STOPPING RULES FOR x_n/n AND RELATED PROBLEMS

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ABSTRACT

The following results illustrate the problems with which this note deals. Let x_n ($n = 1, 2, ..$) be non-negative, independent, identically distributed random variables, let $\beta > 1$ and $Ex_1^\beta < \infty$. Then there exists a stopping rule τ with $P\{\tau < \infty\} = 1$, which maximizes $E\, x_t/t$ among all stopping rules t. Moreover, the same rule maximizes $E \max(x_1, .., x_t)/t$ and $E \max(x_1, .., x_\tau)/\tau = Ex_\tau/\tau$.

0. Summary. This note deals with some problems concerning the existence of optimal stopping rules for rather simple stochastic sequences. The following results, which are very special cases of the theorems proved here, are typical.

Let $x_1, x_2, \cdots, x_n, \cdots$ be independent, identically distributed nonnegative random variables possessing a finite moment of some order greater than one (but which may be arbitrarily close to 1). Then there exists a stopping rule τ which maximizes the expectation of x_t/t among all generalized stopping rules t (this is rather easy) moreover, τ is a genuine stopping rule, i.e., it stops with probability 1 (this is more tricky).

We also consider the problem of maximizing the expectation of $\max(x_1, x_2, \cdots, x_t)/t$. This looks like a much more complicated problem. Also, clearly, for every stopping rule t the expectation of $\max(x_1, \cdots, x_t)/t$ is at least as great as that of x_t/t and it seems plausible that $\sup_t E\max(x_1, \cdots, x_t)/t$ will be larger than Ex_τ/τ. Rather surprisingly, it turns out that τ is the optimal rule for both problems and that $E\max(x_1, \cdots, x_\tau)/\tau = Ex_\tau/\tau$.

1. Introduction. We start by recalling some simple facts about stopping rules and fixing our notations. These facts may be found in the well known references [1] and [2] (for generalized stopping rules see also [3] and [4]).

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $z_1, z_2, \cdots, z_n, \cdots$ be random variables on this space. Put $\mathcal{F}_n = \mathcal{B}(z_1, \cdots, z_n)$, the σ-field generated by $(z_1, \cdots, z_n)$

Research supported in part by Grant GP-5705 of the National Science Foundation, USA.
Received March 6, 1969

($n = 1,2,\cdots$) and $\mathcal{F}_0 = \{\Omega, \emptyset\}$. A *genuine* stopping rule for the sequence (z_n) is a random variable t assuming only non-negative integral values such that

$$\{t = n\} \in \mathcal{F}_n \quad (n = 1,2,\cdots). \tag{1.1}$$

A *generalized* stopping rule, or simply a stopping rule, for the sequence (z_n) is a generalized random variable t, which may assume the value ∞ as well as non-negative integral values, satisfying (1.1). For all stopping rules t we have $\sum_{n \geq 1} P\{t = n\} = 1 - P\{t = \infty\}$ and for genuine rules we also have $P\{t = \infty\} = 0$, and in the sequel any stopping rule satisfying this condition will be called genuine.

For generalized stopping rules we define $z_\infty = 0$. (For certain purposes it is preferable to define z_∞ otherwise, but this definition is the simplest for our purposes.) Then the expectation of z_t is always given by

$$Ez_t = \sum_{n \geq 1}^{\infty} \int_{\{t=n\}} z_n = \int_{\{t<\infty\}} z_t, \tag{1.2}$$

provided the last integral exists (here and everywhere we omit writing dP under the integration sign).

Let C_n ($n = 1,2,\cdots$) be the class of stopping rules for which $P\{t \geq n\} = 1$ (C_1 is the class of all stopping rules). We define

$$\gamma_n = \operatorname*{ess\,sup}_{t \in C_n} E(z_t \mid \mathcal{F}_n) \qquad (n = 1,2,\cdots) \tag{1.3}$$

and

$$v_n = \sup_{t \in C_n} Ez_t \qquad (n = 1,2,\cdots) \tag{1.4}$$

whenever (1.2) is defined for all $t \in C_n$. A stopping rule t is called *optimal* for the sequence (z_n) if v_1 is defined and $Ez_t = v_1$.

Whenever v_1 is defined we define the *natural* rule τ for the sequence (z_n) by

$$\tau = \inf\{n : z_n \geq \gamma_n\} \tag{1.5}$$

where, by convention, the infimum of an empty set is ∞, i.e., $\tau = \infty$ on $\bigcup_{n \geq 1} \{z_n < \gamma_n\}$.

We shall repeatedly use the following lemma. It can be stated for considerably more general sequences of random variables, but the following suffices for our purposes.

LEMMA 1. *Let $z_1, z_2, \cdots, z_n, \cdots$ be non-negative random variables, and let*

(1.6) $$\lim_{n=\infty} v_n = 0.$$

Then the natural rule τ *given by* (1.5) *is optimal for the sequence* $z_1, z_2, \cdots, z_n, \cdots$.

Proof. Since the z_n are non-negative Ez_t is defined for all stopping rules t. Hence all v_n are defined and the natural rule is also defined.

From (1.3), (1.4) and (1.5) we have

(1.7) $$\gamma_n = \max(z_n, E(\gamma_{n+1} \mid \mathcal{F}_n)), \quad v_n = E\gamma_n \qquad (n = 1,2,\cdots).$$

Therefore,

$$v_1 = \int_{\{\tau \geqq 1\}} \gamma_1 = \int_{\{\tau=1\}} z_1 + \int_{\{\tau>1\}} \gamma_2 = \int_{\{\tau=1\}} z_1 + \int_{\{\tau=2\}} z_2 + \int_{\{\tau>2\}} \gamma_3 = \cdots$$

$$= \sum_{i=1}^{n} \int_{\{\tau=i\}} z_i + \int_{\{\tau>n\}} \gamma_{n+1}.$$

But the last integral is non-negative and $\leqq v_{n+1}$. Hence (1.6) implies $v_1 = Ez_\tau$ as claimed.

Applying this lemma to the sequence $z_n, z_{n+1}, \cdots$ we see that, under the above conditions

(1.8) $$\tau_n = \inf\{m : m \geqq n \text{ and } z_m \geqq \gamma_m\},$$

the natural rule in the class C_n is optimal within this class, i.e.,

(1.9) $$Ez_{\tau_n} = v_n.$$

The natural rules may, of course, be generalized rules. The main difficulty in obtaining our results is precisely in showing that for the situations considered here these rules are genuine stopping rules.

The above expressions simplify somewhat when the random variables are independent. Indeed, if $z_1, \cdots, z_n, \cdots$ are independent (1.7) simplifies to $\gamma_n = \max(z_n, v_{n+1})$ and the natural rule τ of (1.5) is defined by

(1.10) $$\tau = \inf\{n : x_n \geqq v_{n+1}\}$$

and a similar remark holds for τ_n given by (1.8).

In Section 3 we shall need also the following facts. Let C_n^N be the class of stopping rules t satisfying $P\{n \leqq t \leqq N\} = 1$ and put

$$\gamma_n^N = \operatorname*{ess\,sup}_{t \in C_n^N} E(z_t \mid \mathcal{F}_n)$$

then, when $E\ \sup z_n < \infty$ we have

(1.11) $$\lim_{N=\infty} \gamma_n^N = \gamma_n \text{ a.s.} \qquad n = 1,2,\cdots).$$

Moreover, γ_n^N can be calculated recursively via

(1.12) $$\gamma_N^N = z_N,\ \gamma_{n-1}^N = \max(z_{n-1}, E(\gamma_n \mid \mathcal{F}_{n-1})), \qquad (n = N, N-1, \cdots, 2)$$

2. Scaled independent random variables. In this section we deal with a sequence of independent random variables which are identically distributed, up to a scale factor. The main result of this section is the following.

THEOREM 1. *Let $x_1, x_2, \cdots, x_n, \cdots$ be non-negative, independent identically distributed random variables satisfying*

(2.1) $$Ex_n^\beta < \infty$$

for some $\beta \geqq 1$. Let $a_1, a_2, \cdots, a_n, \cdots$ be a sequence of positive constants satisfying

(2.2) $$\sum_{n=1}^{\infty} a_n^\beta < \infty.$$

Put

(2.3) $$\lambda = \liminf_{n=\infty} \frac{a_{2n} + a_{2n+1}}{a_n}$$

and

(2.4) $$r_n = \left(\sum_{i=0}^{\infty} a_{n+i}^\beta\right)^{1/\beta}, \quad \rho = \limsup_{n=\infty} \frac{r_{2n}}{r_n}.$$

Then, if

(2.5) $$\lambda > \rho$$

the natural rule τ given by (1.12) *is optimal for $z_n = a_n x_n$ and it is a geniune stopping rule.*

Proof. The optimality of τ follows immediately from (2.1), (2.2), Lemma 1 and the estimate

(2.6) $$v_n \leqq E \sup_{m \geqq n} a_m x_m = (E \sup_{m \geqq n} a_m^\beta x_m^\beta)^{1/\beta} \leqq r_n (Ex_n^\beta)^{1/\beta}.$$

It thus remains to prove only that τ is a genuine stopping rule, i.e., that

(2.7) $$P\{\tau < \infty\} = 1.$$

To this end we first note that (2.4) and (2.6) imply

$$\liminf_{n=\infty} \frac{v_{2n}}{v_n} \leqq \rho. \tag{2.8}$$

Indeed, (2.4) implies $\limsup(r_{2n})^{1/n} \leqq \rho$ while the negation of (2.8) would imply $\liminf(v_{2n})^{1/n} > \rho$.

In order to complete the proof we shall show that the negation of (2.7) implies

$$\liminf_{n=\infty} \frac{v_{2n}}{v_n} \geqq \lambda. \tag{2.9}$$

Since (2.8) and (2.9) together are incompatible with (2.5) this contradiction would prove (2.7).

We proceed to show that the negation of (2.7) indeed implies (2.9). By the above and Lemma 1

$$v_n = \sum_{i=0}^{\infty} a_{n+i} \int_{\{\tau_n = n+i\}} x_{n+i}, \qquad (n = 1,2,\cdots). \tag{2.10}$$

Put

$$P_n = P\{a_n x_n < v_{n+1}\}. \tag{2.11}$$

Since the x_n are independent we have from (1.8), $P\{\tau_n = n+i\} = P_n P_{n+1} \cdots P_{n+i-1}$ and, again by independence, (2.10) becomes

$$v_n = \sum_{i=0}^{n} P_n P_{n+1} \cdots P_{n+i-1} a_{n+i} \int_{\{a_{n+i}x_{n+i} \geqq v_{n+i+1}x_{n+i}\}} x_{n+i}. \tag{2.12}$$

We now define a new stopping rule τ' by

$$\tau' = \inf\{m : m \geqq 2n \text{ and } a_{[m/2]} x_m \geqq v_{[m/2]+1}\} \tag{2.13}$$

where $[m/2]$ denotes the integral part of $m/2$. Then $\tau' \in C_{2n}$ and, therefore,

$$v_{2n} \geqq E a_{\tau'} x_{\tau'} = \sum_{m=2n}^{\infty} a_m \int_{\{\tau' = m\}} x_m. \tag{2.14}$$

But, by (2.13), we have for $m = 2n + 2i$ $(i = 0,1,2,\cdots)$, $\{\tau' = 2n + 2i\} = \{a_n x_{2n} < v_{n+1}, a_n x_{2n+1} < v_{n+1}, \cdots, a_{n+i} x_{2n+2i-2} < v_{n+i}, a_{n+i} x_{2n+2i-1} < v_{n+i}, a_{2n+2i} x_{2n+2i} \geqq v_{n+i+1}\}$ and thus, by (2.11), independence and equidistribution of the x_n, we have

$$\int_{\{\tau' = 2n+2i\}} x_{2n+2i} = P_n^2 P_{n+1}^2 \cdots P_{n+i-1}^2 \int_{\{a_{n+i} x_{2n+2i} \geqq v_{n+i+1}\}} x_{2n+2} \cdot$$

Similarly

$$\int_{\{\tau'=2n+2i+1\}} x_{2n+2i+1} = P_n^2 P_{n+1}^2 \cdots P_{n+i-1}^2 P_{n+i} \int_{\{a_{n+i}x_{2n+2i+1} \geqq v_{n+i+1}\}} x_{2n+2i+1}.$$

By equidistribution each of the integrals on the right side of the last two equations is equal to $\int_{\{a_{n+i}x_{n+i} \geqq v_{n+i+1}\}} x_{n+i}$.

Substituting in (2.14) we obtain

$$Ea_{\tau'}x_{\tau'} = \sum_{i=0}^{\infty} (P_n^2 \cdots P_{n+i}^2 a_{2n+2i} + P_n^2 \cdots P_{n+i}^2 P_{n+i+1} a_{2n+2i+1}) \int_{\{a_{n+i}x_{n+i} \geqq v_{n+i+1}\}} x_{n+i}.$$

From this, (2.12) and the inequality (2.14) we obtain

$$(2.15) \qquad \frac{v_{2n}}{v_n} \geqq \inf_{i \geqq 0} \; (P_n \cdots P_{n+i} a_{2n+2i} + P_n \cdots P_{n+i} P_{n+i-1} a_{2n+2i+1}) / a_{n+i}$$

$$\geqq \prod_{i=0}^{\infty} P_{n+i} \inf_{i \geqq 0} \frac{a_{2n+2} + a_{2n+2i+1}}{a_{n+i}}.$$

But $\prod_{n=1}^{\infty} P_n = P\{\tau = \infty\}$ and the negation of (2.7) implies that this product is convergent, hence the first factor on the right side of (2.15) tends to 1 as $n \to \infty$ and we have

$$\liminf_{n=\infty} \frac{v_{2n}}{v_n} \geqq \liminf_{n=\infty} \inf_{i \geqq 0} \frac{a_{2n+2i} + a_{2n+2i+1}}{a_{n+i}}.$$

But, by (2.3), this is precisely (2.9) and the proof of Theorem 1 is thus completed.

Theorem 1 has many specializations, we confine ourselves to giving one.

COROLLARY 1. *Let* $x_1, x_2, \cdots, x_n, \cdots$ *be independent, identically distributed non-negative random variables, let* $\alpha > 0$ *and* $Ex_n^{\beta} < \infty$ *for some* $\beta > \max(1, 1/\alpha)$ *then the natural stopping rule is a genuine optimal stopping rule for the sequence* x_n/n^{α}.

Indeed, (2.2) is satisfied and we have $\lambda = 2^{1-\alpha}$ while $\rho = 2^{1/\beta - \alpha}$ and thus (2.5 holds and the theorem applies.

3. A problem involving the maximum of random variables. Our aim in this section is to prove the following theorem about stopping rules for $\bar{z}_n = a_n \max(x_1, \cdots, x_n)$. $\bar{v}_n, \bar{\gamma}_n$ etc. will denote for $(\bar{z}_n)$ what v_n, γ_n etc. denote for (z_n).

THEOREM 2. *Let* $x_1, x_2, \cdots, x_n, \cdots$ *and* $a_1, a_2, \cdots, a_n, \cdots$ *satisfy the conditions of Theorem* 1. *Let, moreover,* (a_n) *be a logarithmically concave sequence, i.e. let*

$$a_n/a_{n+1} \leqq a_{n+1}/a_{n+2} \qquad (n = 1,2,\cdots). \tag{3.1}$$

Then the natural rule $\bar\tau$ *for* $\bar z_n = a_n \max(x_1,\cdots,x_n)$ *coincides with the natural rule* τ *for* $z_n = a_n x_n$. *It is thus a genuine stopping rule and we have*

$$\bar v_1 = E a_\tau \max(x_1,\cdots,x_\tau) = E a_\tau x_\tau = v_1. \tag{3.2}$$

Proof. We first show that $\bar\tau$ is optimal. We have

$$\begin{aligned} \bar v_n = \sup_{t\in C_n} E a_t \max(x_1,\cdots,x_t) &\leqq \sup_{t\in C_n} (E a_t^\beta \max(x_1^\beta,\cdots,x_t^\beta))^{1/\beta} \\ &\leqq (E(a_n^\beta(x_1^\beta + \cdots + x_n^\beta) + \sum_{i=1}^{\infty} a_{n+i}{}^\beta x_{n+i}{}^\beta))_{1/\beta} \\ &\leqq \left(n a_n^\beta + \sum_{i=1}^{\infty} a_{n+i}^\beta\right)^{1/\beta} \cdot (E x_n^\beta)^{1/\beta}. \end{aligned} \tag{3.3}$$

From (3.1) and $a_n \to 0$ (which follows from (2.2)) we deduce that (a_n) is monotone. Since a_n^β is monotone it follows from the convergence of the series (2.2) that $n a_n^\beta \to 0$ and therefore

$$\lim_{n=\infty} \bar v_n = 0.$$

In virtue of the lemma this proves the optimality of $\bar\tau$.

In our problem $\bar\gamma_n$ and $\bar\gamma_n^N$ are a.s. functions of $\max(x_1,\cdots,x_n)$. Let us put

$$\bar\gamma_n(u) = E(\bar\gamma_n \mid \max(x_1,\cdots,x_n) = u), \ \ \bar\gamma_n^N(u) = E(\bar\gamma_n^N \mid \max(x_1,\cdots,x_n) = u).$$

By (1.12)

$$\bar\gamma_N^N(u) = a_N u \geqq \bar\gamma_{N+1}^{N+1}(u) = a_{N+1} u.$$

We next show that generally

$$\frac{1}{a_n}\bar\gamma_n^N(u) \leqq \frac{1}{a_{n+1}}\bar\gamma_{n+1}^{N+1}(u) \qquad (n = 1,2,\cdots,N). \tag{3.4}$$

Indeed,

$$\frac{1}{a_n}\bar\gamma_n^N(u) = \sup_{t\in C_n^N} E\frac{a_t}{a_n}\max(u, x_{n+1},\cdots,x_t) \tag{3.5}$$

while

$$\frac{1}{a_{n+1}}\bar\gamma_{N+1}^{N+1}(u) = \sup_{t\in C_{n+1}^{N+1}} E\frac{a_t}{a_{n+1}}\max(u, x_{n+2},\cdots,x_t). \tag{3.6}$$

By idependence we may confine ourselves to rules in which $\{t = i\} \in \mathscr{B}(x_{n+1}, \cdots, x_i)$ in (3.5) and $\{t = i + 1\} \in \mathscr{B}(x_{n+2}, \cdots, x_{n+i+1})$ in (3.6) for $i = n, \cdots, N$ (for $i = n$ the corresponding $\mathscr{B}$ are, by convention, $\{\Omega, \emptyset\}$). There is a canonical one-to-one shift correspondence between these stopping rules of C_n^N and C_{n+1}^{N+1} where to $t \in C_n^N$ corresponds to $t' = t + 1 \in C_{n+1}^{N+1}$. Since, by (3.4), $a_t/a_n \leqq a_{t+1}/a_{n+1}$ for every value of $t \geqq n$ it follows on comparing the right sides of (3.5) and (3.6) that (3.4) holds.

From (3.1) it follows that $E \sup \bar{z}_n < \infty$, therefore, keeping n fixed and letting $N \to \infty$ we obtain from (1.11)

$$\text{(3.7)} \qquad \frac{1}{a_n}\bar{\gamma}_n(u) \leqq \frac{1}{a_{n+1}}\bar{\gamma}_{n+1}(u) \qquad (n = 1, 2, \cdots).$$

Consider now the natural rule $\bar{\tau}$ for $\bar{z}_n$. If $\bar{\tau} = n + 1$ then $a_{n+1} \max(x_1, \cdots, x_{n+1}) \geqq \bar{\gamma}_{n+1}(\max(x_1, \cdots, x_{n+1}))$ while $a_n \max(x_1, \cdots, x_n) < \bar{\gamma}_n(\max(x_1, \cdots, x_n))$. Since $\bar{\gamma}_n(u)$ is a non-decreasing function of u it follows from (3.6) that $\max(x_1, \cdots, x_{n+1}) > \max(x_1, \cdots, x_n)$ or that $x_{n+1} = \max(x_1, \cdots, x_{n+1})$. Since obviously also $x_1 = \max x_1$ we see that $\max(x_1, \cdots, x_{\bar{\tau}}) = x_{\bar{\tau}}$ and therefore $\bar{v}_1 = E\bar{z}_{\bar{\tau}} = Ez_{\bar{\tau}} \leqq v_1$. On the other hand $\bar{v}_1 \geqq v_1$ as remarked earlier, thus the two are equal. Similarly $\bar{v}_n = v_n$ for all n and since $\bar{\tau} = \inf\{n : a_n x_n \geqq \bar{v}_{n+1}\}$ it follows that $\bar{\tau} = \tau$.

We again bring only one example of the application of this theorem.

Corollary 2. *Under the conditions of Corollary* 1 *the natural rules for* x_n/n^α *and* $\max(x_1, \cdots, x_n)/n^\alpha$ *coincide. This common rule* τ *is genuine, optimal for both problems and*

$$E\frac{x_\tau}{\tau^\alpha} = E\frac{\max(x_1, \cdots, x_\tau)}{\tau^\alpha}.$$

4. Remark. Our method can give somewhat more general results than stated. We may, for example, relax the condition about the equidistribution of $x_1, \cdots, x_n, \cdots$. However, there remains a very simple fundamental question which we cannot solve. For all we know there may 'always' exist a genuine optimal rule. For instance if $Ex_n = \infty$ then clearly $t \equiv 1$ is such a rule. It would be extremely interesting to solve this problem even for the case $a_n = 1/n$.

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