sigma}$$

$$= (\sum_{k \in I} p_{ik} M_{kj}(n))^{\sigma} = [P\ M(n)]^{\sigma}.$$

For $s < 1$, we may define an operator

$M = \sum_{n=0}^{\infty} s^n M(n)$. Since $||P^{\sigma}|| = 1$ and $||M(n)|| \leq ||P^{\sigma}||^n = 1$ $\sum_{n=0}^{\infty} s^n M(n)$ converges in the strong operator sense. Thus M is a matrix by (x) section I.

<u>THEOREM 5.2.</u> For $s < 1$, M is the unique bounded solution of

$$M = I + s\ [P\ M]^{\sigma}$$

<u>Proof</u> M is a solution because

$$I + s\ [P\ M]^{\sigma} = I + s \sum_{n=0}^{\infty} s^n\ [P\ M(n)]^{\sigma}$$

$$= I + \sum_{n=1}^{\infty} s^n\ M(n) = M$$

Uniqueness follows in the same manner as in Theorem 3.3.

<u>THEOREM 5.3.</u> If P is the transition matrix of a M.C. and if $[P^k]^{\sigma}\ P = P\ [P^k]^{\sigma}$, then for $s < 1$,

$$M = \exp\left(\sum_{1}^{\infty} \frac{s^k (P^k)^{\sigma}}{k}\right)$$

<u>Proof</u> Let $L = (I - s\ P) = - \sum_{k=1}^{\infty} \frac{s^k\ P^k}{k}$

We know that M is the unique bounded solution of

$$I = [(I - s\ P)\ M]^{\sigma}$$

Since $\exp(-L^{\sigma})$ is a bounded solution for $s < 1$, the proof is finished.

If the M.C. is a random walk then $[P^k]^{\sigma} P = P[P^k]^{\sigma}$ is satisfied and Theorem 5.3 becomes Spitzer's identity.

VI OTHER FUNCTIONALS

The analysis performed in Section III made use of the first jump to obtain a recurrence relation involving the supremum functional. This type of analysis may be used for other functionals as will be described in this section.

However, there are functionals whose analysis by the methods of this paper would involve a last jump. For instance, $P_{i,0}\ [X_s \geq i \text{ for } 0 \leq s \leq t]$ is such a functional. The analysis by last jump presents two problems. First, there may be no last jump before time t. Second, even if we restrict ourselves to processes with only a finite number of jumps in any finite interval of time, we encounter the problem that the process $\{X_t,\ t \geq 0\}$ is not stationary. That is, an arbitrary instant of time need not be a regeneration point of a semi-Markov process, whereas the time of the first jump is a regeneration point.

In characterizing the functionals below we follow Baxter [1] and define an operator "+" on matrices as follows. DEFINITION. If A is a matrix with elements a_{ij}, then we define A^+ to be the matrix with elements

$$a^+_{ij} = \begin{cases} a_{ij} \text{ if } i > j \\ 0 \text{ if } i \le j \end{cases}$$

It is easy to see that the analogs of properties (i) through (ix) in Section I hold for the "+" operator where $A^- = A - A^+$.

DEFINITION.

$$L_{ij}(t) = P_{i,0}\,[Z_t = j \text{ and } j > X_s \text{ for } 0 \le s < t - Y_t]$$

$$V_{ij}(t) = P_{i,0}\,[w \le t;\ Z_t = i;\ j = X_t > X_s \text{ for } w \le s < t - Y_t]$$

Note that $L_{ij}(t)$ is the probability that starting at i, the maximum j is reached on the last jump.

<u>THEOREM 6.1</u>. For a S.M.S.P.

$$(6.1)\quad \begin{aligned} L(t) &= (\delta_{ij}(1 - F_i(t))) + (Q(t) * L(t))^+ \\ V(t) &= (Q(t) * L(t))^- \end{aligned}$$

<u>Proof</u> If $i > j$, then $L_{ij}(t) = 0$, and if $i = j$, then $L_{ii}(t) = 1 - F_i(t)$. If $j > i$, then by a proof similar to that of Theorem 3.1,

$$L_{ij}(t) = \sum_{k \le j} \int_0^t L_{kj}(t-s)\, d\, Q_{ik}(s) = \sum_{k \in I} \int_0^t L_{kj}(t-s)\, d\, Q_{ik}(s)$$

For the proof of the second equality in (6.1) we see that if $j > i$, then $V_{ij}(t) = 0$. If $j \le i$, then

$$V_{ij}(t) = \sum_k \int_0^t L_{kj}(t-s)\, d\, Q_{ik}(s)$$

<u>COROLLARY 4.1.</u> If $\ell_{ij}(s) = \int_0^\infty e^{-st} L_{ij}(t)\, dt$ and $v_{ij}(s) = \int_0^\infty e^{-st} V_{ij}(t)\, dt$, then

$$\ell(s) = g(s) + (q(s)\ \ell(s))^+$$

$$v(s) = (q(s)\ \ell(s))^- \tag{6.2}$$

We state the following theorems without proof since their proofs are similar to those in Section III.

THEOREM 6.2. If , in a S.M.S.P., $Q'_{ij}(t)$ is continuous for all i and j then $L_{ij}(t)$ and $V_{ij}(t)$ are continuous for all i and j.

<u>THEOREM 6.3.</u> For a S.M.S.P., $||q(s_0)|| < 1$ for some s_0 implies that

$$\ell(s) = g(s) + (q(s)\ \ell(s))^+$$

$$v(s) = (q(s)\ \ell(s))^-$$

have unique bounded solutions for $s \geq s_0$.

<u>COROLLARY 6.3.</u> For a S.M.S.P., $\sum_k Q'_{ik}(t) \leq B$ for all i implies that equations 6.2 have $\ell(s)$ and $v(s)$ for their unique bounded solutions.

<u>THEOREM 6.4.</u> Suppose that we have a S.M.S.P. such that $F_i(t) = F(t)$ for all i and that $\left(q^k(s)\right)^+ q(s) = q(s)\left(q^k(s)\right)^+$ for all $k \geq 0$. If $||q(s_0)|| < 1$ for some s_0, then for $s \geq s_0$ let

$$N = \log(I - q(s)) = -\sum_{1}^{\infty} q^k(s)/k$$

and
$$\eta(s) = \int_0^{\infty} e^{-st}(1 - F(t))\, dt.$$

Then for $s \geq s_0$

$$\ell(s) = \eta(s) \exp\left(\sum_{1}^{\infty} (q^k(s))^+/k\right)$$

Again we can derive some special results in the case of a continuous parameter M.C. For instance,

THEOREM 6.5. For a continuous parameter stationary M.C. with a standard transition matrix and no instantaneous states

$$\ell(s) = \left(\delta_{ij}/s\right) + \left(A\ \ell(s)/s\right)^+.$$

VII AN EXAMPLE

Let us consider a class of semi-Markov processes in which the state space is the non negative integers and the process can only go up or down one step at a time. We further assume that except for the state 0, the process is homogeneous in space. A single server queue with Poisson input and negative exponential service time is an example of such a process.

Let

$$Q_0(t) = P_{0,0}\,[w \le t \text{ and } X_w = 1]$$

$$Q_1(t) = P_{i,0}\,[w \le t \text{ and } X_w = i + 1] \qquad (i \ge 1)$$

$$Q_{-1}(t) = P_{i,0}\,[w \le t \text{ and } X_w = i - 1] \qquad (i \ge 1)$$

Let $q_0(s)$, $q_1(s)$ and $q_{-1}(s)$ be the Laplace Stieltjes transforms of the measures $Q_0(t)$, $Q_1(t)$ and $Q_{-1}(t)$ respectively. The matrix q(s) of Corollary 3.1 becomes

$$\begin{pmatrix} 0 & q_1(s) & 0 & 0 & \cdots \\ q_{-1}(s) & 0 & q_1(s) & 0 & \cdots \\ 0 & q_{-1}(s) & 0 & q_1(s) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix}$$

Since $||q(s)|| = \max \{q_0(s),\ q_{-1}(s) + q_1(s)\}$, we see that for s large enough $||q(s)|| < 1$. So we may apply Theorem 3.3 to assure that m(s) is the unique solution of equation (3.3) which we may write in the form

$$\left((I - q(s))\ m(s)\right)^{\sigma} = g(s) \tag{7.1}$$

We note that g(s) is a diagonal matrix for which $g_{00}(s) = g_0(s)$ and $g_{ii}(s) = g(s)$ $(i \geq 1)$ for some functions $g_0(s)$ and g(s).

Let $B(s) = (I - q(s))\ m(s)$, then

$$B_{ij}(s) = \begin{cases} -q_{-1}(s)\ m_{i-1,j} + m_{ij} - q_1(s)\ m_{i+1,j} & (i \geq 1) \\ m_{0j} - q_0(s)\ m_{1j} & (i = 0) \end{cases}$$

Equation (7.1) becomes, $[B(s)]^{\sigma} = g(s)$, which yields the following system of simultaneous equations:

$$g_0(s) = m_{00}(s)$$

$$0 = m_{oj} - q_0\ m_{ij}$$

and for $i \geq 1$,

$$0 = -q_{-1}\ m_{i-1,j} + m_{ij} - q_1\ m_{i+1,j} \quad \text{(when } j > i\text{)}$$

$$g(s) = -q_{-1}(m_{i-1,i} + m_{i-1,i-1}) + m_{ii}$$

The above equations may be regrouped to obtain

$$g_0(s) = m_{00}(s)$$

and for $n \geq 1$,

$$0 = m_{0n} - q_0 \, m_{1n}$$

$$0 = - q_{-1} \, m_{0n} + m_{1,n} - q_1 \, m_{2,n}$$

$$\vdots$$

$$0 = - q_{-1} \, m_{n-2,n} + m_{n-1,n} - q_1 \, m_{nn}$$

$$g(s) + q_{-1} \, m_{n-1,n-1} = - q_{-1} \, m_{n-1,n} + m_{nn}$$

Let Δ_n be the $(n+1) \times (n+1)$ matrix one obtains by deleting all but the first n+1 rows and columns of $I-q(s)$. Let $\Delta_{j,n}$ $(0 \leq j \leq n)$ be the matrix obtained by replacing the j^{th} column of Δ_n by

$(0, 0, \ldots 0, g(s) + q_{-1} \, m_{n-1,n-1})$. For large S

Det $(\Delta_n) \neq 0$ (see below) So,

$$m_{j,n}(s) = \text{Det } (\Delta_{j,n})/\text{Det } (\Delta_n) \quad 0 \leq j \leq n$$

Clearly if $j > n$, $m_{j,n}(s) \equiv 0$. Thus we have a recursive method of calculating $m_{ij}(s)$. To find the probability $M_{ij}(t)$ it still remains to invert its Laplace transform $m_{ij}(s)$.

We can solve a difference equation for $\mathrm{Det}(\Delta_n) = D_n$. and show that for large s, $\mathrm{Det}(\Delta) \neq 0$. Let $\mathrm{Det}(\Delta) = D_n$. Then for $n \geq 0$,

$$D_{n+2} = D_{n+1} - q_1\, q_{-1}\, D_n$$

Let $\quad r_1 = (1 + \sqrt{1-4q_1q_{-1}})/2$

and $\quad r_2 = (1 - \sqrt{1-4q_1q_{-1}})/2$

Then, $D_n = a\, r_1^{\,n} + b\, r_2^{\,n}$ where

$$a + b = 1$$

$$a\, r_1 + b\, r_2 = 1 - q_0(s)\, q_{-1}(s)$$

As $s \to \infty$, q_0, q_{-1} and q_1 approach zero. So, for large s $r_1/r_2 > 1$, and by solving the above equations for a and b it can be shown that $-b/a < 1$. Thus $D_n \neq 0$ for large s because $D_n = 0$ implies $1 > -b/a = (^{r_1}/r_2) > 1$.

If we formally define $D(z) = \sum_{n=0}^{\infty} D_n\, z^n$, then

$$D(z) = \frac{1 + z\, q_0\, q_1}{1 - z + z^2\, q_1\, q_{-1}}$$

In the same spirit we may define $F(z) = \sum_{n=0}^{\infty} D_{nn}\, z^n$ where

$$D_{00} = g_0(s)$$

$$D_{nn} = \mathrm{Det}(\Delta_{n,n}) \qquad (n \geq 1)$$

Then for $n \geq 0$,

$$D_{n+1,n+1} = g(s)\ D_n + q_{-1}\ m_{n,n}\ D_n$$

$$= g(s)\ D_n + q_{-1}\ D_{nn}$$

Multiplying both sides of the above relation by z^{n+1} and summing we obtain

$$F(z) - g_0(s) = z\ g(s)\ D(z) + z\ q_{-1}\ F(z)$$

So that

$$F(z) = \frac{g_0(s) + z\ g(s)\ D(z)}{(1 - z\ q_{-1})}.$$

Similarly if we define $D_{0,n} = \mathrm{Det}(\Delta_{0,n})$ for $n \geq 1$, then for $n \geq 0$ we obtain

$$D_{0,n+1} = (g(s) + q_{-1}\ m_{nn})\ q_0\ q_1{}^n(-1)^{n+1}$$

$$= (g(s) + q_{-1}\ \frac{D_{nn}}{D_n})\ q_0\ q_1{}^n(-1)^{n+1}$$

So,

$$\sum_{n=0}^{\infty} (-1)^{n+1} q_0^{-1}\ q_1^{-n}\ D_n D_{0,n+1}\ z^n = g(s)\ D(z) + q_{-1}\ F(z).$$

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This paper deals with the problem of finding the distribution of $z(t) = \sup [x(s) = 0 \leq s \leq t]$ for a wide class of semi-Markov processes $\{x(t) = t \geq 0\}$.

Semi-Markov processes are analyzed by considering the time and place of the first jump in order to obtain the main result which is a recurrence relation for the distribution of $z(t)$. Define $m_{ij}(s)$ to be the Laplace transform of $P[Z(t) = j | X(0) = i]$ and let $m(s) = (m_{ij}(s))$. The recurrence relation which is obtained for $m(s)$ is $m(s) = g(s) + (q(s)m(s))^{\sigma}$ where $q(s)$ and $g(s)$ are matrices whose elements are Laplace transforms of distributions which occur in the definition of the semi-Markov process and σ is an operator on matrices.

Under a condition on the matrix $q(s)$ which guarantees that the process makes only a finite number of jumps in any finite interval of time, $m(s)$ is the unique solution of the above equation. In the case of spatial homogeneity, it is possible to solve the recurrence relation by a Wiener-Hopf type factorization.

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