# SEQUENTIAL SELECTION OF THE BEST OF k POPULATIONS*

by

J. J. Deely

Sandia Corporation

and

Purdue University

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series No. 91

August, 1966

---

* This work was supported in part by Contract AF 33(657)11737 at Purdue University with the Aerospace Research Laboratories and the United States Atomic Energy Commission. 

Sequential Selection of the Best of k Populations

J. J. Deely
Sandia Laboratory and Purdue University

## 0. Introduction:

In recent years many papers have appeared concerning themselves with various aspects of the following problem. Let $\pi_1, \pi_2, \cdots, \pi_k$ denote k-populations (categories, varieties, processes, etc.) from each of which observations are taken on a random variable whose distribution depends upon an unknown parameter $\theta_i$. This parameter is used to define the "best" population; e.g. the population with largest mean could be defined as "best". Based upon the observations, one is interested in determining (in some optimal sense) which population is "best". There are basically two aspects of this problem: (1) seek optimal procedures which select only one population, or (2) seek optimal procedures which select a random subset of the k-populations.

The basic approach to (1) has been to select only one population, so as to guarantee with probability P* the selected population is best provided some other condition on the parameters is satisfied. Among the contributions to this problem are Bechhofer [2], Bechhofer and Sobel [3], Bechhofer, Sobel, and Dunnett [4].

The basic approach to (2) has been to select a subset of random size but with expected size "reasonably small" and also guaranteeing that the selected subset includes the best population with probability P*, regardless of the true configuration of the parameters. Among contributions to this problem are Paulson [16], Gupta [10], [12], Gupta and Sobel [11]. Currently work is being done to find a procedure which <u>minimizes</u> the expected size of the selected subset while still assuring the P* condition, regardless of the true values of the parameters. (Studden [17]).

There are many other contributions to each of these problems as well as some variations. For a list of references the reader is referred to Gupta [13]. However, the work to date has left the sequential analysis of this problem practically untouched. We point out that there are two possible sequential views. First, one could sample sequentially from <u>observation</u> to <u>observation</u> (each observation being a vector consisting of one observation from each of the k-populations) deciding at each stage either to stop and make a selection (one or more populations) or take another observation. The second view is to sample from <u>population</u> to <u>population</u>, each sample from a population consisting of a fixed number of observations from that population. Very little has been done with the first view (See Bechhofer [5] and Birnbaum [6]) and nothing has been done with the second. It is the purpose of this paper to initiate such a study. Besides the interesting mathematical problems, there are two very practical reasons why such a study should be considered. First, the number k may be so large as to physically prevent one from sampling simultaneously from them all; and secondly, there are actual physical situations in which the populations appear sequentially and the option of simultaneous observation is not given. As each population appears, one must either accept it as best or reject it, never to be able to recall that population at a later time. It should be noted that in the special case in which the components of the vector, $\theta = (\theta_1, \cdots, \theta_k)$, can themselves be observed instead of a random variable, then this second view is precisely the well known "Secretary Problem" and various optimal stopping rules have been given. (See Chow, Moriguti, Robbins, Samuels [8], and Lindley [15]). For a detailed discussion of various aspects of this problem the reader is referred to Gilbert and Mosteller [9]. The question of optimal stopping rules in a more general framework has been discussed by Chow and Robbins [7]. They treat the case in which k approaches infinity and show how the corresponding optimal stopping rule is obtained from the truncated optimal stopping rules. In this paper we will use their results to obtain the following:

(i) A stopping rule which maximizes the minimum expected payoff assuming only that each $\theta_i$ is bounded below by some real number d;

(ii) A stopping rule which maximizes the expected payoff assuming each $\theta_i$ has the same a priori distribution G.

(iii) A stopping rule which maximizes the expected payoff assuming each $\theta_i$ has an a priori distribution $G_i$.

The explicit rules are given for two specific payoff functions along with examples.

Finally, reference should be made to a recent contribution by Haggstrom [14] on another variant of the sequential problem. Instead of assuming the populations appear in a random order (or some prescribed order), at each stage of sampling a decision is made not only to stop or continue, but exactly what population is to be observed next if another sample is to be taken.

## 1. Mathematical Formulation and Definitions

Let $\pi_1, \pi_2, \ldots, \pi_k$ be k populations from which we observe sequentially random variables $Y_1, Y_2, \ldots$ until we stop; always stopping at $Y_k$. Assume $Y_i$ has a density $f(y|\theta_i)$ with a monotone likelihood ratio; i.e. $f(y|\theta_i)/f(z|\theta_i)$ is nonincreasing in $\theta_i$ for $y < z$. Let $\omega = (\theta_1, \theta_2, \ldots, \theta_k)$ denote the vector of unknown parameters, and define the best population as that one associated with the largest $\theta_i$. (In case of a tie, one of the possible candidates is tagged as best.) We assume the ordering of the components of $\omega$ to be random so that even if the entries into the vector $\omega$ were known precisely, the location of the largest one would still be unknown. Thus by the joint density of $Y_1, \ldots, Y_n$ given $\omega$, we will mean the "expected" density taken over the k! equally likely permutations of the given vector $\omega$. That is, letting $\omega^{(j)} = \left(\theta_1^{(j)}, \ldots, \theta_k^{(j)}\right)$ denote a typical random permutation of $\omega$, we write:

(1.1) $$f(y_1, \ldots, y_n|\omega) = \frac{1}{k!} \sum_{j=1}^{k!} f\left(y_1, \ldots, y_n|\omega^{(j)}\right) .$$

We further assume the $Y_i$'s to be conditionally independent; i.e.

(1.2) $$f(y_i, y_j|\theta_i, \theta_j) = f(y_i|\theta_i) f(y_j|\theta_j) \quad ,$$

for $i, j = 1.2, \ldots, k$, $i \neq j$. Then the density (1.1) can be written as:

(1.3) $$f(y_1, \ldots, y_n|\omega) = \frac{1}{k!} \sum_{j=1}^{k!} \prod_{i=1}^{n} f\left(y_i|\theta_i^{(j)}\right) \quad ,$$

and the conditional density of $Y_{n+1}$ given $y_1, \ldots, y_n$ and $\omega$ becomes

(1.4)
$$h(y_{n+1}|y_1,\cdots,y_n;\omega) = \frac{\sum_{j=1}^{k!} \prod_{i=1}^{n+1} f\left(y_i|\theta_i^{(j)}\right)}{\sum_{j=1}^{k!} \prod_{i=1}^{n} f\left(y_i|\theta_i^{(j)}\right)} \quad .$$

Let $x_n = g_n(y_1,\cdots,y_n)$ represent the gain (payoff) at the $n^{th}$ observation, and $C_k$, the class of all stopping rules which stop at stage k. A stopping rule $s_k$ in $C_k$ will be called optimal in $C_k$ if the expected gain using $s_k$ is no smaller than the expected gain using any other rule t in $C_k$. That is, if

(1.5)
$$E[X_{s_k}] \geq \sup_{t \in C_k} E[X_t] \quad .$$

Assuming only that expected payoff exists at each stage, the optimal rule $s_k$ may be described as follows: at each stage take another observation if and only if the expected gain by doing so is larger than the gain currently attained. For a given $\omega$ this rule can be prescribed mathematically by a vector of real numbers, $\beta'(k) = \left(\beta_1^k,\cdots,\beta_k^k\right)$, the components of which are obtained via a "backwards induction" technique. (See Arrow, Blackwell, and Girshick [1], and Chow and Robbins [7].) That is, set

$$\beta_k^k = x_k \quad ,$$

and compute the expected gain from taking the $k^{th}$ observation given that we are at the (k-1) stage which is given by

$$E\left[\beta_k^k|y_1,\cdots,y_{k-1}\right] = \int_{-\infty}^{\infty} x_k \; h(y_k|y_1,\cdots,y_{k-1};\omega)dy_k \quad ;$$

where $h(\cdot|y_1,\cdots,y_k)$ is given by (1.4). Clearly we stop with $x_{k-1}$ if and only if

(1.8) $$x_{k-1} \geq E\left[\beta_k^k | y_1, \cdots, y_{k-1}\right] .$$

Then set

(1.9 $$\beta_{k-1}^k = \max\left\{x_{k-1}, E\left[\beta_k^k | y_1, \cdots, y_{k-1}\right]\right\}$$

and similar to (1.7) compute

(1.10) $$E\left[\beta_{k-1}^k | y_1, \cdots, y_{k-2}\right]$$

$$= \int_{-\infty}^{\infty} \beta_{k-1}^k \, h(y_{k-1} | y_1, \cdots, y_{k-2}) \, dy_{k-1}$$

We continue "backwards" inductively defining:

(1.11) $$\beta_n^k = \max\left\{x_n, E\left[\beta_{n+1}^k | y_1, \cdots, y_n\right]\right\} , \qquad (n=1,2,\cdots,k-1),$$

in which

(1.12) $$E\left[\beta_{n+1}^k | y_1, \cdots, y_n\right] = \int_{-\infty}^{\infty} \beta_{n+1}^k \, h(y_{n+1} | y_1, \cdots, y_n) \, dy_{n+1} .$$

Then the optimal rule $s_k$ in $C_k$ is defined by:

Stop the first time $x_n = \beta_n^k$, $n=1,2,\cdots,k-1$.

Otherwise stop with $x_k$.

Since the joint density functions appearing in the expections used in defining $\beta'(k)$ depend upon $\omega$, the optimal procedure thus defined is also a function of $\omega$ and we write

(1.13) $$\beta'(k,\omega) = \left(\beta_1^k(\omega), \cdots, \beta_{k-1}^k(\omega)\right) .$$

If the components of $\omega$ are unknown, then the optimal procedure is not available to us. In this situation a "maximin" procedure is meaningful. Let $\Omega$ denote the set of possible vectors $\omega$. We say a stopping rule $t^*$ in $C_k$ is <u>maximin</u> (with respect to $\Omega$ and $C_k$) if

(1.14) $$\inf_{\omega\in\Omega} E[X_{t^*} | \omega] \geq \sup_{t\in C_k} \inf_{\omega\in\Omega} E[X_t | \omega] .$$

We obtain such a maximin rule in Section 2 restricting $\Omega$ to $\Omega_d = \{\omega: \theta_i \geq d$ for some real number d, $i=1,2,\cdots,k\}$ and for a payoff function which is nondecreasing in any argument. The explicit rules as functions of the underlying density function are given for the two payoff functions:

(1.15) $$\tilde{x}_n = y_n - cn \quad , \quad \text{and}$$

(1.16) $$x_n = \max(y_1,\cdots,y_n) - cn.$$

A specific case in which $f(y_i|\theta_i)$ is taken as a normal density with unknown mean $\theta_i$ and unit variance is worked out for illustrative purposes. We remark that the payoff function (1.15) reflects the situation in which a population, once rejected, can no longer be selected; whereas the function (1.16) reflects the situation in which it is possible to recall a previously rejected population.

In many situations it is meaningful and possible to assume that $\theta_i$ itself is a random variable. For example, suppose a company is receiving sequentially one lot (say, containing 1000 items) from each of k suppliers. The $i^{th}$ lot has a true fraction defective $p_i = 1-\theta_i$ $(0 < \theta_i < 1)$ and based upon a sample from that lot, the company either accepts the lot and stops further shipments from other suppliers or rejects and orders from the next supplier. In this situation it is extremely realistic to assume $\theta_i$ to be a random variable being distributed according to an overall quality distribution of the $i^{th}$ supplier. Other examples can be given substantiating this view. With this in mind we consider the situation in which $\theta_i$ has a distribution $G_i$; the $\theta_i$'s being independent random variables.

Then using the above technique with an appropriately modified payoff function, a Bayes (optimal) stopping rule can be obtained. In Section 3 such rules are obtained in two cases: (a) $G_i = G$, $i = 1,2,\cdots,k$; (b) $G_i$'s distinct. The payoff function is taken analogously to (1.15) and (1.16) as either:

(1.17) $$\tilde{x}_n = E[\theta_n | y_1,\cdots,y_n]-cn \quad , \qquad \text{or}$$

(1.18) $$x_n = \max_{1\le i\le n} \left\{ E[\theta_i | y_1,\cdots,y_n] \right\} -cn \quad .$$

Note that by the independence of the $\theta_i$'s and the conditional independence of the $Y_i$'s (see (1.2)), we can write

(1.19) $$E[\theta_i | y_1,\cdots,y_n] = E[\theta_i | y_i] = \frac{\int \theta\, f(y_i | \theta) dG_i(\theta)}{\int f(y_i | \theta) dG_i(\theta)} \quad ,$$

the a posteriori mean of the $i^{th}$ population. Thus the two payoff functions reduce to

(1.20) $$\tilde{x}_n = E[\theta_n | y_n]-cn \qquad , \text{ and}$$

(1.21) $$x_n = \max_{1\le i\le n} \{E[\theta_i | y_i]\}-cn,$$

respectively. The normal case is again worked out as an example.

## 2. Maximin Stopping Rule and Minimax Configuration

Viewed as a two person zero sum game we have the following problem. "Nature" selects a vector $\omega = (\theta_1 \cdots, \theta_k)$ from some set $\Omega$ of "allowable" vectors and then gives a random permutation to the statistician. But the statistician can only observe sequentially the vector $(Y_1, \cdots, Y_k)$ of random variables whose densities depend not only upon the vector $\omega$ but the particular permutation of its components as well. The statistician seeks the location of the largest $\theta_i$ as soon as possible. Thus a strategy for nature is really a particular configuration of the parameter space, and a strategy for the statistician is a stopping rule. Given a particular payoff function the statistician seeks a strategy (stopping rule) $t^*$ such that the least he can expect is maximized no matter what strategy in $\Omega$ is employed by nature; i.e. a maximum stopping rule. Whereas nature seeks a strategy (configuration) $\omega^*$ in $\Omega$ such that the most she can expect to pay is minimized regardless of the statisticians strategy; i.e. a minimax configuration. As in Section 1 we let $X_t$ denote the statistician's gain associated with the strategy t, and $E[X_t|\omega]$ denote the statistician's expected gain when using strategy t given that nature's strategy is $\omega$. Formally, we make the following definitions.

Definition 2.1: A stopping rule $t^*$ in $C_k$ is called maximin if

$$\inf_{\omega\epsilon\Omega} E[X_{t*}|\omega] \geq \sup_{t\epsilon C_k} \inf_{\omega\epsilon\Omega} E[X_t|\omega].$$

Definition 2.2: A configuration $\omega^*$ in $\Omega$ is called minimax if

$$\sup_{t\epsilon C_k} E[X_t|\omega^*] \leq \inf_{\omega\epsilon\Omega} \sup_{t\epsilon C_k} E[X_t|\omega] \quad .$$

The following notation and results will be useful in the theorem below. Let $t_\omega$ indicate the optimal rule associated with the vector $\beta'(k,\omega)$ obtained via backwards induction as given in Section 1. It is shown in Chow and Robbins [7] that for a given $\omega$

(2.1) $$\sup_{t\in C_k} E[X_t|\omega] = E[X_{t_\omega}|\omega] = E\left[\beta_1^k(\omega)|\omega\right] .$$

We remark that the last equality above says that the expected payoff is precisely the expectation of the first component of the vector $\beta'(k,\omega)$. Thus for $\omega_o,\omega$ in $\Omega$ we have

(2.2) $$E[X_{t_{\omega_o}}|\omega] = E\left[\beta_1^k(\omega_o)|\omega\right] .$$

A further result in [7] states that if the $Y_i$'s are independent and identically distributed with density $f(y)$, then the optimal stopping rule for the payoff function (1.16) is defined by:

Stop the first $n$ $(n=1,2,\cdots,k-1)$ for which $y_n \geq \gamma$;
$\gamma$ a real number such that $\int_\gamma^\infty (y-\gamma)f(y)dy = c$.
Otherwise stop at $n = k$.

We now state and prove the main theorem; recalling first that the following assumptions have been made:

(i) $f(y|\theta_i)$ has monotone likelihood ratio, $i = 1,2,\ldots k$.

(ii) $Y_i$'s are conditionally independent (see 1.2).

(iii) the payoff function $g_n(y_1,\cdots,y_n)$ is nondecreasing in each of its arguments.

(iv) $\Omega_d = \{\omega=(\theta_1,\cdots,\theta_k): \theta_i\geq d,\ i=1,2,\cdots,k \text{ for a real number } d\}$.

<u>Theorem</u> Let $\Omega = \Omega_d$ for some real number d and let $\omega^* = (d,d,\cdots,d)$. Then the game described above has the value

(2.3) $$v = E[X_{t_{\omega^*}}|\omega^*]$$

Remark: This theorem says $t_{\omega^*}$ is the maximin stopping rule and $\omega^*$ is the minimax configuration.

Proof: It suffices to prove that

(2.4) $$\inf_{\omega\in\Omega_d} E[X_{t_{\omega^*}}|\omega] \geq E[X_{t_{\omega^*}}|\omega^*].$$

For if the above is true, it then follows that

(2.5) $$\sup_{t\in C_k} \inf_{\omega\in\Omega_d} E[X_t|\omega] \geq \inf_{\omega\in\Omega_d} E[X_{t_{\omega^*}}|\omega] \geq E[X_{t_{\omega^*}}|\omega^*] \geq \inf_{\omega\in\Omega_d} E[X_{t_\omega}|\omega]$$
$$\geq \inf_{\omega\in\Omega_d} \sup_{t\in C_k} E[X_t|\omega] \geq \sup_{t\in C_k} \inf_{\omega\in\Omega_d} E[X_t|\omega]$$

Thus the game has a value v given by

(2.6) $$v = \sup_{t\in C_k} \inf E[X_t|\omega] = \inf \sup E[X_t|\omega] = E[X_{t_{\omega^*}}|\omega^*],$$

which is the stated result.

To prove (2.4), we note that using (2.2) it remains to show that

(2.7) $$\inf_{\omega\in\Omega_d} E\left[\beta_1^k(\omega^*)|\omega\right] \geq E\left[\beta_1^k(\omega^*)|\omega^*\right],$$

where $\beta'(k,\omega^*) = \left(\beta_1^k(\omega^*),\cdots,\beta_k^k(\omega^*)\right)$ is obtained via the backwards induction and defines the optimal rule when $\omega^*$ is the true configuration. That is,

(2.8) $$\beta_k^k(\omega^*) = x_k = g_k(y_1,\cdots,y_k) ;$$

(2.9) $$\beta_n^k(\omega^*) = \max\left\{g_n(y_1,\cdots,y_n);\ E\left[\beta_{n+1}^k(\omega^*)|y_1,\cdots,y_n;\omega^*\right]\right\}$$

for $n = 1,2,\cdots,k-1$; and we define recurssively

(2.10) $$E\left[\beta_{n+1}^k(\omega^*)|y_1,\cdots,y_n;\omega^*\right]$$
$$=\int_{\infty}^{\infty}\beta_{n+1}^k(\omega^*)\ h(y_{n+1}|y_1,\cdots,y_n;\omega^*)dy_{n+1} \quad .$$

But from (1.4) with $\omega = \omega^*$ we have that

(2.11) $$h(y_{n+1}|y_1, \cdots, y_n; \omega^*) = f(y_{n+1}|d)$$

(i.e. when $\omega = \omega^*$, the $Y_i$'s are independent and identically distributed), hence (2.10) can be written as

(2.12) $$E\left[\beta^k_{n+1}(\omega^*)|y_1, \cdots, y_n; \omega^*\right] = \int_{-\infty}^{\infty} \beta^k_{n+1}(\omega^*)\ f(y_{n+1}|d)dy_{n+1}\ ,$$

$n = 1, 2, \cdots, k-1$. We will use (2.12) to make the following inductive argument. For $n=k-1$, it is seen that assumption (iii) implies $E\left[\beta^k_k(\omega^*)|y_1, \cdots, y_{k-1}; \omega^*\right]$ is non-decreasing in each of its arguments; (denoted by simply NDA hereafter). But from (2.9) this in turn implies $\beta^k_{k-1}(\omega^*)$ is NDA. Then from (2.12) with $n=k-2$, the preceeding statement implies $E\left[\beta^k_{k-2}(\omega^*)|y_1, \cdots, y_{k-3}; \omega^*\right]$ is NDA. Proceeding backwards inductively we obtain the fact that $\beta^k_1(\omega^*)$ is NDA.

Next, observe that for any $\omega$ in $\Omega_d$ we have

(2.13) $$E\left[\beta^k_1(\omega^*)|\omega\right] = \int_{-\infty}^{\infty} \beta^k_1(\omega^*)\ h(y_1|\omega)dy_1 = \int_{-\infty}^{\infty} \beta^k_1(\omega^*)\ f(y_1|\omega)dy_1,$$

in which, by (1.3),

(2.14) $$f(y_1|\omega) = \frac{1}{k}\sum_{i=1}^{k} f(y_1|\theta_i)\quad .$$

Thus we can write (2.13) as

(2.15) $$E\left[\beta^k_1(\omega^*)|\omega\right] = \frac{1}{k}\ \sum_{i=1}^{k}\int_{-\infty}^{\infty}\beta^k_1(\omega^*)\ f(y_1|\theta_i)dy_1\ .$$

But $\beta^k_1(\omega^*)$ has already been shown to be non-decreasing in $y_1$; thus by assumption (i) we have

(2.16) $$\int_{-\infty}^{\infty}\beta^k_1(\omega^*)\ f(y_1|\theta_i)dy_1 \geq \int_{-\infty}^{\infty}\beta^k_1(\omega^*)\ f(y_1|d)$$

for every $i = 1,2,\cdots,k$. Therefore

(2.17) $$E\left[\beta_1^k(\omega^*)\,|\,\omega\right] \geq \int_{-\infty}^{\infty} \beta_1^k(\omega^*)\; f(y_1|d) = E\left[\beta_1^k(\omega^*)\,|\,\omega^*\right]$$

for every $\omega\in\Omega_d$; which gives (2.7) thereby completing the proof of the theorem.

As observed in the proof of the above, the $Y_i$'s are independent and identically distributed with density $f(y|d)$ when $\omega=\omega^*$. Thus using the remark preceeding the theorem, the following Corollary is immediate.

Corollary 1 For the payoff function (1.16), the maximin stopping rule over $\Omega_d$ is given by:

Stop the first n $(n=1,2,\cdots,k-1)$ for which $y_n \geq \gamma$; where $\gamma$ is a real number such that $\int_{\gamma}^{\infty} (y-\gamma)\; f(y|d) = c$. Otherwise stop at $n=k$.

Corollary 2 For the payoff function (1.15), the maximin stopping rule over $\Omega_d$ is given by:

Stop the first $n(n=1,2,\cdots,k-1)$ for which $y_n \geq d_{k-n} -c$ where

$$d_j = \int_{d_{j-1}-c}^{\infty} y\; f(y|d)dy + (d_{j-1}-c) \int_{-\infty}^{d_{j-1}-c} f(y|d)dy \text{ for } j=2,3,\cdots,k-1$$

and $d_1 = \int_{-\infty}^{\infty} y\; f(y|d)dy$. Otherwise, stop at $n=k$.

Proof: From the definition of the sequence $\beta'(k,\omega^*)$ we compute

(2.18) $$E\left[\beta_k^k(\omega^*)\,|\,\omega^*\right] = \int_{-\infty}^{\infty} y\; f(y|d)dy - ck = d_1 - ck \quad ,$$

and set

(2.19) $$\beta_{k-1}^k(\omega^*) = \max\,\{y_{k-1} - c(k-2), d_1 + c - ck\} = \max\,\{y_{k-1} + c,\; d_1\} - ck \quad ;$$

and we stop at $y_{k-1}$ if and only if $y_{k-1} \geq d_1 - c$.

Then we compute

(2.20) $$E\left[\beta_k^k(\omega^*)\,|\,\omega^*\right] = \int_{d_1-c}^{\infty} y\, f(y|d)dy + (d_1-c)\int_{-\infty}^{d_1-c} f(y|d)dy + c\ -ck$$
$$= d_2 + c\ -ck\ ;$$

and set

(2.21) $$\beta_{k-1}^k(\omega^*) = \max\{y_{k-2}-c(k-2), d_2+c-ck\} = \max\{y_{k-2}+c, d_2\} -ck +c$$

and we stop at $y_{k-2}$ if and only if $y_{k-2} \geq d_2 - c$. Proceeding backwards inductively we obtain the desired result.

Example: Let $f(y_i|\theta_i)$ be the normal density with mean $\theta_i$ and variance one, and suppose each $\theta_i \geq d$. Then for the payoff function (1.16) the maximin stopping rule given by Corollary 1 is:

> Stop the first n $(n=1,2,\cdots,k-1)$ for which $y_n \geq z+d$, where z is such that $\varphi(z) - z(1-\Phi(z)) = c$; $\varphi$ and $\Phi$ being the standard normal density and cumulative distribution functions respectively. Otherwise stop at $n = k$.

For the payoff function (1.15), the maximin stopping rule given by Corollary 2 is:

> Stop the first $n(n=1,2,\cdots,k-1)$ for which $y_n \geq d_{k-n} - c$
>
> where
>
> $$d_j = \varphi(d_{j-1}-c-d) + d\left(1-\Phi(d_{j-1}-c-d)\right) + (d_{j-1}-c)\Phi(d_{j-1}-c-d)$$
>
> for $j=2,3,\cdots,k-1$ and $d_1 = d$; $\varphi$ and $\Phi$ as above. Otherwise stop at $n=k$.

## 3. Bayes Stopping Rules

In this section we assume that $\theta_i$ is itself a random variable distributed according to $G_i$ over some set $\Theta$ of real numbers; $\theta_i$'s being independent. We consider two cases: (1) $G_i = G$ for $i=1,2,\cdots,k$; and (2) $G_i$'s distinct. Two payoff functions are considered in each case; namely, (1.20) and (1.21). We shall use the notation $u^+ = \max(u,0)$.

Case 1 $G_i = G$ for $i=1,2,\cdots,k$.

In this case we obtain the unconditional density of $Y_i$ by:

(3.1) $$f(y) = \int_\Theta f(y|\theta_i) d\, G(\theta_i)$$

and note that the $Y_i$'s are identically distributed. Furthermore the independence of the $Y_i$'s follows from the independence of $\theta_i$'s and the conditional independence of the $Y_i$'s, (see (1.2)). Thus if we define

(3.2) $$Z_i = E[\theta_i|Y_i] \quad , i=1,2,\cdots,k \quad ,$$

then the $Z_i$'s are independent and identically distributed and the payoff functions (1.20) and (1.21) become

(3.3) $$\tilde{x}_n = z_n - cn$$

and

(3.4 $$x_n = \max(z_1,\cdots,z_n) - cn$$

respectively. For payoff function (3.4) the remark preceeding the Theorem in Section 2 is applicable and the Bayes (optimal) stopping rule with respect to G is given by:

Stopping Rule (B.1) Stop the first n $(n=1,2,\cdots,k-1)$ for which $z_n \geq \gamma$, where $\gamma$ is a real number such that $\int_{-\infty}^{\infty}(E[\theta|y]-\gamma)^+ f(y)dy = c$. Otherwise stop at $n=k$.

For the payoff function (3.3) the Bayes stopping rule is obtained by generating the vector $\beta'(k)$ as indicated in Section 1. Note that the independence of the $Y_i$'s reduces the expression (1.12) to

(3.5) $$E\left[\beta_n^k | y_1,\cdots,y_{n-1}\right] = \int_{-\infty}^{\infty} \beta_n^k \, f(y_n) dy_n .$$

Thus

(3.6) $$E\left[\beta_k^k | y_1,\cdots,y_{k-1}\right] = E\{E[\theta|Y_k]-ck\} = \lambda - ck,$$

where we have set

(3.7) $$\lambda = \int_{\Theta} \theta \, d\, G(\theta) \quad .$$

Then

(3.8) $$\beta_{k-1}^k = \max\,\{z_{k-1}+c,\lambda\} -ck \quad , \quad \text{and}$$

(3.9) $$E\left[\beta_{k-1}^k | y_1\cdots,y_{k-2}\right] = \int_{-\infty}^{\infty} \beta_{k-1}^k \, f(y_{k-1}) dy_{k-1} - ck$$

$$= \int_{-\infty}^{\infty} \max\,\left\{E\left[\theta|y\right]+c,\lambda\right\} \, f(y)dy$$

$$= \lambda + c + \int_{-\infty}^{\infty}\left(\lambda - c - E\left[\theta|y\right]\right)^+ f(y)dy$$

In general we obtain

(3.10) $$\beta_k^k = z_k - ck;$$

(3.11) $$\beta_n^k = \max\,\{z_n+c,\ \alpha_{n+1}\} - ck + \big(k-(n+1)\big)\,c; \; n=1,2,\cdots,k-1;$$

in which;

(3.12) $$\alpha_k = \lambda$$

(3.13 $$\alpha_j = \lambda + \int_{-\infty}^{\infty}(\alpha_{j+1}-c-E[\theta|y])^+ f(y)dy; \; j=1,2,\cdot\cdot,k-1.$$

Thus the Bayes (optimal) rule as defined in Section 1 becomes

<u>Stopping Rule (B.2)</u> Stop the first n $(n=1,2,\cdots,k-1)$ for which $E[\theta|y_n] \geq \alpha_{n+1}-c$, the $\alpha_j$'s being defined above. Otherwise stop at $n=k$.

Example. To illustrate Stopping Rule (B.1), consider the normal density $f(y|\theta_i)$ with mean $\theta_i$ and variance one and let $\theta_i$ be distributed normally with mean $\lambda$ and unit variance. Then

(3.14) $$E[\theta|y_i] = \frac{\lambda+y_i}{2} \quad \text{, and}$$

the density (3.1) of each $Y_i$ is normal with mean $\lambda$ and variance two. Using (3.14) in (B.1) we obtain the Bayes stopping rule for the payoff function (3.4):

Stop the first n $(n=1,2,\cdots,k-1)$ for which $y_n \geq \sqrt{2}\,\delta + \lambda$, where $\delta$ is a real number such that $\varphi(\delta) - \delta\left(1-\Phi(\delta)\right) = \sqrt{2}\,c$, $\varphi$ and $\Phi$ as before. Otherwise stop at $n=k$.

Case 2 $G_i$'s distinct.

If each $\theta_i$ has a distribution $G_i$ not necessarily the same for all $i=1,2,\cdots,k$, then the unconditional density for $Y_i$ becomes

(3.15) $$f_i(y) = \int_\Theta f(y|\theta)d\,G_i(\theta)$$

If we set

(3.16) $$\lambda_i = \int_\Theta \theta\, d\,G_i(\theta) \qquad (i=1,2,\cdots,k),$$

then for the payoff function (3.3) with $Z_n$ defined by (3.2), the Bayes stopping rule is quite similar to (B.2). By making the appropriate changes in (3.8) - (3.11), we obtain the Bayes stopping rule:

Stopping Rule (C.1) Stop the first n $(n=1,2,\cdots,k-1)$ for which

$E[\theta_n|y_n] \geq \tilde{\alpha}_{n+1} - c$ where $\tilde{\alpha}_k = \lambda_k$ and

$$\tilde{\alpha}_j = \lambda_j + \int_{-\infty}^{\infty}(\tilde{\alpha}_{j+1} - c - E[\theta_j|y])^+ f_j(y)dy \quad , j=1,2,\cdots,k-1.$$

Otherwise stop at $n=k$.

If we use the payoff function (3.4), the Bayes stopping rule in this case ($G_i$'s distinct) cannot be obtained using the technique of Case 1 because the $Y_i$'s are no longer identically distributed. Of course the backwards induction technique would work but becomes quite tedious and complicated. However, by making some further restrictions on the $G_i$'s and densities $f(y_i|\theta_i)$, a further result of Chow and Robbins [7] can be used to give the Bayes stopping rule quite simply. Their result is as follows: let $\gamma_n$ denote the real number such that

(3.17) $$E\left[(Z_{n+1}-\gamma_n)^+\right] = c, \qquad (n=1,\cdots,k-1),$$

where $Z_{n+1}$ is defined by (3.2). If the sequence $\{\gamma_n\}$ is non-increasing, i.e. $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{k-1}$, then the Bayes (optimal) stopping rule is given by:

Stopping Rule (C.2) Stop the first n $(n=1,2,\cdots,k-1)$ for which

$$\max_{1\leq i\leq n} \{E[\theta_i|y_i] \geq \gamma_n. \quad \text{Otherwise stop at } n=k.$$

Thus any combination of $f(y_i|\theta_i)$ and $G_i$ which gives rise to a non-increasing sequence of $\gamma_n$'s will have the above as its Bayes stopping rule. One condition which guarantees the sequence $\{\gamma_n\}$ to be non-increasing is the following. Assume $f_i(y)$ given by (3.15) is a member of some parametric family indexed by $\lambda_i$ in (3.16); i.e.

(3.18) $$f_i(y) = f_{\lambda_i}(y) \quad , \qquad i=1,2,\cdots,k; \text{ and}$$

assume that $f_{\lambda_i}(y)$ has a monotone likelihood ratio in $\lambda_i$. Let the populations be arranged so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. If $E[\theta_n|y_n]$ is an increasing function of $y_n$ and $\lambda_n$, then from (3.17) it follows that $\{\gamma_n\}$ is non-increasing. We conclude with an example illustrating Stopping Rule (C.2).

Example: Let $f(y|\theta_i)$ be normal with mean $\theta_i$ and unit variance and assume $\theta_i$ to be normally distributed with mean $\lambda_i$ and unit variance. Assume further that $\lambda_i \geq \lambda_2 \geq \cdots \geq \lambda_k$, (which would be the intuitive way of sampling the populations anyway.). Then for (3.18) we write

$$f_i(y) = f_{\lambda_i}(y) = \frac{1}{\sqrt{2\pi}\ \sqrt{2}}\ e^{-\frac{1}{2}\left(\frac{y-\lambda_i}{\sqrt{2}}\right)} \tag{3.19}$$

which has monotone likelihood ratio in $\lambda_i$; and the a posteriori mean becomes

$$E[\theta_i|y_i] = \frac{\lambda_i+y_i}{2}, \tag{3.20}$$

which is strictly increasing in $y_i$ and $\lambda_i$. Thus the Bayes stopping rule is:

> Stop the first n $(n=1,2,\cdots,k-1)$ for which $\max_{1\leq i\leq n}\left\{\frac{\lambda_i+y_i}{2}\right\} \geq \frac{\delta}{\sqrt{2}} + \lambda_{n+1}$ where $\delta$ is a real number such that $\varphi(\delta)-\delta(1-\Phi(\delta)) = \sqrt{2}\ c$, with $\varphi$ and $\Phi$ as before. Otherwise stop at $n = k$.

ACKNOWLEDGEMENT: The author wishes to thank Professor S. S. Gupta who encouraged the author to work in this area and whose gracious invitation to visit Purdue allowed such research to be performed. The author also expresses his sincere appreciation to Professor W. J. Studden of Purdue for the many profitable discussions and his helpful suggestions during the preparation of this paper.

## References


1. Arrow, K. J., Blackwell, D., Girshick, M.A. (1949) Bayes and minimax solutions of sequential decisions problems. Econometrica (17), 213 - 244.

2. Bechhofer, R. E. (1954) A single-sample multiple decision procedure for ranking means of normal populations with known variances. Ann. Math. Statist. (25), 16 - 29.

3. Bechhofer, R. E. and Sobel, M. (1954) A single sample decision procedure for ranking variances of normal populations. Ann. Math. Statist. (25), 273 - 289.

4. Bechhofer, R. E., Dunnett, C. W. and Sobel, M. (1954). A two-sample multiple decision procedure for ranking means of normal populations with a common unknown variance. Biometrika (41), 170 - 176.

5. Bechhofer, R. E. (1958) A sequential multiple-decision procedure for selecting the best one of several normal populations with a common unknown variance and its use with various experimental designs. Biometrics (14), 408 - 429.

6. Birnbaum, Allan (1958) Sequential tests for variance ratios and components of variance. Ann. Math. Statist. (29), 504 - 514.

7. Chow, Y. S., Robbins, H. (1963) On optimal stopping rules. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete (2), 33 - 49.

8. Chow, Y. S., Moriguiti, S., Robbins, H., Samuels, S.M., (1964) Optimal selection based on relative rank (the "Secretary Problem"). Israel Journal of Mathematics (2), 81 - 90.

9. Gilbert, J. P., Mosteller, F. (1966) Recognizing the maximum of a sequence. J. Amer. Statist. Assoc. (61), 35 - 73.

10. Gupta, S. S. (1956) On a decision rule for a problem in ranking means. Mimeograph Series No. 150, Institute of Statistics, University of North Carolina, Chapel Hill, N.C.

11. Gupta, S. S. and Sobel, M. (1957) On a statistic which arises in selection and ranking problems. Ann. Math. Statist. (28) 957 - 967.

12. Gupta, S. S. (1963b) On a selection and ranking procedure for gamma populations. Ann. Inst. Statist. Math. Tokyo (14), 199 - 216.

13. Gupta, S. S. (1965) On some multiple decision (selection and ranking) rules, Technometrics (7).

14. Haggstrom, G.W. (1966). Optimal stopping and experimental design. Ann. Math. Statist. (37), 7-29.

15. Lindley, D.V. (1961). Dynamic programming and decision theory Applied Statistics (10), 39-51.

16. Paulson, E. (1949). A multiple decision procedure for certain problems in analysis of variance. Ann. Math. Statist. (20), 95-98.

17. Studden, William (1966). On the Problem of Ranking and Selection from k Fixed Populations. Mimeograph Series #89, Department of Statistics, Purdue University.

Unclassified

Security Classification

## DOCUMENT CONTROL DATA - R&D

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| | |
|---|---|
| 1. ORIGINATING ACTIVITY *(Corporate author)* Purdue University | 2a. REPORT SECURITY CLASSIFICATION Unclassified |
| | 2b. GROUP |

3. REPORT TITLE

Sequential Selection of the Best of k Populations.

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Technical Report - August 1966

5. AUTHOR(S) *(Last name, first name, initial)*

Deely, J. J.

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| August 1966 | 21 | 17 |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO. AF 33(657)11737 | 9a. ORIGINATOR'S REPORT NUMBER(S) Mimeograph Series Number 91 |
| b. PROJECT NO. | |
| c. | 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* |
| d. | |

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

| | |
|---|---|
| 11. SUPPLEMENTARY NOTES Also supported by the U. S. Atomic Energy Commission | 12. SPONSORING MILITARY ACTIVITY Aerospace Research Laboratories Wright-Patterson AFB Dayton, Ohio |


13. ABSTRACT

Let $\pi_1, \pi_2 \ldots, \pi_k$ be k populations and let $\omega = (\theta_1, \theta_2, \ldots, \theta_k)$ be the vector of unknown parameters. We assume the ordering to be a random permutation so that even if the entries of $\omega$ were known, the exact location of the largest one would still be unknown. We observe sequentially the random variables $Y_1, Y_2, \ldots$ until we stop; always stopping at $Y_k$. $Y_i$ is assumed to have a density $f(y|\theta_i)$ having a monotone likelihood ratio. Using results of Chow and Robbins (Z. Wahrscheinlichkeitstheorie 1963), maximum stopping rules are obtained over the set $\Omega_d = \{\omega: \theta_i \geq d, 1, 2, \ldots, k\}$ for a payoff function which is increasing in each of its arguments. Further, if there exists an a priori distribution $G_i$ on $\theta_i$, then a Bayes stopping rule is obtained. The case $G_i = G$, $i=1,2,\ldots,k$ is also treated. In each of the cases mentioned above, the specific rules are obtained for two payoff functions: $\tilde{X} = y_n - cn$ and $x_n = \max(y_1, y_2, \ldots, y_n) - cn$. An example using the normal density is also given.


DD FORM 1 JAN 64 1473

Unclassified

Security Classification

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Best Populations<br>Sequential Rules<br>Selection and Ranking | | | | | | |

## INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
_________________________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
_________________________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
_________________________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.