98

Convergence of sums of squares of martingale differences[1]

by

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1. Introduction and notation. Let $(\Omega,\mathcal{F},P)$ be a probability space. A stochastic basis $(\mathcal{F}_n, n \geq 1)$ is a monotonically increasing sequence of σ-fields of measurable sets. A stochastic sequence $(y_n,\mathcal{F}_n,n \geq 1)$ consists of a stochastic basis $(\mathcal{F}_n,n \geq 1)$ and a sequence of random variables $(y_n,n \geq 1)$ such that y_n is $\mathcal{F}_n$-measurable. For a stochastic sequence $(x_n,\mathcal{F}_n,n \geq 1)$, we put (here as well as in following sections)

$$x_0 = 0, \quad \mathcal{F}_0 = \{\Phi,\Omega\}, \quad d_n = x_n - x_{n-1} \text{ for } n \geq 1, \quad s_n = (\Sigma_{k=1}^n d_k^2)^{1/2},$$
$$x^* = \sup_{n \geq 1}|x_n|, \quad d^* = \sup_{n \geq 1}|d_n|, \quad s = \lim_{n \to \infty} s_n,$$

and I_A = indicator function of set A. If $(x_n,\mathcal{F}_m,n \geq 1)$ is a martingale, then $(d_n,\mathcal{F}_n, n \geq 1)$ is called a martingale difference sequence. For a given stochastic basis $(\mathcal{F}_n, n \geq 1)$, a stopping time t is an extended positive integral valued measurable function such that $[t = n] \in \mathcal{F}_n$ for each n. For a stopping time t and a measurable function y, $E_t y$ is defined as $\int_{[t < \infty]} y \, dP$ (or $\int_{[t < \infty]} y$, in short), if it exists.

Let $(x_n,\mathcal{F}_n,n \geq 1)$ be a martingale. Austin [1] recently proves that if $\sup_{n \geq 1} E|x_n| < \infty$, $s < \infty$ a.e.; also Burkholder [2] proves that if $E\, s < \infty$ x_n converges a.e. and that if $\sup_{n \geq 1}E|x_n| < \infty$, then $\Sigma_{k=1}^\infty \varphi_k d_k$ converges a.e. for every stochastic sequence $(\varphi_k,\mathcal{F}_{k-1},n \geq 1)$ for which $\sup_{n \geq 1}|\varphi_n| < \infty$ a.e.; Gundy [6] proves that if $(d_n,n \geq 1)$ is an orthonormal sequence such that

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each d_n assumes at most two nonzero values with positive probability, and if the σ-field generated by $d_1, \ldots, d_n$ consists of exactly n atoms, such that

$$\inf_{n \geq 1} \min(P[d_n > 0], P[d_n < 0]) / P[d_n \neq 0] > 0 ,$$

then for every sequence a_n of real numbers, $\Sigma_{n=1}^{\infty} a_n^2 d_n^2 < \infty$ if and only if $\Sigma_{n=1}^{\infty} a_n d_n$ converges .

Let $(\mathcal{F}_n, n \geq 1)$ be a stochastic basis if for each n, $\mathcal{F}_n$ is generated by atoms of $\mathcal{F}_n$, then $(\mathcal{F}_n, n \geq 1)$ is said to be atomic. For a σ-field $\mathcal{G}$ of measurable sets and $A \in \mathcal{F}$, a $\mathcal{G}$-measurable cover of A is a set $C \in \mathcal{G}$ such that $P(A - C) = 0$ and that if $B \in \mathcal{G}$ and $P(A - B) = 0$, then $P(C - B) = 0$. For $A \in \mathcal{F}$, let $C_n(A)$ be the $\mathcal{F}_n$-measurable cover of A. If there exists $M > 0$ such that $P\, C_n(A) \leq M\, P\, A$ for every $A \in \mathcal{F}_n$, $n = 1,2,\ldots,$ then $(\mathcal{F}_n, n \geq 1)$ is said to be regular.

Let $(x_n, \mathcal{F}_n, n \geq 1)$ be a submartingale and $E|x_n| < \infty$ for each n. If $(\mathcal{F}_n, n \geq 1)$ is an atomic, regular stochastic basis, then [3] x_n converges a.e. on the set $[\sup x_n < \infty]$. In [5], Doob extends this result to the non-atomic cases: if for $K > 0$, there exist $M \geq K$ and $\delta > 0$ such that

$$P\{[\max_{k \leq n} x_k < K] - ([E(x_{n+1} \geq M|\mathcal{F}_n) = 0] \cup [E(x_{n+1} \geq K|\mathcal{F}_n) > \delta])\} = 0 ,$$

then x_n converges a.e. on the set $[\sup_{n \geq 1} x_n < K]$.

In this paper, we will give new proofs of those theorems mentioned above and in some cases extend them, by method of stopping times. The results of Gundy, Austin and Burkholder are unified into Theorems 3 and 5. Theorem 5 extends a result of Doob [4; 320] to regular stochastic basis.

2. A proof of Austin's theorem.

Theorem 1 (Austin [1]). If $(x_n, \mathscr{F}_n, n \geq 1)$ is a martingale and $\sup_{n \geq 1} E|x_n| = M < \infty$, then $s < \infty$ a.e. .

Proof. Let $\delta > 0$ and $K > 1$. Put

$$g_0 = 1, \qquad g_n = \Pi_{k=1}^{n} (1 + d_k K^{-1}) \text{ for } n \geq 1 ,$$

$$t = t_K = \inf \{n \,|\, |g_n| \geq 1 + \delta \text{ or } |x_n| \geq \log K\} .$$

Set $h_n = g_{\min(t,n)}$. Then

$$|h_n| \leq (1 + \delta) I_{[t > n]} + (1 + \delta)(1 + |d_t| K^{-1}) I_{[t \leq n]} .$$

Since

$$\int_{[t \leq n]} |d_t| \leq \int_{[t \leq n]} (|x_t| + |x_{t-1}|)$$

$$\leq \log K + E|x_{\min(t,n)}| \leq \log K + M .$$

Hence $E_t|d_t| \leq \log K + M$, where $E_t|d_t| = \int_{[t < \infty]} |d_t|$. Therefore, $E h^* < \infty$ and $(h_n, \mathscr{F}_n, n \geq 1)$ is a martingale. By Doob's martingale convergence theorem [4, p.319], h_n tends to h_∞ a.e. and in L_1.

$$1 = E h_\infty = \int_{[t < \infty]} h_t + \int_{[t = \infty]} h_\infty$$

$$\leq (1 + \delta) \int_{[t < \infty]} (1 + |d_t| K^{-1}) + (1 + \delta) P[t = \infty, h_\infty > 0] .$$

Let $\epsilon > 0$. Since $E_t|d_t| \leq \log K + M$, $E_t|d_t| \leq \epsilon K$ for all large K and $(1 + \delta) P[t_K = \infty, h_\infty \leq 0] \leq \delta + (1 + \delta)\epsilon$. Since x_n converges a.e., we have that $\lim g_n = g_\infty$ exists a.e., $\lim_{K \to \infty} P[t_K = \infty] = 1$, and that $g_\infty > 0$

for all large K if and only if $s < \infty$. Hence $P[g_\infty \leq 0] \leq \delta + 2\epsilon$ and $P[s = \infty] \leq \delta + 3\epsilon$, if K is large enough. Therefore $P[s = \infty] = 0$.

3. A proof of Burkholder's theorem.

Theorem 2 (Burkholder [2]). If $(x_n, \mathcal{F}_n, n \geq 1)$ is a margingale and $E\, s < \infty$, then x_n converges a.e. .

Proof. If $d_n \neq 0$, then $d_n(s_n - s_{n-1})^{-1} = s_n + s_{n-1} \leq 2s$. For $K > 1$, let

$$t = t_K = \inf\{n \mid 0 < d_n^2 \geq K(s_n - s_{n-1})\} \quad .$$

Since $s < \infty$ a.e., $\lim_{K \to \infty} P[t_K = \infty] = 1$. Put $e_k = d_k I_{[t \geq k,\, d_k^2 \leq 1]}$ and $z_n = \Sigma_{k=1}^n (e_k - E(e_k|\mathcal{F}_{k-1}))$. Then $(z_n, \mathcal{F}_n, n \geq 1)$ is a martingale, and

$$E\, z_n^2 \leq \Sigma_{k=1}^n E\, e_k^2 \leq \Sigma_{k=1}^n \left(\int_{[t > k]} d_k^2 + \int_{[t = k]} |d_k|\right)$$

$$\leq K\, \Sigma_{k=1}^n E(s_k - s_{k-1}) + \int_{[t < \infty]} |d_t| \leq (K + 1)\, E\, s \quad .$$

Hence z_n converges a.e. . Now

$$|E(e_k|\mathcal{F}_{k-1})| = |E(I_{[t \geq k,\, d_k^2 > 1]}\, d_k|\mathcal{F}_{k-1})|$$

$$\leq E(I_{[t = k]}\, |d_t|\, \mathcal{F}_{k-1}) + E(I_{[t > k]}\, d_k^2|\mathcal{F}_{k-1})$$

$$\leq E(I_{[t = k]}\, |d_t|\mathcal{F}_{k-1}) + K\, E(s_k - s_{k-1}|\mathcal{F}_{k-1}) \quad .$$

Hence $E\, \Sigma_{k=1}^\infty |E(e_k|\mathcal{F}_{k-1})| \leq E_t|d_t| + K\, E\, s < \infty$ and $\Sigma_{k=1}^\infty E(e_k|\mathcal{F}_{k-1})$ converges a.e. . Therefore $\Sigma_{k=1}^\infty e_k$ converges a.e. . Since $\lim d_k = 0$ a.e., x_n converges a.e. on $[t_k = \infty]$. Hence x_n converges a.e. .

Alternatively, we can also prove Theorem 2 as follows. Put $a_k = d_k I_{[t \geq k]}$. Then $\Sigma_{k=1}^{n} a_k$ is a martingale and

$$\sum_{k=1}^{\infty} E(a_k^2 I_{[a_k^2 \leq 1]} + |a_k| I_{[a_k^2 \geq 1]}) < \infty ,$$

which can be proved by the same method that was used in the preceding proof. By a theorem of Loéve [10], $\Sigma_{k=1}^{\infty} a_k$ converges a.e. and then x_n converges a.e. .

4. An induced stopping time.

Let $(\mathcal{F}_n, n \geq 1)$ be a stochastic basis and t be a stopping time. Let $(B_n, n \geq 1)$ be a sequence of measurable sets such that $B_n \in \mathcal{F}_n$ for each n. Let $A_{n+1} = B_n[t = n + 1]$ and $C_n = C_n(A_{n+1})$ be the $\mathcal{F}_n$-measurable cover of A_{n+1}. For $m = 1,2,\ldots,$ define

$$\tau = \tau_m = \inf \{n \geq m \mid \omega \in C_n\} ,$$

and

(1) $$t^* = t_m^* = \min (t, \tau) .$$

Then the stopping time t^* is said to be induced by $\{t, (B_n, n \geq 1), m\}$.

Lemma 1. Let t be a stopping time and $(B_n, n \geq 1)$ be a sequence of measurable sets such that $B_n \in \mathcal{F}_n$. For $m = 1,2,\ldots,$ define t^* by (1). Then t^* is a stopping time, $t^* \leq t$ a.e., $[t^* = t = k] \subset [t = k] - B_{k-1}$ for $m < k < \infty$, and if

(2) $$P[C_n, \text{ i.o.}] = \lim_{n \to \infty} P(\bigcup_{k=n}^{\infty} C_k) = 0 ,$$

then

(3) $$\lim_{m \to \infty} P[t_m^* < t = \infty] = 0 .$$

Proof. Obviously t^* is a stopping time and $t^* \leq t$ a.e. . Since $[t > n] \supset C_n \subset B_n$, $\tau < t$ if $\tau < \infty$. For $m < k < \infty$, if $t^*(\omega) = t(\omega) = k$, then $\omega \notin \bigcup_{j=m}^{\infty} C_j$, $\omega \notin C_{k-1} \supset B_{k-1}[t = k]$ and $\omega \in [t = k] - B_{k-1}$. If (2) holds, then

$$\lim_{m \to \infty} P[t_m^* < t = \infty] \leq \lim P[\tau_m < t] = \lim P(\bigcup_{k=m}^{\infty} C_k) = 0 \quad ,$$

which yields (3).

In most applications, in Lemma 1, we either put $B_n = \phi$ for every n or put $B_n = \Omega$ for every n. In the former case, (2) is automatically satisfied; in the latter case, if $(\mathcal{F}_n, n \geq 1)$ is regular, (2) is always satisfied, since for some $M > 0$,

$$(4) \qquad P(\bigcup_{k=n}^{\infty} C_k) \leq \sum_{k=n}^{\infty} P\, C_k \leq M \sum_{k=n}^{\infty} P\, A_{k+1} \leq M\, P[n < t < \infty] \quad .$$

5. Main results.

Theorem 3. Let $(x_n, \mathcal{F}_n, n \geq 1)$ be a martingale, $E|x_n| < \infty$ and for $K > 0$, put

$$(5) \qquad t = \inf \{n | x_n \geq K\} \quad .$$

Let $(y_n, \mathcal{F}_n, n \geq 1)$ be a stochastic sequence such that $y_n \geq 0$ a.e. . For $n \geq 1$, let $B_n \in \mathcal{F}_n$ and

$$(6) \qquad B_n \supset [E(I_{[t = n+1]}(x_t - y_t) \mid \mathcal{F}_n) > 0] \quad .$$

Let $A_{n+1} = B_n[t = n + 1]$ and C_n be the $\mathcal{F}_n$-measurable cover of A_{n+1}. If (2) holds and

(7) $$\sum_{k=2}^{\infty} \int_{[t = k] - B_{k-1}} y_t = M < \infty \quad ,$$

then

(8) $$P\,[s = \infty, \ \sup x_n < K] = 0 \quad .$$

Proof. For $m = 1,2,\ldots,$ define $t^* = t^*_m$ by (1). Put $z_n = \Sigma_{k=1}^n d_k\, I_{[t^* \geq k]}$. Then $(z_n, \mathcal{F}_n, n \geq 1)$ is a martingale and

$$\begin{aligned} z_n &\leq K && , \text{ if } t^* > n \text{ or } t > t^* < n \ , \\ &= x_t > 0 && , \text{ if } m < t^* = t = k \leq n \ , \\ &\leq |d_1| + \ldots + |d_m| && , \text{ if } t^* \leq m \quad . \end{aligned}$$

Since

$$\sum_{k=m+1}^{\infty} \int_{[t^*=t=k]} x_t \leq \sum_{k=m+1}^{\infty} \int_{[t=k]-B_{k-1}} ((x_t - y_t) + y_t)$$

$$\leq M + \sum_{k=m+1}^{\infty} \int_{\Omega - B_{k-1}} E(I_{[t=k]}\,(x_t - y_t)\,|\,\mathcal{F}_{k-1}) \leq M \quad ,$$

then $\sup E\, z_n^+ < \infty$ and thus $\sup E|z_n| < \infty$. By Austin's theorem,

$$\sum_{k=1}^{\infty} d_k^2 \ I_{[t^* \geq k]} < \infty \quad \text{a.e.} \quad .$$

Therefore $P[s = \infty, \ t^*_m = \infty] = 0$. By Lemma 1, $P[s = \infty, \ t = \infty] = 0$. Hence $P[s = \infty, \ \sup x_n < K] = 0$.

Theorem 4. Let $(x_n, \mathcal{F}_n, n \geq 1)$ be a martingale, $E\, x_n^2 < \infty$, and for $K > 0$, put

$$t = \inf\{n \mid |x_n| \geq K\} \quad . \tag{9}$$

Assume that $(y_n, \mathcal{F}_n, n \geq 1)$ is a stochastic sequence such that $y_n \geq 0$ a.e. . For $n \geq 1$, let $B_n \in \mathcal{F}_n$ and

$$B_n \supset [E(I_{[t=n+1]}(x_t^2 - y_t) \mid \mathcal{F}_n) > 0] \quad . \tag{10}$$

Let $A_{n+1} = B_n[t = n+1]$ and C_n be the $\mathcal{F}_n$-measurable cover of A_{n+1}. If (2) and (7) hold, then

$$P\left[\sum_{k=1}^{\infty} E(d_{k+1}^2 \mid \mathcal{F}_k) = \infty \,,\; x^* < K\right] = 0, \tag{11}$$

$$P\left[x_n \text{ diverges}, \sum_{k=1}^{\infty} E(d_{k+1}^2 \mid \mathcal{F}_k) < \infty\right] = 0 \quad . \tag{12}$$

Proof. For $m = 1,2,\ldots,$ define $t^* = t_m^*$ by (1). Put $z_n = \Sigma_{k=1}^{n} d_k I_{[t^* \geq k]}$. Then $(z_n, \mathcal{F}_n, n \geq 1)$ is a martingale and

$$\begin{aligned} |z_n| &\leq K &&, \text{ if } t^* > n \text{ or } t > t^* < n \;, \\ &= |x_k| &&, \text{ if } m < t^* = t = k \leq n \;, \\ &\leq |d_1| + \ldots + |d_m|, && \text{ if } t^* \leq m \;. \end{aligned}$$

As in the proof of Theorem 3, we have

$$\sum_{k=m+1}^{\infty} \int_{[t^*=t=k]} x_t^2 \leq \sum_{k=m+1}^{\infty} \int_{[t=k]-B_{k-1}} ((x_t^2 - y_t) + y_t)$$

$$\leq M \ ;$$

hence $\sup E\, z_n^2 < \infty$ and $E\, \Sigma_{k=1}^{\infty} d_k^2\, I_{[t^* \geq k]} < \infty$. Therefore $\Sigma_{k=1}^{\infty} E(d_k^2 \mid \mathcal{F}_{k-1})\, I_{[t^* \geq k]} < \infty$ a.e. and

$$P[\ \Sigma_{k=1}^{\infty} E(d_k^2 \mid \mathcal{F}_{k-1}) = \infty\ ,\ t_m^* = \infty] = 0 \ .$$

By Lemma 1, (11) holds.

In [6; p.320], Doob stated that if $(x_n, \mathcal{F}_n, n \geq 1)$ is a martingale and $E(d^*)^2 < \infty$, then x_n converges if and if $\Sigma\, E(d_{k+1}^2 | \mathcal{F}_k) < \infty$. However, his proof of the "if" part requires only the assumption that $E(d_{k+1}^2 | \mathcal{F}_k) < \infty$. Hence (12) is a special case of Doob's theorem.

Theorem 5. Let $(x_n, \mathcal{F}_n, n \geq 1)$ be a martingale, $E|x_n| < \infty$, and for $K > 0$, define

$$t = \inf \{n \mid s_n \geq K\} \ . \tag{13}$$

Assume $(y_n, \mathcal{F}_n, n \geq 1)$ is a stochastic sequence such that $y_n \geq 0$, a.e. . For $n \geq 1$, let $B_n \in \mathcal{F}_n$ and

$$B_n \supset [E(I_{[t=n+1]}\ (s_t - y_t | \mathcal{F}_n) > 0] \ . \tag{14}$$

Let $A_{n+1} = B_n[t=n+1]$ and C_n be the $\mathcal{F}_n$-measurable cover of A_{n+1} . If (2) and (7) hold, then

(15) $$P[x_n \text{ diverges}, \ s < K] = 0 \ .$$

Proof. For $m = 1,2,\ldots,$ define $t = t_m^*$ by (1). Put $e_k = d_k I_{[t^* \geq k]}$ and $z_n = \Sigma_{k=1}^n e_k$. Then $(z_n, \mathscr{F}_n, n \geq 1)$ is a martingale, and

$$E(\sum_{k=1}^n e_k^2)^{1/2} \leq \int_{[t^* > n]} s_n + \int_{[t^* \leq m]} s_m + \sum_{k=m+1}^n \int_{[t^* = k \leq t]} s_k$$

$$\leq K + E\, s_m + \sum_{k=m+1}^\infty \int_{[t^* = t = k]} s_k \ .$$

As before, we have

$$\sum_{k=m+1}^\infty \int_{[t^* = t = k]} s_k \leq \sum_{k=m+1}^\infty \int_{[t=k-B_{k-1}} (s_t - y_t + y_t) \leq M \ .$$

Hence $E(\Sigma_{k=1}^\infty e_k^2)^{1/2} < \infty$ and by Burkholder's theorem, z_n converges a.e. . Hence $P[x_n \text{ diverges}, \ t^* = \infty] = 0$. By Lemma 1, we have (15).

6. Application and Corollaries.

In order to apply Theorems 3, 4 and 5, we need some identification of a measurable cover, which is furnished by the following lemma.

Lemma 2. Let $\mathcal{G}$ be a σ-field of measurable sets and let C be the $\mathcal{G}$-measurable cover of a measurable set A. Then $C = [P(A|\mathcal{G}) > 0]$.

Proof. First, $P(A-C) = PA - P(AC) = PA - \int_C P(A|\mathcal{G}) = PA - E(P(A|\mathcal{G})) = 0$. Now, let $B \in \mathcal{G}$ and $P(A - B) = 0$. Then $P(A - ABC) = 0$ and $P(A(C - B)) = 0$. Hence $\int_{C-B} P(A|\mathcal{G}) = P(A(C - B)) = 0$. Since $P(A|\mathcal{G}) > 0$ a.e. on C, $P(C - B) = 0$.

Theorem 6. Let $(x_n, \mathcal{F}_n, n \geq 1)$ be a martingale with $E|x_n| < \infty$.

(i) If $(\mathcal{F}_n, n \geq 1)$ is a regular stochastic basis, then except on a null set, the following statements are equivalent:

(16) $$s < \infty ,$$

(17) $$x_n \text{ converges},,$$

(18) $$\sum_{k=1}^{\infty} E(d_{k+1}^2 \mid \mathcal{F}_k) < \infty .$$

(ii) If $E_t\, d_t^2 < \infty$ for every stopping time t of the form $t = \inf\{n \mid |x_n| \geq K\}$, then, except on a null set, (17) and (18) are equivalent.

(iii) For $K > 0$, put $t = \inf\{n | x_n \geq K\}$ and $\tau = \inf\{n | s_n \geq K\}$. If $E_t\, x_t < \infty$, then $P[s = \infty, \sup x_n < K] = 0$, and if $E_\tau\, s_\tau < \infty$, then $P[x_n$ diverges, $s < K] = 0$. In particular, if $E_\sigma |d_\sigma| < \infty$ for every stopping time σ, then, except on a null set, (16) and (17) are equivalent.

(iv) For $K > 0$, put $t = \inf\{n | x_n \geq K\}$ and $\tau = \inf\{n | s_n \geq K\}$. Let $M \geq K$ and $\delta > 0$. If

(19) $$P\{[t \geq n] - ([P(x_{n+1} \geq M|\mathcal{F}_n) = 0] \cup [P(x_{n+1} \geq K|\mathcal{F}_n) \geq \delta])\} = 0 ,$$

then $P[s = \infty, \sup x_n < K] = 0$, and if

(20) $$P\{[s \geq n] - ([P(s_{n+1} \geq M|\mathcal{F}_n) = 0] \cup [P(s_{n+1} \geq K|\mathcal{F}_n) \geq \delta])\} = 0 ,$$

then $P[x_n$ diverges, $s < K] = 0$.

Proof. (i) Put $y_n = 0$ and $B_n = \Omega$ for each n. Then $A_n = [t = n]$ and for some $M > 0$,

$$P(\cup_{k=n}^{\infty} C_k) \leq \sum_{k=n}^{\infty} P\,C_k \leq M \sum_{k=n+1}^{\infty} P\,A_k = M\,P[n < t < \infty] \quad .$$

Hence (2) holds. The equivalence of (16), (17) and (18) follows immediately from Theorems 3,4,5 and Lemma 1.

(ii) For $K > 0$, put $y_n = 2K + 2d_n^2$. On the set $[t = n+1]$,

$$x_t^2 \leq 2(x_{t-1}^2 + d_t^2) \leq 2K + 2d_t^2 = y_t \quad .$$

Let $B_n = \phi$ for each n in (10). Theorem 4 implies that (17) and (18) are equivalent.

(iii) Assume that $E_t\, x_t < \infty$. Put $y_n = \max(0, x_n)$. Then $E_t\, y_t < \infty$. Let $B_n = \phi$ for each n in (6). By Theorem 3, $P[s = \infty, \sup x_n < K] = 0$.

Assume that $E_\tau\, s_\tau < \infty$. Put $y_n = s_n$. Then $E_\tau\, y_\tau < \infty$. Let $B_n = \phi$ for each n in (14). By Theorem 5, $P[x_n$ diverges, $s < K] = 0$.

If $E_\sigma |d_\sigma| < \infty$ for every stopping time σ, then for every $K > 0$, $E_t\, x_t \leq K + E_t|d_t| < \infty$ and $E_\tau s_\tau \leq K + E_\tau|d_\tau| < \infty$. Hence (16) and (17) are equivalent.

(iv) For the first part, we will apply Theorem 3. Put $y_n = M$ for each n and $B_n = [E(I_{[t=n+1]}(x_t - M)|\mathcal{F}_n) > 0]$. From Lemma 2,

$$\begin{aligned} P\,C_n &= P\,[P(A_{n+1}|\mathcal{F}_n) > 0] \\ &= P\,[t > n,\ P(x_{n+1} \geq K|\mathcal{F}_n) > 0,\ E(I_{[x_{n+1} \geq K]}\,(x_{n+1} - M)|\mathcal{F}_n) > 0] \\ &\leq P\,[t > n,\ E(I_{[x_{n+1} \geq M]}\,(x_{n+1} - M)|\mathcal{F}_n > 0] \quad . \end{aligned}$$

Since $E|x_{n+1}| < \infty$, by monotone convergence theorem for conditional expectations,

$$[E(I_{[x_{n+1} \geq M]}\,(x_{n+1} - M)|\mathcal{F}_n) > 0] \subset [P(x_{n+1} \geq M|\mathcal{F}_n) > 0] \quad .$$

Hence

$$P\,C_n \le P\,[t > n,\ P(x_{n+1} \ge M|\mathcal{F}_n) > 0]$$

$$\le P\,[t > n,\ P(x_{n+1} \ge K|\mathcal{F}_n) \ge \delta]$$

$$\le \delta^{-1} \int_{[t > n]} P(x_{n+1} \ge K|\mathcal{F}_n) = \delta^{-1}\, P\,[t = n+1] \quad .$$

Therefore

$$P(U_{k=n}^{\infty} C_k) \le \sum_{k=n}^{\infty} P\,C_k \le \delta^{-1}\, P\,[n < t < \infty] \to 0$$

as $n \to \infty$, and by Theorem 3, $P[s = \infty,\ \sup x_n < K] = 0$.

For the second part, we only need to replace Theorem 3, x_n and t respectevely by Theorem 5, s_n and τ in the preceding proof.

The equivalence of (16) and (17) in Theorem 6(i) extends Gundy's result (mentioned in the first section) to non-atomic cases. Theorem 6(ii) was due to Doob [4; p.322-323]. Theorem 6(iii) was due to Burkholder [2, Theorem 4]. Condition (19) was introduced by Doob [5] to ensure a submartingale $(x_n,\mathcal{F}_n,n \ge 1)$ converges a.e. on the set $[\sup x_n < K]$. His method in [5] can not be applied here.

As an application, we prove the following Burkholder's martingale transform convergence theorem.

Theorem 7 (Burkholder [2]). Let $(x_n,\mathcal{F}_n,n \ge 1)$ be a martingale and $\sup E|x_n| < \infty$. If $(g_n,\mathcal{F}_{n-1},n \ge 1)$ is a stochastic sequence such that $\sup |g_n| < \infty$ a.e., then $\Sigma_{k=1}^{\infty} g_k\, d_k$ converges a.e. .

Proof. By Doob's martingale convergence theorem, x_n converges a.e. . Let τ be a stopping time and $\tau_n = \min(\tau, n)$. Then [6; p.300]

$$E_\tau |x_\tau| \leq \lim_{n \to \infty} E\, x_{\tau_n}| \leq \lim_{n \to \infty} E|x_n| < \infty \quad .$$

By Theorem 6(iii), $s < \infty$ a.e. . Note that without loss of generality we can assume that $|g_n| \leq 1$ a.e. for each n. Then $\Sigma_{k=1}^{\infty} g_k^2 d_k^2 < \infty$ a.e. . For $K > 0$, put $t = t_K = \inf\{n \mid |x_n| \geq K\}$, $e_k = I_{[t \geq k]}\, g_k\, d_k$, and $z_n = \Sigma_{k=1}^{n} e_k$. Then $(z_n, \mathcal{F}_n, n \geq 1)$ is a martingale and $\Sigma_{k=1}^{\infty} e_k^2 < \infty$ a.e. . Since $|e_k| \leq I_{[t \geq k]}\, |x_k - x_{k-1}| \leq 2K + |x_t|\, I_{[t < \infty]}$, $E(\sup |e_k|) < \infty$. By Theorem 6(iii) (or [2, Theorem 4]), $\Sigma_{k=1}^{\infty} e_k$ converges a.e. and $P[\Sigma_{k=1}^{\infty} g_k d_k$ diverges, $t = \infty] = 0$. Since $\lim_{K \to \infty} P[t_K = \infty] = 1$, $\Sigma_{k=1}^{\infty} g_k d_k$ converges a.e. .

7. A submartingale convergence theorem.

Theorem 8. Let $(x_n, \mathcal{F}_n, n \geq 1)$ be a submartingale, $E|x_n| < \infty$, and for $K > 0$, define

$$t = \inf\{n \mid x_n \geq K\} \quad .$$

Assume that $(y_n, \mathcal{F}_n, n \geq 1)$ is a stochastic sequence such that $y_n > 0$ a.e. . For $n \geq 1$, let $B_n \in \mathcal{F}_n$ and

$$B_n \supset [E(I_{[t = n+1]}\,(x_t - y_t) | \mathcal{F}_n > 0] \quad .$$

Let $A_{n+1} = B_n[t = n+1]$ and C_n be the $\mathcal{F}_n$-measurable cover of A_{n+1}. If (2) and (7) hold, then $P[x_n$ diverges, $\sup x_n < K] = 0$.

Proof. For $m = 1,2,\ldots,$ define $t^* = t_m^*$ by (1). Then t^* is a stopping time and $t^* \leq t$ a.e. . Put $z_n = \Sigma_{k=1}^{n} I_{[t^* \geq k]}\, d_k$, where $d_1 = x_1$ and

$d_n = x_n - x_{n-1}$ for $n \geq 2$. Then $(z_n, \mathcal{F}_n, n \geq 1)$ is a submartingale and

$$\begin{aligned} z_n &\leq K &&, \text{ if } t^* > n \text{ or } t > t^* \leq n , \\ &= x_t &&, \text{ if } m < t^* = t = k \leq n , \\ &\leq |x_1| + \dots + |x_m| &&, \text{ if } t^* \leq m . \end{aligned}$$

Since

$$\sum_{k=m+1}^{\infty} \int_{[t^* = t = k]} x_t \leq \sum_{k=m+1}^{\infty} \int_{[t=k]-B_{k-1}} (x_t - y_t + y_t)$$

$$\leq M + \sum_{k=m+1}^{\infty} \int_{\Omega - B_{k-1}} E(I_{[t=k]}(x_t - y_t)|\mathcal{F}_{k-1}) \leq M ,$$

$\sup E z_n^+ < \infty$. Since $E z_1 > -\infty$, $\sup E|z_n| < \infty$. By Doob's submartingale convergence theorem, z_n converges a.e. . Hence $P[x_n$ diverges, $t_m^* = \infty] = 0$. By Lemma 1, $P[x_n$ diverges, $t = \infty] = 0$, i.e., $P[x_n$ diverges, $\sup x_n < K] = 0$.

If condition (19) holds, then the conditions of Theorem 8 are satisfied, as in the proof of Theorem 6(iv), by letting $y_n = M$ and

$$B_n = [E(I_{[t = n+1]}(x_t - M)|\mathcal{F}_n) > 0] .$$

Hence Theorem 8 includes Doob's theorem as a special case and the classical submartingale convergence theorem can be obtained from Theorem 8. If the stochastic basis $(\mathcal{F}_n, n \geq 1)$ is regular, the conditions of Theorem 8 are satisfied by letting $y_n = 0$ and $B_n = \Omega$. Then the preceding proof of Theorem 8 becomes rather simple.

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