On the distribution of an elementary symmetric function of the roots of a matrix

by

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0. Summary. A lemma is proved in this paper using some results on zonal polynomials which evaluates in terms of multivariate gamma functions the kth moment of $V^{(s)}_{s-1,m,n}$, the (s-1)th elementary symmetric function (E.S.F.) in the s characteristic roots, $0 < \theta_1 \leq \cdots \leq \theta_s < 1$, jointly distributed according to Fisher-Girshick-Hsu-Roy distribution. Alternate expressions for the first four moments of $V^{(s)}_{s-1,m,n}$ are also given employing a different method and upper 5 and 1 per cent points computed for s = 3,4 and 5 using the moment quotients. An example is given to illustrate the use of this criterion.

1. Introduction. Many of the distribution problems in multivariate analysis are based on the distribution of the non-null characteristic roots of a matrix derived from sample observations taken from multivariate normal populations. This distribution (under certain null hypotheses) given by Fisher (1939), Girshick (1939), Hsu (1939) and Roy (1939) is of the form

$$(1.1) \qquad f(\theta_1,\ldots,\theta_s) = C(s,m,n) \prod_{i=1}^{s} \{\theta_i^m(1-\theta_i)^n\} \prod_{i>j} (\theta_i - \theta_j)$$

$$(0 < \theta_1 \leq \cdots \leq \theta_s < 1) \quad ,$$

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where

$$(1.2)\qquad C(s,m,n) = \pi^{\frac{1}{2}s} \prod_{i=1}^{s} \{\Gamma(m+n+\tfrac{1}{2}s+\tfrac{1}{2}i+1) \;/\; \Gamma(m+\tfrac{1}{2}i+\tfrac{1}{2})\Gamma(m+\tfrac{1}{2}i+\tfrac{1}{2})\Gamma(\tfrac{1}{2}i)\}$$

and m and n are defined differently for various situations described by Pillai (1955, 1960).

The studies on the first E.S.F. in the θ's have been carried out by Pillai (1954, 1956, 1960) and Pillai and Mijares (1959). In this paper,,the moments of $V^{(s)}_{s-1,m,n}$, the $(s-1)$th E.S.F., are studied.

2. Moments of $V^{(s)}_{s-1,m,n}$. First let us consider that a positive definite symmetric matrix variate $\underset{\sim}{S}(s \times s)$ has a multivariate beta distribution,

$$(2.1)\qquad \{\Gamma_s(\tfrac{1}{2}f_2+\tfrac{1}{2}f_1) \;/\; \Gamma_s(\tfrac{1}{2}f_2)\Gamma_s(\tfrac{1}{2}f_1)\}\; |\underset{\sim}{S}|^{\frac{1}{2}(f_2-s-1)}\, |\underset{\sim}{I} - \underset{\sim}{S}|^{\frac{1}{2}(f_1-s-1)}\, d\,\underset{\sim}{S} \quad ,$$

where the multivariate gamma function (James, 1964) is defined by

$$(2.2)\qquad \Gamma_s(a) \;=\; \pi^{\frac{1}{4}s(s-1)} \prod_{i=1}^{s} \Gamma(a-\tfrac{1}{2}(i-1)) \quad .$$

If the characteristic roots of $\underset{\sim}{S}$ are assumed to be $\theta_1,\ldots,\theta_s$ considered earlier, then it is well known that (1.1) can be derived from (2.1) by suitable transformation of variables and $f_2 = 2m+s+1$ and $f_1 = 2n+s+1$. Further it has been shown (James, 1964) that

$$(2.3)\qquad (\mathrm{tr}\,\underset{\sim}{S})^k \;=\; \sum_{\kappa} C_\kappa(\underset{\sim}{S}) \quad ,$$

where $C_\kappa(\underset{\sim}{S})$ is the zonal polynomial of degree k (James, 1964) corresponding to the partition $\kappa = (k_1,k_2,\ldots,k_p)$, $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$, $k_1 + \ldots + k_p = k$, of k into not more than s parts. Further for any positive definite matrix $\underset{\sim}{B}$ and $f_2 \geq s + k_1$, Khatri (1966) has shown that

$$(2.4) \qquad E\{C_\kappa(\underset{\sim}{B}\,\underset{\sim}{S}^{-1})\} = \frac{\Gamma_s(\tfrac{1}{2}f_2+\tfrac{1}{2}f_1)\,\Gamma_s(\tfrac{1}{2}f_2, -\kappa)\,C_\kappa(\underset{\sim}{B})}{\Gamma_s(\tfrac{1}{2}f_2)\,\Gamma_s(\tfrac{1}{2}f_2+\tfrac{1}{2}f_1, -\kappa)} \quad ,$$

where

$$(2.5) \qquad \Gamma_s(a, -\kappa) = \pi^{\frac{1}{4}s(s-1)} \prod_{i=1}^{s} \Gamma(a-k_i-\tfrac{1}{2}s+\tfrac{1}{2}i) \quad .$$

In view of the above results we now prove the following lemma:

<u>Lemma 1</u>. If $\mu_k'\{V^{(s)}_{s-1,m,n}\}$ denotes the kth moment of $V^{(s)}_{s-1,m,n}$, then

$$(2.6) \qquad \mu_k'\,\{V^{(s)}_{s-1,m,n}\} = \frac{\Gamma_s(\tfrac{1}{2}f_1+\tfrac{1}{2}f_2)}{\Gamma_s(\tfrac{1}{2}f_2)} \sum_{\kappa} \left\{\frac{\Gamma_s(\tfrac{1}{2}f_2+k, -\kappa)\,C_\kappa(\underset{\sim}{I}_s)}{\Gamma_s(\tfrac{1}{2}f_1+\tfrac{1}{2}f_2+k, -\kappa)}\right\} \quad .$$

Proof. Note that $\mathrm{tr}(\underset{\sim}{S}^{-1}) = \sum_{i=1}^{s} (1/\theta_i)$ and hence $V^{(s)}_{s-1,m,n} = (\prod_{i=1}^{s} \theta_i)\mathrm{tr}\,\underset{\sim}{S}^{-1}$.

Further, using (2.3) we get

$$(2.7) \qquad \{V^{(s)}_{s-1,m,n}\}^k = (\prod_{i=1}^{s} \theta_i)^k \sum_{\kappa} C_\kappa(\underset{\sim}{S}^{-1}) \quad .$$

Now put $B = I_s$ and replace $\tfrac{1}{2}f_2$ by $\tfrac{1}{2}f_2+k$ in $\Gamma_s(\cdot, -\kappa)$ in (2.4) to obtain $E\{V^{(s)}_{s-1,m,n}\}^k$ with the help of (2.1). We get the right side of (2.6) by summing over all partitions, κ. Hence the lemma.

Pillai (1965) had given a lemma obtaining the moments of $V^{(s)}_{s-1,m,n}$ in terms of the moments of $U^{(s)}_1 = \sum_{i=1}^{s} \{\theta_i/(1-\theta_i)\}$. Unlike that lemma, the present one gives direct results and hence is more useful.

The first four moments of $V^{(s)}_{s-1,m,n}$ presented below are derived alternately using the lemma of Pillai (1965).

$$(2.8)\qquad \mu_1'\{V^{(s)}_{s-1,m,n}\} = [sa/\{2(m+1)\}] \prod_{i=1}^{s} \{(2m+i+1)/(a+i-1)\} \quad ,$$

where $a = 2m+2n+s+3$,

$$(2.9)\qquad \mu_2'\{V^{(s)}_{s-1,m,n}\} = \frac{s(a+2)}{4(m+2)^2}\left\{\frac{(2n+s+1)(2m+s+4)}{(m+1)(2m+5)} + s(a+2)\right\} \cdot A_1$$

where

$$A_1 = \prod_{i=1}^{s} [(2m+i+3)(2m+i+1) / \{(a+i+1)(a+i-1)\}] \quad ,$$

$$(2.10)\qquad \mu_3'\{V^{(s)}_{s-1,m,n}\} = \left\{\frac{s(a+4)}{8(m+3)^3}\right\}\left\{\frac{(2n+s+1)(2m+s+6)}{(m+2)(2m+7)}\left[\frac{4(2n+m+s+4)(2m+2s+6)}{(m+1)(2m+8)} + 3s(a+4)\right] + s^2(a+4)^2\right\} B_1$$

where

$$B_1 = \prod_{i=1}^{s} \{(2m+i+5)\,A_1 / (a+i+3)\}$$

and

$$(2.11)\qquad \mu_4'\{V^{(s)}_{s-1,m,n}\} = \left\{\frac{s(a+6)}{16(m+4)^4}\right\}\left\{\frac{48(2n+s+1)\ [C+Dn(m+n+s+5)]}{(m+1)(m+2)(m+3)(m+5)(2m+7)(2m+9)(2m+11)} + E\right\}\cdot F_1$$

where C and D are given in (3.6) of Pillai (1965),

$$E = s(a+6)\left\{\frac{2(2n+s+1)(2m+s+8)}{(m+3)(2m+9)}\left[\frac{4(2n+m+s+5)(2m+2s+8)}{(m+2)(m+5)} + 3s(a+6)\right] + s^2(a+6)^2\right\}$$

and

$$F_1 = \sqrt[s]{\prod_{i=1}^{s} \{(2m+i+7) B_1\} / (a+i+5)} \quad .$$

3. Upper percentage points of $V^{(s)}_{s-1,m,n}$. Using (2.8) - (2.11) the moment quotients, β_1 and β_2, were computed on IBM 7094 accurate to five decimals for values of $s = 3$, 4 and 5, $m = -\frac{1}{2}(\frac{1}{2})5,7,10(5)50,60,80,100,130,160,200,$ 300,500 and 1000 and n starting from 20, but otherwise as for m. The upper percentage points ($\alpha = .05, .01$) were computed manually using "Table of percentage points of Pearson Curves, for given $\sqrt{\beta_1}$ and β_2, expressed in Standard measure" (Johnson et al, (1963)). Tables 1 - 6 give the percentage points thus computed.

4. An example. The use of the tabulations may be demonstrated by an example. The criterion $V^{(s)}_{s-1,m,n}$ is being suggested for different kinds of hypotheses in multivariate analysis, for example, (i) that of equality of the covariance matrices of two p-variate normal populations, (ii) that of equality of the p-dimensional mean vectors of ℓ p-variate normal populations and, (iii) that of independence between a p-set and a q-set of variates in a (p+q)-variate normal population. For a test of hypothesis (ii), the data from Rao (1952, p.263) may be considered which relates to the measurements of 140 school boys of almost the same age belonging to six different schools in an Indian city. Three characters studied were: (1) head length, (2) height, and (3) weight. We may test the hypothesis of equality of mean characters from the different schools using $V^{(3)}_{2,m,n}$. In this example, as shown earlier (see Pillai (1965) and Pillai and Samson (1959)), $m = 0.5$ and $n = 65$. Further, from the computations presented in Pillai and Gupta (1967) the value of $V^{(3)}_{2,0.5,65}$ obtained from the data is 0.010340 . This may be seen from Tables 1 and 2 to be

significant at the upper 5% level but not significant at the upper 1% level. This agrees with the findings based on the test $U^{(3)}_{1, 0.5, 65}$ defined in Section 3 (See Pillai and Samson (1959)) and on the test $U^{(3)}_{2,0.5,65}$ (See Pillai 1965)) and also with those of Rao, who examined the data using the Λ criterion of Pearson and Wilks (1933). Foster (1957), however, showed that the largest root is not significant at the upper 5% level but only at the upper 15% level.

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Table 1. Upper 5% points of $V^3_{2,m,n}$

M \ N	20	25	30	35	40	45	50	60	80	100	130	160	200	300	500	1000
-0.5	.0250	.0172	.0126	.00930	.00729	.00588	.00490	.00348	.00202	.00131	$.0^3786$	$.0^3524$	$.0^3339$	$.0^3152$	$.0^4556$	$.0^4140$
0	.0418	.0288	.0215	.0161	.0126	.0105	.00818	.00598	.00353	.00231	.00139	$.0^3914$	$.0^3592$	$.0^3267$	$.0^4971$	$.0^4248$
0.5	.0617	.0428	.0314	.0240	.0190	.0153	.0127	.00907	.00524	.00350	.00211	.00141	$.0^3909$	$.0^3411$	$.0^3149$	$.0^4375$
1.0	.0829	.0579	.0427	.0328	.0260	.0211	.0176	.0125	.00734	.00482	.00292	.00196	.00127	$.0^3574$	$.0^3210$	$.0^4530$
1.5	.105	.0845	.0550	.0424	.0337	.0274	.0227	.0164	.00963	.00634	.00385	.00259	.00168	$.0^3760$	$.0^3278$	$.0^4703$
2.0	.129	.0913	.0681	.0527	.0420	.0342	.0284	.0205	.0121	.00801	.00488	.00328	.00213	$.0^3968$	$.0^3354$	$.0^4897$
2.5	.153	.109	.0819	.0636	.0508	.0415	.0346	.0250	.0149	.00983	.00600	.00404	.00263	.00120	$.0^3438$	$.0^3111$
3.0	.178	.128	.0963	.0750	.0601	.0493	.0411	.0298	.0178	.0118	.00721	.00486	.00317	.00144	$.0^3530$	$.0^3134$
3.5	.204	.147	.111	.0869	.0699	.0574	.0479	.0349	.0209	.0139	.00851	.00574	.00375	.00171	$.0^3629$	$.0^3160$
4.0	.229	.167	.127	.0993	.0800	.0658	.0551	.0402	.0242	.0161	.00988	.00668	.00437	.00200	$.0^3735$	$.0^3187$
4.5	.255	.187	.142	.112	.0904	.0746	.0625	.0458	.0276	.0182	.0112	.00768	.00502	.00230	$.0^3848$	$.0^3216$
5.0	.281	.207	.158	.125	.101	.0836	.0702	.0516	.0312	.0209	.0129	.00872	.00566	.00260	$.0^3959$	$.0^3247$
7	.385	.289	.225	.179	.147	.122	.103	.0766	.0469	.0317	.0197	.0134	.00884	.00408	.00152	$.0^3388$
10	.536	.412	.327	.265	.220	.185	.158	.119	.0741	.0506	.0319	.0219	.0145	.00676	.00253	$.0^3652$
15	.765	.608	.495	.411	.346	.295	.255	.196	.126	.0876	.0562	.0391	.0262	.0124	.00470	.00122
20	.962	.785	.652	.551	.471	.407	.355	.278	.183	.130	.0845	.0594	.0402	.0193	.00742	.00195
25	1.130	.942	.796	.682	.590	.516	.455	.361	.243	.175	.116	.0823	.0562	.0274	.0107	.00283
30	1.275	1.080	.926	.803	.702	.620	.550	.443	.304	.222	.149	.107	.0739	.0365	.0143	.00384
35	1.400	1.203	1.044	.914	.807	.718	.642	.523	.366	.270	.184	.133	.0928	.0465	.0185	.00500
40	1.509	1.312	1.150	1.016	.904	.810	.729	.600	.427	.319	.220	.161	.113	.0572	.0230	.00628
45	1.604	1.409	1.246	1.110	.995	.896	.811	.674	.487	.367	.256	.189	.134	.0687	.0279	.00768
50	1.689	1.496	1.334	1.196	1.078	.977	.889	.745	.545	.416	.293	.218	.156	.0807	.0332	.00921
60	1.832	1.645	1.486	1.348	1.228	1.123	1.031	.877	.657	.511	.368	.277	.201	.106	.0446	.0126
80	2.044	1.873	1.722	1.588	1.469	1.363	1.269	1.106	.861	.689	.513	.397	.295	.162	.0708	.0207
100	2.191	2.036	1.896	1.770	1.655	1.552	1.458	1.293	1.037	.850	.651	.514	.390	.223	.100	.0303
130	2.344	2.209	2.085	1.971	1.866	1.768	1.677	1.517	1.257	1.058	.838	.679	.529	.316	.149	.0470
160	2.447	2.330	2.219	2.116	2.020	1.929	1.845	1.691	1.436	1.234	1.002	.829	.660	.410	.202	.0662
200	2.544	2.443	2.347	2.256	2.170	2.089	2.011	1.866	1.625	1.426	1.189	1.006	.821	.532	.275	.0948
300	2.683	2.609	2.537	2.468	2.401	2.337	2.276	2.159	1.949	1.768	1.539	1.352	1.151	.808	.460	.176
500	2.803	2.755	2.708	2.661	2.616	2.571	2.528	2.444	2.288	2.145	1.953	1.786	1.594	1.230	.795	.357
1000	2.899	2.873	2.848	2.823	2.797	2.773	2.748	2.700	2.606	2.517	2.391	2.273	2.129	1.822	1.378	.782

Table 2. Upper 1% points of $V^3_{2,m,n}$

M \ N	20	25	30	35	40	45	50	60	80	100	130	160	200	300	500	1000
-0.5	.0434	.0306	.0224	.0175	.0138	.0111	.00921	.00655	.00380	.00248	.00149	$.0^{3}997$	$.0^{3}644$	$.0^{3}290$	$.0^{3}106$	$.0^{4}266$
0	.0663	.0461	.0339	.0262	.0204	.0164	.0138	.00985	.00578	.00378	.00228	.00154	$.0^{3}999$	$.0^{3}451$	$.0^{3}164$	$.0^{4}426$
0.5	.0901	.0629	.0464	.0356	.0282	.0229	.0189	.0136	.00787	.00528	.00319	.00214	.00139	$.0^{3}628$	$.0^{3}233$	$.0^{4}589$
1.0	.117	.0823	.0610	.0470	.0373	.0304	.0254	.0181	.0107	.00702	.00427	.00286	.00186	$.0^{3}842$	$.0^{3}308$	$.0^{4}779$
1.5	.145	.120	.0765	.0592	.0471	.0385	.0320	.0231	.0136	.00900	.00548	.00368	.00239	.00109	$.0^{3}398$	$.0^{3}101$
2.0	.173	.124	.0926	.0720	.0575	.0470	..0391	.0284	.0168	.0111	.00679	.00457	.00298	.00135	$.0^{3}496$	$.0^{3}126$
2.5	.202	.145	.109	.0853	.0684	.0560	.0467	.0340	.0202	.0134	.00821	.00553	.00361	.00164	$.0^{3}603$	$.0^{3}153$
3.0	.231	.167	.127	.0991	.0796	.0654	.0547	.0398	.0238	.0158	.00971	.00656	.00428	.00195	$.0^{3}719$	$.0^{3}183$
3.5	.261	.190	.144	.113	.0913	.0752	.0629	.0460	.0276	.0184	.0113	.00765	.00500	.00229	$.0^{3}842$	$.0^{3}214$
4.0	.290	.212	.162	.128	.103	.0852	.0715	.0524	.0316	.0211	.0130	.00880	.00576	.00264	$.0^{3}973$	$.0^{3}248$
4.5	.320	.235	.180	.143	.116	.0955	.0803	.0590	.0357	.0236	.0146	.0100	.00655	.00301	.00111	$.0^{3}283$
5.0	.349	.258	.199	.158	.128	.106	.0893	.0658	.0400	.0268	.0166	.0113	.00732	.00336	.00124	$.0^{3}321$
7	.463	.350	.273	.220	.180	.150	.127	.0949	.0584	.0395	.0247	.0169	.0111	.00514	.00191	$.0^{3}490$
10	.625	.484	.386	.315	.261	.220	.189	.142	.0894	.0613	.0387	.0266	.0177	.00827	.00310	$.0^{3}800$
15	.862	.691	.565	.471	.398	.341	.295	.228	.147	.103	.0661	.0461	.0309	.0147	.00558	.00146
20	1.062	.873	.729	.618	.530	.460	.403	.316	.209	.149	.0973	.0686	.0466	.0224	.00864	.00227
25	1.231	1.032	.877	.754	.655	.574	.507	.404	.274	.197	.131	.0935	.0641	.0313	.0122	.00325
30	1.374	1.171	1.009	.878	.770	.681	.607	.490	.338	.248	.167	.120	.0833	.0413	.0163	.00437
35	1.496	1.292	1.127	.990	.877	.782	.701	.573	.403	.299	.204	.149	.104	.0520	.0208	.00563
40	1.602	1.400	1.233	1.093	.975	.876	.790	.653	.467	.350	.242	.178	.125	.0636	.0257	.00702
45	1.694	1.495	1.328	1.187	1.066	.963	.874	.729	.529	.401	.281	.208	.147	.0759	.0309	.00854
50	1.775	1.580	1.414	1.272	1.150	1.044	.952	.801	.589	.451	.320	.238	.170	.0887	.0366	.0102
60	1.915	1.724	1.215	1.422	1.299	1.190	1.095	.935	.704	.549	.397	.300	.218	.116	.0488	.0138
80	2.120	1.946	1.793	1.657	1.537	1.428	1.332	1.164	.911	.731	.547	.424	.316	.175	.0764	.0224
100	2.261	2.105	1.963	1.834	1.718	1.614	1.518	1.350	1.088	.894	.687	.544	.414	.237	.107	.0325
130	2.400	2.270	2.146	2.031	1.924	1.825	1.733	1.571	1.307	1.103	.876	.712	.556	.334	.158	.0500
160	2.501	2.385	2.275	2.171	2.074	1.983	1.897	1.742	1.484	1.278	1.041	.864	.690	.430	.213	.0700
200	2.591	2.491	2.396	2.306	2.219	2.138	2.060	1.912	1.670	1.469	1.228	1.042	.852	.554	.288	.0996
300	2.717	2.645	2.574	2.506	2.440	2/377	2.315	2.199	1.989	1.806	1.576	1.387	1.183	.833	.476	.183
500	2.825	2.779	2.733	2.687	2.642	2.599	2.557	2.474	2.318	2.176	1.984	1.816	1.623	1.256	.814	.368
1000	2.911	2.886	2.862	2.837	2.812	2.788	2.764	2.717	2.625	2.536	2.411	2.294	2.151	1.844	1.397	.795

Table 3. Upper 5% points of $V^{4}_{3,m,n}$

M \ N	20	25	30	35	40	45	50	60	80	100	130	160	200	300	500	1000
-0.5	.00308	.00177	.00110	$.0^{3}735$	$.0^{3}514$	$.0^{3}373$	$.0^{3}280$	$.0^{3}169$	$.0^{4}750$	$.0^{4}397$	$.0^{4}186$	$.0^{4}102$	$.0^{5}529$	$.0^{5}160$	$.0^{6}352$	$.0^{7}446$
0	.00645	.00378	.00233	.00156	.00109	$.0^{3}794$	$.0^{3}597$	$.0^{3}362$	$.0^{3}161$	$.0^{4}854$	$.0^{4}402$	$.0^{4}220$	$.0^{4}115$	$.0^{5}348$	$.0^{6}768$	$.0^{7}974$
0.5	.0110	.00632	.00406	.00274	.00192	.00140	.00106	$.0^{3}649$	$.0^{3}290$	$.0^{3}155$	$.0^{4}370$	$.0^{4}397$	$.0^{4}207$	$.0^{5}643$	$.0^{5}142$	$.0^{6}177$
1.0	.0165	.00980	.00633	.00425	.00302	.00221	.00168	$.0^{3}959$	$.0^{3}470$	$.0^{3}252$	$.0^{3}119$	$.0^{4}641$	$.0^{4}335$	$.0^{4}102$	$.0^{5}226$	$.0^{6}292$
1.5	.0238	.0141	.00911	.00618	.00429	.00324	.00245	.00153	$.0^{3}684$	$.0^{3}367$	$.0^{3}176$	$.0^{4}971$	$.0^{4}505$	$.0^{4}154$	$.0^{5}342$	$.0^{6}438$
2.0	.0320	.0193	.0124	.00865	.00613	.00450	.00344	.00210	$.0^{3}967$	$.0^{3}521$	$.0^{3}247$	$.0^{3}141$	$.0^{4}700$	$.0^{4}215$	$.0^{5}488$	$.0^{6}621$
2.5	.0414	.0252	.0164	.0113	.00811	.00601	.00458	.00283	.00130	$.0^{3}712$	$.0^{3}338$	$.0^{3}186$	$.0^{4}978$	$.0^{4}303$	$.0^{5}669$	$.0^{6}855$
3.0	.0519	.0318	.0209	.0145	.0104	.00774	.00592	.00368	.00170	$.0^{3}921$	$.0^{3}442$	$.0^{3}245$	$.0^{3}129$	$.0^{4}398$	$.0^{5}884$	$.0^{5}113$
3.5	.0633	.0392	.0259	.0180	.0130	.00972	.00744	.00464	.00216	.00117	$.0^{3}565$	$.0^{3}314$	$.0^{3}166$	$.0^{4}512$	$.0^{4}114$	$.0^{5}147$
4.0	.0758	.0473	.0314	.0220	.0159	.0119	.00916	.00574	.00268	.00146	$.0^{3}707$	$.0^{3}393$	$.0^{3}208$	$.0^{4}645$	$.0^{4}144$	$.0^{5}186$
4.5	.0890	.0560	.0375	.0263	.0192	.0144	.0111	.00697	.00327	.00179	$.0^{3}867$	$.0^{3}484$	$.0^{3}257$	$.0^{4}797$	$.0^{4}179$	$.0^{5}230$
5.0	.103	.0654	.0439	.0310	.0227	.0171	.0132	.00832	.00393	.00216	.00105	$.0^{3}586$	$.0^{3}311$	$.0^{4}969$	$.0^{4}218$	$.0^{5}281$
7	.166	.108	.0746	.0535	.0396	.0302	.0235	.0151	.00726	.00404	.00199	.00112	$.0^{3}599$	$.0^{3}188$	$.0^{4}427$	$.0^{5}553$
10	.275	.186	.132	.0967	.0730	.0564	.0445	.0291	.0144	.00816	.00409	.00233	.00126	$.0^{3}402$	$.0^{4}922$	$.0^{4}121$
15	.474	.337	.248	.188	.145	.115	.0923	.0622	.0321	.0187	.00962	.00559	.00307	.00100	$.0^{3}234$	$.0^{4}311$
20	.675	.498	.378	.293	.232	.187	.153	.106	.0568	.0337	.0178	.0105	.00587	.00196	$.0^{3}466$	$.0^{4}628$
25	.867	.659	.513	.407	.328	.265	.222	.157	.0869	.0530	.0286	.0172	.00973	.00331	$.0^{3}804$	$.0^{3}110$
30	1.045	.815	.648	.523	.428	.355	.297	.214	.122	.0761	.0420	.0256	.0147	.00510	.00126	$.0^{3}175$
35	1.209	.962	.778	.638	.529	.444	.376	.276	.162	.103	.0577	.0357	.0207	.00735	.00184	$.0^{3}259$
40	1.358	1.101	.904	.751	.630	.534	.456	.341	.204	.132	.0757	.0474	.0279	.0101	.00257	$.0^{3}366$
45	1.495	1.229	1.023	.860	.729	.623	.537	.407	.249	.164	.0956	.0605	.0361	.0133	.00344	$.0^{3}497$
50	1.619	1.349	1.135	.964	.825	.711	.618	.474	.297	.198	.117	.0751	.0453	.0169	.00446	$.0^{3}653$
60	1.837	1.564	1.341	1.159	1.007	.881	.775	.608	.394	.270	.165	.108	.0666	.0257	.00698	.00105
80	2.176	1.910	1.684	1.492	1.328	1.187	1.064	.856	.594	.425	.274	.186	.119	.0487	.0140	.00221
100	2.428	2.174	1.955	1.763	1.595	1.448	1.318	1.100	.789	.585	.392	.275	.182	.0784	.0237	.00393
130	2.700	2.470	2.263	2.080	1.915	1.767	1.634	1.404	1.059	.817	.575	.420	.288	.133	.0432	.00764
160	2.893	2.685	2.495	2.322	2.164	2.020	1.889	1.658	1.296	1.032	.754	.568	.403	.197	.0681	.0128
200	3.076	2.893	2.723	2.565	2.419	2.283	2.158	1.932	1.567	1.287	.979	.761	.561	.292	.109	.0220
300	3.347	3.208	3.076	2.950	2.830	2.717	2.609	2.409	2.066	1.784	1.449	1.193	.935	.550	.236	.0560
500	3.589	3.496	3.406	3.318	3.233	3.150	3.070	2.917	2.639	2.395	2.081	1.819	1.534	1.038	.539	.162
1000	3.787	3.736	3.686	3.637	3.588	3.540	3.493	3.400	3.223	3.058	2.830	2.623	2.377	1.880	1.235	.527

Table 4. Upper 1% points of $V^{4}_{3,m,n}$

M \ N	20	25	30	35	40	45	50	60	80	100	130	160	200	300	500	1000
-0.5	.00669	.00384	.00240	.00160	.00112	$.0^{3}816$	$.0^{3}612$	$.0^{3}369$	$.0^{3}164$	$.0^{4}870$	$.0^{4}408$	$.0^{4}223$	$.0^{4}116$	$.0^{5}352$	$.0^{6}774$	$.0^{7}981$
0	.0132	.00784	.00492	.00330	.00240	.00175	.00132	$.0^{3}800$	$.0^{3}343$	$.0^{3}182$	$.0^{4}858$	$.0^{4}470$	$.0^{4}245$	$.0^{5}745$	$.0^{5}164$	$.0^{6}209$
0.5	.0193	.0117	.00753	.00508	.00359	.00263	.00198	.00126	$.0^{3}565$	$.0^{3}309$	$.0^{3}147$	$.0^{4}800$	$.0^{4}418$	$.0^{4}130$	$.0^{5}287$	$.0^{6}294$
1.0	.0276	.0164	.0107	.00739	.00516	.00389	.00286	.00182	$.0^{3}817$	$.0^{3}441$	$.0^{3}209$	$.0^{3}118$	$.0^{4}619$	$.0^{4}189$	$.0^{5}418$	$.0^{6}438$
1.5	.0363	.0222	.0145	.00987	.00713	.00527	.00406	.00248	.00114	$.0^{3}613$	$.0^{3}299$	$.0^{3}163$	$.0^{4}884$	$.0^{4}267$	$.0^{5}593$	$.0^{6}744$
2.0	.0471	.0286	.0185	.0130	.00930	.00701	.00546	.00332	.00153	$.0^{3}836$	$.0^{3}393$	$.0^{3}216$	$.0^{3}116$	$.0^{4}355$	$.0^{5}791$	$.0^{5}103$
2.5	.0595	.0365	.0240	.0166	.0119	.00887	.00677	.00420	.00198	.00106	$.0^{3}521$	$.0^{3}288$	$.0^{3}152$	$.0^{4}478$	$.0^{4}104$	$.0^{5}134$
3.0	.0731	.0452	.0299	.0208	.0150	.0112	.00857	.00535	.00248	.00135	$.0^{3}650$	$.0^{3}361$	$.0^{3}191$	$.0^{4}589$	$.0^{4}131$	$.0^{5}167$
3.5	.0876	.0547	.0364	.0254	.0185	.0138	.0106	.00665	.00310	.00169	$.0^{3}818$	$.0^{3}455$	$.0^{3}241$	$.0^{4}746$	$.0^{4}167$	$.0^{5}215$
4.0	.103	.0650	.0435	.0305	.0222	.0167	.0129	.00809	.00380	.00208	.00101	$.0^{3}562$	$.0^{3}298$	$.0^{4}926$	$.0^{4}208$	$.0^{5}267$
4.5	.120	.0759	.0511	.0361	.0264	.0199	.0153	.00969	.00458	.00251	.00122	$.0^{3}683$	$.0^{3}363$	$.0^{3}113$	$.0^{4}254$	$.0^{5}327$
5.0	.137	.0875	.0593	.0420	.0309	.0233	.0180	.0114	.00543	.00299	.00146	$.0^{3}818$	$.0^{3}435$	$.0^{3}136$	$.0^{4}306$	$.0^{5}395$
7	.212	.140	.0967	.0697	.0519	.0396	.0310	.0199	.00966	.00539	.00266	.00150	$.0^{3}806$	$.0^{3}254$	$.0^{4}578$	$.0^{5}751$
10	.337	.230	.164	.121	.0917	.0711	.0563	.0370	.0184	.0105	.00527	.00301	.00163	$.0^{3}522$	$.0^{3}120$	$.0^{4}158$
15	.556	.399	.295	.225	.181	.139	.112	.0758	.0394	.0230	.0119	.00693	.00382	.00125	$.0^{3}293$	$.0^{4}390$
20	.771	.574	.438	.342	.272	.220	.180	.125	.0675	.0404	.0214	.0127	.00710	.00238	$.0^{3}568$	$.0^{4}768$
25	.971	.745	.583	.465	.376	.309	.256	.182	.102	.0623	.0338	.0203	.0115	.00395	$.0^{3}961$	$.0^{3}132$
30	1.155	.907	.725	.588	.483	.402	.338	.245	.141	.0880	.0488	.0298	.0172	.00599	.00148	$.0^{3}207$
35	1.321	1.059	.861	.709	.591	.497	.422	.312	.184	.117	.0663	.0411	.0240	.00853	.00215	$.0^{3}303$
40	1.472	1.200	.990	.826	.696	.592	.507	.381	.230	.149	.0860	.0540	.0319	.0116	.00296	$.0^{3}424$
45	1.608	1.331	1.113	.939	.799	.686	.592	.451	.278	.184	.108	.0685	.0409	.0151	.00394	$.0^{3}571$
50	1.732	1.451	1.227	1.046	.898	.777	.676	.521	.328	.220	.131	.0843	.0510	.0191	.00507	$.0^{3}745$
60	1.946	1.665	1.435	1.244	1.085	.951	.839	.661	.432	.297	.182	.120	.0741	.0287	.00785	.00118
80	2.277	2.007	1.776	1.579	1.409	1.262	1.135	.926	.640	.460	.297	.203	.130	.0536	.0155	.00246
100	2.522	2.265	2.042	1.847	1.676	1.524	1.390	1.165	.840	.625	.421	.296	.196	.0852	.0259	.00431
130	2.787	2.554	2.343	2.159	1.992	1.842	1.706	1.471	1.114	.863	.610	.447	.308	.143	.0466	.00828
160	2.972	2.763	2.571	2.396	2.236	2.091	1.958	1.723	1.354	1.081	.793	.599	.427	.210	.0729	.0138
200	3.147	2.964	2.794	2.635	2.487	2.350	2.223	1.995	1.624	1.338	1.021	.797	.589	.308	.115	.0234
300	3.402	3.264	3.133	3.007	2.888	2.774	2.666	2.465	2.118	1.834	1.494	1.233	.968	.572	.247	.0589
500	3.626	3.535	3.446	3.359	3.275	3.193	3.113	2.961	2.684	2.438	2.122	1.858	1.570	1.066	.556	.168
1000	3.807	3.758	3.709	3.661	3.613	3.566	3.519	3.428	3.253	3.088	2.861	2.654	2.407	1.909	1.258	.539

Table 5. Upper 5% points of $V^{5}_{4,m,n}$

M \ N	20	25	30	35	40	45	50	60	80	100	130	160	200	300	500	1000
-0.5	$.0^{3}41$	$.0^{3}20$	$.0^{3}11$	$.0^{4}63$	$.0^{4}39$	$.0^{4}26$	$.0^{4}18$	$.0^{5}91$	$.0^{5}31$	$.0^{5}13$	$.0^{6}49$	$.0^{6}22$	$.0^{7}93$	$.0^{7}19$	$.0^{8}25$	$.0^{9}16$
0	$.0^{3}98$	$.0^{3}48$	$.0^{3}26$	$.0^{3}16$	$.0^{4}98$	$.0^{4}65$	$.0^{4}44$	$.0^{4}23$	$.0^{5}79$	$.0^{5}34$	$.0^{5}13$	$.0^{6}57$	$.0^{6}24$	$.0^{7}49$	$.0^{8}65$	$.0^{9}42$
0.5	.00171	$.0^{3}948$	$.0^{3}52$	$.0^{3}31$	$.0^{3}197$	$.0^{3}131$	$.0^{4}901$	$.0^{4}47$	$.0^{4}16$	$.0^{5}71$	$.0^{5}26$	$.0^{5}12$	$.0^{6}50$	$.0^{6}10$	$.0^{7}14$	$.0^{9}88$
1.0	.00329	.00167	$.0^{3}945$	$.0^{3}555$	$.0^{3}353$	$.0^{3}234$	$.0^{3}162$	$.0^{4}845$	$.0^{4}295$	$.0^{4}129$	$.0^{5}478$	$.0^{5}216$	$.0^{6}913$	$.0^{6}188$	$.0^{7}253$	$.0^{8}164$
1.5	.00522	.00264	.00147	$.0^{3}889$	$.0^{3}574$	$.0^{3}382$	$.0^{3}265$	$.0^{3}138$	$.0^{4}497$	$.0^{4}214$	$.0^{5}823$	$.0^{5}373$	$.0^{5}158$	$.0^{6}320$	$.0^{7}431$	$.0^{8}278$
2.0	.00758	.00389	.00219	.00136	$.0^{3}878$	$.0^{3}576$	$.0^{3}401$	$.0^{3}215$	$.0^{4}758$	$.0^{4}333$	$.0^{4}125$	$.0^{5}573$	$.0^{5}244$	$.0^{6}508$	$.0^{7}680$	$.0^{8}438$
2.5	.0105	.00545	.00312	.00194	.00124	$.0^{3}836$	$.0^{3}587$	$.0^{3}313$	$.0^{3}111$	$.0^{4}501$	$.0^{4}189$	$.0^{5}840$	$.0^{5}358$	$.0^{6}747$	$.0^{6}101$	$.0^{8}653$
3.0	.0142	.00740	.00430	.00273	.00167	.00116	$.0^{3}826$	$.0^{3}434$	$.0^{3}158$	$.0^{4}696$	$.0^{4}263$	$.0^{4}121$	$.0^{5}514$	$.0^{5}108$	$.0^{6}146$	$.0^{8}942$
3.5	.0183	.00973	.00571	.00354	.00230	.00157	.00110	$.0^{3}589$	$.0^{3}209$	$.0^{4}951$	$.0^{4}367$	$.0^{4}166$	$.0^{5}710$	$.0^{5}149$	$.0^{6}205$	$.0^{7}133$
4.0	.0232	.0125	.00733	.00457	.00298	.00207	.00145	$.0^{3}778$	$.0^{3}283$	$.0^{3}127$	$.0^{4}485$	$.0^{4}230$	$.0^{5}936$	$.0^{5}202$	$.0^{6}274$	$.0^{7}178$
4.5	.0287	.0156	.00923	.00580	.00382	.00262	.00185	.00100	$.0^{3}375$	$.0^{3}166$	$.0^{4}635$	$.0^{4}293$	$.0^{4}125$	$.0^{5}267$	$.0^{6}365$	$.0^{7}238$
5.0	.0348	.0192	.0114	.00720	.00477	.00328	.00233	.00127	$.0^{3}470$	$.0^{3}212$	$.0^{4}806$	$.0^{4}376$	$.0^{4}165$	$.0^{5}347$	$.0^{6}472$	$.0^{7}310$
7	.0657	.0375	.0229	.0148	.00997	.00696	.00500	.00278	.00106	$.0^{3}485$	$.0^{3}189$	$.0^{4}884$	$.0^{4}385$	$.0^{5}826$	$.0^{5}114$	$.0^{7}753$
10	.129	.0771	.0489	.0325	.0224	.0160	.0117	.00666	.00262	.00123	$.0^{3}493$	$.0^{3}234$	$.0^{3}103$	$.0^{4}225$	$.0^{5}317$	$.0^{6}211$
15	.268	.171	.114	.0789	.0563	.0412	.0309	.0183	.00762	.00371	.00154	$.0^{3}746$	$.0^{3}336$	$.0^{4}758$	$.0^{4}110$	$.0^{6}744$
20	.433	.290	.201	.145	.106	.0792	.0606	.0372	.0162	.00817	.00349	.00173	$.0^{3}798$	$.0^{3}185$	$.0^{4}274$	$.0^{5}190$
25	.609	.424	.304	.223	.168	.129	.100	.0632	.0288	.0150	.00659	.00333	.00157	$.0^{3}373$	$.0^{4}567$	$.0^{5}400$
30	.786	.565	.416	.313	.240	.187	.148	.0960	.0455	.0243	.0110	.00569	.00272	$.0^{3}665$	$.0^{3}103$	$.0^{5}742$
35	.959	.708	.534	.410	.320	.253	.203	.135	.0662	.0362	.0168	.00887	.00431	.00108	$.0^{3}172$	$.0^{4}126$
40	1.125	.850	.654	.511	.405	.325	.264	.179	.0906	.0507	.0242	.0130	.00640	.00165	$.0^{3}267$	$.0^{4}199$
45	1.283	.989	.774	.614	.493	.401	.329	.227	.119	.0677	.0331	.0180	.00904	.00238	$.0^{3}395$	$.0^{4}300$
50	1.432	1.123	.892	.717	.583	.479	.397	.279	.149	.0871	.0435	.0241	.0123	.00330	$.0^{3}559$	$.0^{4}432$
60	1.703	1.374	1.119	.921	.764	.639	.539	.390	.219	.132	.0687	.0391	.0205	.00578	.00102	$.0^{4}814$
80	2.150	1.806	1.527	1.299	1.112	.957	.828	.629	.381	.244	.135	.0811	.0447	.0136	.00258	$.0^{3}221$
100	2.498	2.157	1.871	1.630	1.426	1.253	1.105	.869	.558	.374	.219	.137	.0788	.0257	.00523	$.0^{3}476$
130	2.891	2.568	2.287	2.043	1.830	1.643	1.480	1.209	.829	.587	.367	.241	.146	.0522	.0117	.00116
160	3.182	2.879	2.611	2.372	2.160	1.971	1.801	1.513	1.089	.803	.529	.362	.230	.0884	.0214	.00231
200	3.467	3.192	2.943	2.717	2.513	2.327	2.158	1.862	1.407	1.082	.751	.537	.357	.150	.0400	.00475
300	3.899	3.683	3.480	3.290	3.113	2.947	2.791	2.510	2.044	1.681	1.273	.981	.710	.349	.113	.0166
500	4.297	4.148	4.005	3.867	3.734	3.606	3.484	3.254	2.846	2.499	2.071	1.731	1.378	.819	.341	.0688
1000	4.631	4.548	4.467	4.386	4.308	4.230	4.155	4.008	3.731	3.477	3.134	2.832	2.482	1.816	1.036	.333

Table 6. Upper 1% points of $V^{5}_{4,m,n}$

M \ N	20	25	30	35	40	45	50	60	80	100	130	160	200	300	500	1000
-0.5	$.0^{3}91$	$.0^{3}44$	$.0^{3}24$	$.0^{3}14$	$.0^{4}89$	$.0^{4}58$	$.0^{4}40$	$.0^{4}21$	$.0^{5}70$	$.0^{5}30$	$.0^{5}11$	$.0^{6}50$	$.0^{6}21$	$.0^{7}43$	$.0^{8}57$	$.0^{9}37$
0	.0021	.0010	$.0^{3}57$	$.0^{3}34$	$.0^{3}21$	$.0^{3}14$	$.0^{4}97$	$.0^{4}50$	$.0^{4}17$	$.0^{5}75$	$.0^{5}28$	$.0^{5}12$	$.0^{6}52$	$.0^{6}11$	$.0^{7}14$	$.0^{9}92$
0.5	.00405	.00202	.0011	$.0^{3}66$	$.0^{3}421$	$.0^{3}279$	$.0^{3}193$	$.0^{3}10$	$.0^{4}35$	$.0^{4}15$	$.0^{5}56$	$.0^{5}25$	$.0^{5}11$	$.0^{6}22$	$.0^{7}30$	$.0^{8}19$
1.0	.00661	.00338	.00194	.00116	$.0^{3}740$	$.0^{3}511$	$.0^{3}354$	$.0^{3}185$	$.0^{4}623$	$.0^{4}272$	$.0^{4}101$	$.0^{5}458$	$.0^{5}193$	$.0^{6}398$	$.0^{7}535$	$.0^{8}344$
1.5	.00914	.00486	.00274	.00166	.00111	$.0^{3}739$	$.0^{3}526$	$.0^{3}281$	$.0^{4}999$	$.0^{4}354$	$.0^{4}168$	$.0^{5}763$	$.0^{5}324$	$.0^{6}668$	$.0^{7}900$	$.0^{8}579$
2.0	.0129	.00662	.00390	.00236	.00153	.00106	$.0^{3}738$	$.0^{3}394$	$.0^{3}141$	$.0^{4}619$	$.0^{4}232$	$.0^{4}110$	$.0^{5}469$	$.0^{6}958$	$.0^{6}131$	$.0^{8}846$
2.5	.0167	.00903	.00520	.00328	.00216	.00144	$.0^{3}995$	$.0^{3}528$	$.0^{3}197$	$.0^{4}871$	$.0^{4}328$	$.0^{4}154$	$.0^{5}656$	$.0^{5}137$	$.0^{6}185$	$.0^{7}120$
3.0	.0214	.0117	.00689	.00418	.00277	.00191	.00133	$.0^{3}720$	$.0^{3}264$	$.0^{3}120$	$.0^{4}453$	$.0^{4}204$	$.0^{5}894$	$.0^{5}183$	$.0^{6}257$	$.0^{7}167$
3.5	.0270	.0144	.00879	.00559	.00358	.00247	.00174	$.0^{3}934$	$.0^{3}345$	$.0^{3}157$	$.0^{4}590$	$.0^{4}274$	$.0^{4}118$	$.0^{5}247$	$.0^{6}346$	$.0^{7}222$
4.0	.0335	.0183	.0108	.00676	.00442	.00309	.00223	.00121	$.0^{3}446$	$.0^{3}195$	$.0^{4}767$	$.0^{4}351$	$.0^{4}154$	$.0^{5}324$	$.0^{6}450$	$.0^{7}293$
4.5	.0408	.0225	.0134	.00844	.00559	.00384	.00273	.00149	$.0^{3}557$	$.0^{3}256$	$.0^{4}982$	$.0^{4}455$	$.0^{4}197$	$.0^{5}418$	$.0^{6}573$	$.0^{7}379$
5.0	.0489	.0272	.0163	.0103	.00688	.00475	.00338	.00185	$.0^{3}691$	$.0^{3}313$	$.0^{3}119$	$.0^{4}552$	$.0^{4}245$	$.0^{5}530$	$.0^{6}728$	$.0^{7}486$
7	.0880	.0507	.0313	.0203	.0137	.00963	.00695	.00388	.00148	$.0^{3}685$	$.0^{3}268$	$.0^{3}126$	$.0^{4}548$	$.0^{4}118$	$.0^{5}164$	$.0^{6}108$
10	.165	.0995	.0636	.0425	.0295	.0211	.0155	.00887	.00352	.00166	$.0^{3}667$	$.0^{3}317$	$.0^{3}140$	$.0^{4}308$	$.0^{5}435$	$.0^{6}290$
15	.326	.210	.141	.0981	.0703	.0517	.0389	.0232	.00973	.00477	.00198	$.0^{3}966$	$.0^{3}437$	$.0^{4}988$	$.0^{4}143$	$.0^{6}975$
20	.509	.344	.240	.175	.128	.0961	.0737	.0455	.0200	.0101	.00436	.00217	.00100	$.0^{3}234$	$.0^{4}347$	$.0^{5}241$
25	.699	.491	.355	.262	.198	.152	.119	.0756	.0348	.0181	.00804	.00409	.00192	$.0^{3}461$	$.0^{4}702$	$.0^{5}497$
30	.887	.643	.477	.361	.278	.218	.173	.113	.0539	.0289	.0132	.00683	.00328	$.0^{3}806$	$.0^{3}126$	$.0^{5}907$
35	1.068	.795	.604	.466	.365	.290	.234	.156	.0772	.0424	.0199	.0105	.00512	.00129	$.0^{3}206$	$.0^{4}152$
40	1.240	.944	.731	.574	.457	.368	.299	.204	.104	.0587	.0282	.0152	.00752	.00195	$.0^{3}317$	$.0^{4}238$
45	1.402	1.089	.857	.683	.551	.449	.370	.257	.135	.0776	.0381	.0209	.0105	.00279	$.0^{3}464$	$.0^{4}354$
50	1.554	1.226	.980	.792	.646	.533	.443	.313	.169	.0990	.0497	.0276	.0141	.00383	$.0^{3}652$	$.0^{4}506$
60	1.827	1.483	1.214	1.003	.836	.701	.593	.432	.244	.148	.0775	.0443	.0223	.00661	.00117	$.0^{4}940$
80	2.272	1.918	1.629	1.391	1.194	1.031	.894	.682	.416	.268	.150	.0899	.0497	.0152	.00291	$.0^{3}250$
100	2.614	2.267	1.974	1.725	1.514	1.334	1.179	.931	.602	.405	.239	.150	.0865	.0284	.00581	$.0^{3}530$
130	2.998	2.670	2.386	2.137	1.919	1.727	1.559	1.278	.882	.627	.394	.260	.158	.0568	.0128	.00127
160	3.283	2.975	2.704	2.463	2.248	2.055	1.882	1.586	1.147	.850	.562	.386	.246	.0952	.0232	.00251
200	3.558	3.282	3.031	2.803	2.597	2.408	2.237	1.936	1.469	1.134	.790	.567	.378	.160	.0429	.00513
300	3.973	3.758	3.555	3.365	3.186	3.019	2.862	2.578	2.107	1.737	1.321	1.021	.741	.366	.119	.0176
500	4.349	4.202	4.060	3.923	3.791	3.664	3.542	3.311	2.902	2.553	2.121	1.776	1.417	.845	.354	.0718
1000	4.660	4.579	4.499	4.420	4.343	4.266	4.192	4.046	3.771	3.518	3.175	2.871	2.520	1.849	1.059	.342

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