Two dimensional spectral analysis

by

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Errata sheet for

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Page 1	Line 7 up	comma after "vehicles"
4	7	comma after "transformation"
5	8	"suto -" should be "auto-"
	3 up	"lot" should be "let"
7	3	"$\sqrt{\int}$(s(..." should be "$\sqrt{\int}$|s(... "
	5	"an" should be "a"
8	4 up	"2πifd" should be "2πifd"
12	7	"value" should be "valves"
14	3 up	"W'_{ab} should be "w'_{ab}"
25	last line	colon after "each"
29		units should be cpf (cycles per foot) on each axis.

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Introduction

A quick sketch of multidimensional stationary stochastic processes was given by Bartlett [1] sec. 6.5. The theoretical aspects are a special case of the work of Karhunen [5], Parzen [7], and others. A skeleton discussion of the statistical estimation problem was given by the author and others in [8], but more complete work is, to our knowledge, lacking. We present a theoretical sketch of two dimensional stationary processes with some possible uses of their spectra. Then we will give details of the statistical estimation of two dimensional spectra. We proceed to discuss measurement errors and finally give a summary of the formulas used in computation.

The major part of this work was sponsored by the Land Locomation Laboratory of the Army Tank-Automotive Center, Warren, Michigan and appears in a report [6]. The measured spectra were obtained for part of a study on the traversability of open ground by vehicles sponsored by the Land Locomotion Laboratory. Most of the practical problems discussed here were met in the processing of this data. Results of this study will appear in a future report of the Laboratory.

Representation

Suppose $h(x,y)$ is a stationary stochastic process having second moments. Stationarity will mean that for any finite set of points, $\{(x_i,y_i): i = 1,2,\ldots,k\}$, the random variables $\{h(x_i,y_i)\}$ have a joint distribution which is invariant

under translation, i.e. it equals that of the random variables $\{h(x_i + h, y_i + k)\}$ for any (h,k). There is no difficulty applying Karhunen's analysis, [5] p.42, to this case to arrive at the representation

$$(1) \qquad h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i(\lambda x + \mu y)} \, dW(\lambda,\mu) \quad .$$

In this $W(\lambda,\mu)$ is a complex valued orthogonal increments process of zero mean, to be specific, if Δ_{ab} denotes a difference operator and complex conjugates are denoted by $*$,

$$(2) \qquad E\{(\Delta_{ab}W(\lambda,\mu))\} = E\{(W(\lambda+a,\mu+b) - W(\lambda+a,\mu) - W(\lambda,\mu+b) + W(\lambda,\mu))\} = 0$$

$$E\{\Delta_{ab}W(\lambda_1,\mu_1)\Delta_{cd}W(\lambda_2,\mu_2)\} = 0 \quad \text{for} \quad \begin{matrix} \lambda_1 + a \le \lambda_2 \\ \mu_1 + b \le \mu_2 \end{matrix}$$

$$E\{|\Delta_{ab}W(\lambda,\mu)|^2\} = \Delta_{ab}F(\lambda,\mu) \quad .$$

This representation is useful in applications to vibration problems because of the choice of orthogonal functions $e^{2\pi i(\lambda x + \mu y)}$ which fit with the differential equations of vibratory motion. We assume in this paper that $h(x,y)$ has real values so that

$$(3) \qquad h(x,y) = \tfrac{1}{2}[h(x,y) + h^*(x,y)]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i(\lambda x + \mu y)} d\tfrac{1}{2}[W(\lambda,\mu) + V(\lambda,\mu)]$$

where $V(\lambda,\mu) = W^*(-\lambda,-\mu)$. The random function defined on the half plane, $\mu \ge 0$, by $\frac{1}{2}[W(\lambda,\mu) + V(\lambda,\mu)]$ is an orthogonal increments process whose variances are

$$E\{\tfrac{1}{4}|\Delta_{ab}[W(\lambda,\mu) + V(\lambda,\mu)]|^2\} = \tfrac{1}{4}[E\{|\Delta_{ab}W(\lambda,\mu)|^2\} + E\{\Delta_{ab}W(\lambda,\mu)\Delta_{ab}V^*(\lambda,\mu)\}$$

$$+ E\{\Delta_{ab}W^*(\lambda,\mu)\Delta_{ab}V(\lambda,\mu)\} + E\{|\Delta_{ab}V(\lambda,\mu)|^2\}] \ .$$

We have used the notation $V(\lambda,\mu)$ to make it clear that

$$\Delta_{ab}V(\lambda,\mu) = \Delta_{-a-b}W^*(-\lambda,-\mu) \ .$$

From this we see that the two central terms are zero and that the required variances are

$$\tfrac{1}{2}[\Delta_{ab}F(\lambda,\mu) + \Delta_{-a-b}F(-\lambda,-\mu)] \ . \tag{4}$$

When $F(\lambda,\mu)$ is continuous for $\mu = 0$ there is no ambiguity, but otherwise we must define the half plane so as to include the positive λ-axis and exclude the negative. We will henceforth assume that $F(\lambda,\mu)$ is absolutely continuous everywhere so that it has a density function which may be written $|s(\lambda,\mu)|^2$ since $F(\lambda,\mu)$ has the same properties as a distribution function. We may take $s(\lambda,\mu)$ to be the non=negative square root, whereupon $dW(\lambda,\mu)$ may be replaced by $s(\lambda,\mu)dZ(\lambda,\mu)$. We will say that the orthogonal increments process $Z(\lambda,\mu)$ has unit variance. The representation (1) becomes

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(\lambda x + \mu y)}s(\lambda,\mu)\ dZ(\lambda,\mu) \ . \tag{1a}$$

Equation (3) suggests that because of $h(x,y)$ being real valued we may restrict ourselves to orthogonal increments functions that are conjugate-symmetric through the origin. In truth the integral must be defined in a peculiar way with the differentials going in opposite directions below the λ-axis. If we collapsed

the region of integration to a half-plane this difficulty would not be present, but then in changing variables we would have to pay attention to the shape of the region of integration. We may work with the conjugate-symmetric $W(\lambda,\mu)$ (which is only orthogonal increments in the half plane) if we restrict ourselves to transformations which preserve symmetry about the origin, and also use the entire plane as a region of integration. Let us consider a linear transformation which will suffice for our purposes.

We note that an orthogonal increments process is really given by its differences $\Delta_{ab}W(\lambda,\mu)$, or equivalently by its integrals over measurable sets, $\int_A dW(\lambda,\mu)$. We take $W(\lambda,\mu)$ to be conjugate-symmetric so that its integral over a set is the sum of the integrals over the part of the set that is in the half plane with the part of its reflection through the origin in the half plane with the conjugate taken of the latter. Now consider a non-singular linear transformation

$$\lambda = a_{11}\xi + a_{12}\eta \qquad \xi = b_{11}\lambda + b_{12}\mu$$
$$\mu = a_{21}\xi + a_{22}\eta \qquad \eta = b_{21}\lambda + b_{22}\mu \quad .$$

The (ξ,η) plane may be represented on the (λ,μ) plane by use of an oblique set of axes; $b_{21}\lambda + b_{22}\mu = 0$, the ξ-axis etc. A rectangle in the (ξ,η)-plane becomes a parallelogram in the (λ,μ)-plane. If a set lies in the upper half plane of the (ξ,η)-plane then it lies on one side of the ξ-axis as represented in the (λ,μ)-plane. Consequently it is disjoint from its reflection through the origin and the integral of $dW(\lambda,\mu)$ over it is the sum of random variables which are uncorrelated. Using this it is easy to show that the process which we shall denote $W(a_{11}\xi + a_{12}\eta,\ a_{21}\xi + a_{22}\eta)$ and which is defined as having differences equal to the integral of $dW(\lambda,\mu)$ over the images of the rectangles,

is an orthogonal increments process in the upper half (ξ,η)-plane. Its variances are given by differences of the function gotten in a similar way from $F(\lambda,\mu)$, which we denote $F(a_{11}\xi + a_{12}\eta,\ a_{21}\xi + a_{22}\eta)$. This new orthogonal increments process is also conjugate-symmetric.

We will assume that our stationary process is given in terms of a conjugate-symmetric orthogonal increments process with absolutely continuous $F(\lambda,\mu)$. In representation (1a), the process $Z(\lambda,\mu)$ is conjugate symmetric, and the spectral function $|s(\lambda,\mu)|^2$ will be symmetric. The mean and suto-covariance functions of $h(x,y)$ are derived as usual:

$$E\{h(x,y)\} = 0 \tag{5}$$

$$R(s,t) = E\{h(x+s,\ y+t)h(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |s(\lambda,\mu)|^2\, e^{2\pi i(\lambda s+\mu t)}\, d\lambda\, d\mu \quad .$$

Because of the symmetry of $|s(\lambda,\mu)|^2$ through the origin, the integrals may be written as cosine transforms and/or taken over the upper half plane and doubled. $R(s,t)$ is also real and symmetric and the inverse transformation formula holds

$$|s(\lambda,\mu)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(s,t)e^{2\pi i(\lambda s+\mu t)}\, ds\, dt \quad . \tag{6}$$

In use the independent variables (x,y) could be two space coordinates, as when dealing with roughness of a surface. They could be space and a time coordinate as for a line process that is changing with time. Parallel tracks across the surface or points moving on the line with equal uniform speed will give correlated one variable stationary processes. Using the rough surface example, lot a line be given by the parametric equations in terms of distance from a point (x_o,y_o):

$$(x,y) = (x_o,y_o) + (\alpha,\beta)s \qquad \alpha^2 + \beta^2 = 1 \quad .$$

The surface heights on this line are

$$(7) \qquad k(s) = h(x_o + \alpha s, y_o + \beta s)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\lambda,\mu)\exp\{2\pi i[(x_o+\alpha s)\lambda + (y_o+\beta s)\mu]\}d\, Z(\lambda,\mu)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\lambda,\mu)\exp\{2\pi i(x_o\lambda+y_o\mu)\}\exp\{2\pi i(\alpha\lambda+\beta\mu)s\}d\, Z(\lambda,\mu) \quad .$$

Let us transform

$$\xi = \alpha\lambda + \beta\mu \qquad\qquad \lambda = \alpha\xi + \beta\eta$$

$$\eta = \beta\lambda - \alpha\mu \qquad\qquad \mu = \beta\xi - \alpha\eta$$

$$(8) \qquad k(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\alpha\xi + \beta\eta, \beta\xi - \alpha\eta)\exp\{2\pi i[(x_o\alpha + y_o\beta)\xi + (x_o\beta - y_o\alpha)\eta]\}$$

$$e^{2\pi i\xi s} d\, Z(\alpha\xi + \beta\eta,\ \beta\xi - \alpha\eta)$$

$$= \int_{-\infty}^{\infty} e^{2\pi i\xi s} \int_{-\infty}^{\infty} s(\alpha\xi + \beta\eta, \beta\xi-\alpha\eta)\exp\{2\pi i[x_o\alpha+y_o\beta)\xi +$$

$$+ (x_o\beta-y_o\alpha)\eta]\}\ dZ(\alpha\xi+\beta\eta,\ \beta\xi-\alpha\eta) \quad .$$

Consider now the process in one variable, ξ, given by the differences

$$\Delta_a U(\xi) = \int_{-\infty}^{\infty} s(\alpha\xi + \beta\eta,\ \beta\xi - \alpha\eta)\ \exp\{2\pi i[(x_o\alpha + y_o\beta)\xi + (x_o\beta - y_o\alpha)\eta]\}$$

$$d_\eta[Z(\alpha(\xi+a) + \beta\eta,\ \beta(\xi+a) - \alpha\eta) - Z(\alpha\xi + \beta\eta,\ \beta\xi - \alpha\eta)] \quad .$$

The expected value of $\Delta_a U(\xi)$ is zero, and for two non-overlapping intervals $(\xi_1,\ \xi_1 + a_1)$, $(\xi_2, \xi_2 + a_2)$, the integrals are defined by differences in $Z(\lambda,\mu)$ over disjoint sets. Thus the $U(\xi)$ process has orthogonal increments. The variance of $\Delta_a U(\xi)$ is

$$\int_{\xi}^{\xi+a} |s(\alpha\xi + \beta\eta,\ \beta\xi - \alpha\eta)|^2 \, d\eta \, d\xi$$

thus the $U(\xi)$ process may be represented by

$$\sqrt{\int_{-\infty}^{\infty} (s(\alpha\xi + \beta\eta,\ \beta\xi - \alpha\eta)|^2 \, d\eta} \; Z(\xi)$$

where $Z(\xi)$ is a one variable process with orthogonal increments and unit variance (see p.2. This is an conjugate symmetric process in one variable.). From this the stationary process for heights on the line may be represented as

$$(9) \qquad k(s) = \int e^{2\pi i \xi s} \sqrt{\int_{-\infty}^{\infty} |s(\alpha\xi + \beta\eta,\ \beta\xi - \alpha\eta)|^2 d\eta} \; dZ(\xi) \quad .$$

The spectral density of $k(s)$ is, therefore, an integral of the two dimensional spectrum.

Dr. H. Akaike of the Institute of Statistical Mathematics pointed out to me that the two dimensional spectrum may be reconstructed from the set of line spectra for lines in all directions. The relation is most easily seen from the connection between the two dimensional covariance function and the line covariance functions $R_{\theta}(u)$ for lines making an angle θ with the x-axis:

$$R(s,t) = R_{\text{Arctan } t/s} \left(\sqrt{s^2 + t^2}\right) \quad .$$

This result may be useful in estimating two dimensional spectra from the data taken by profilometers or other devices for rapid measurement of line profiles or spectra.

If two parallel lines are traced on the surface

$$(10) \qquad (x,y) = (x_1,y_1) + (\alpha,\beta)s \qquad \alpha^2 + \beta^2 = 1$$

$$(x,y) = (x_2,y_2) + (\alpha,\beta)s, \qquad (x_2,y_2) = (x_1,y_1) + (\beta,-\alpha)d \quad .$$

We have chosen the points (x_1,y_1) and (x_2,y_2) to be perpendicularly opposite on the lines which are a distance d apart.

The cross covariance of the heights on the two lines is

$$R_{12}(u) = E\{k_1(s+u)k_2(s)\} = E\{h(x_1+\alpha(s+u),\ y_1+\beta(s+u))h(x_2+\alpha s,\ y_2+\beta s)\}$$

$$= R(x_1-x_2+\alpha u,\ y_1-y_2+\beta u) = R(-\beta d+\alpha u,\ \alpha d+\beta u)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |s(\lambda,\mu)|^2 \exp\{2\pi i(-\lambda\beta d+\lambda\alpha u+\mu\alpha d+\mu\beta u)\}d\lambda\ d\mu$$

$$= \int\int |s(\lambda,\mu)|^2 \exp\{2\pi i(-\lambda\beta+\mu\alpha)d + 2\pi i(\lambda d+\mu\beta)u\}\ d\lambda\ d\mu \quad .$$

Letting

$$\lambda\alpha + \mu\beta = \xi \qquad \xi\alpha - \eta\beta = \lambda$$

$$-\lambda\beta + \mu\alpha = \eta \qquad \xi\beta + \eta\alpha = \mu$$

$$R_{12}(u) = \int\int |s(\xi\alpha-\eta\beta,\ \xi\beta+\eta\alpha)|^2 e^{2\pi i(\eta d+\xi u)} d\xi\ d\eta$$

$$= \int e^{2\pi i\xi u}\left[\int |s(\xi\alpha-\eta\beta,\ \xi\beta+\eta\alpha)|^2\ e^{2\pi i\xi d}\ d\eta\right] d\xi \quad .$$

The quantity in the square parentheses is a formula for the co-spectrum of the simultaneous stationary random processes, $k_1(s)$, $k_2(s)$, in terms of the two dimensional spectrum of the surface roughness. This function need not be real,

but it is conjugate-symmetric. The representations of the two line processes both look like (9), but there are two simultaneous orthogonal increments processes $Z_1(\xi)$, $Z_2(\xi)$ which have uncorrelated differences over non-overlapping intervals, but those differences over the same interval $(\xi,\xi+a)$ have covariance

$$\int_{\xi}^{\xi+a} \int_{-\infty}^{\infty} |s(\xi\alpha-\eta\beta,\ \xi\beta+\eta\alpha)|^2\ e^{2\pi i\eta d}\ d\eta d\xi \quad .$$

This rapid sketch is an attempt to indicate how stationary processes in two parameters can be represented and manipulated for practical work. The simple problems used here for illustration give stationary processes in one variable whose spectra and co-spectra may be computed from $|s(\lambda,\mu)|^2$. Curved lines and non-uniform velocities will of course, lead to non-stationary processes. In the Gaussian case these will be given completely by the two dimensional spectrum and simple questions may find answers in manipulations such as these.

Estimation of spectra

Ours will be a two dimensional analogue of the usual method of spectral estimation from evenly spaced measurements as given, e.g. by Blackman and Tukey [2]. The data will be gotten by measuring $h(x,y)$ at points

$$(x_o + (i-1)\Delta_x,\ y_o + (j-1)\Delta_y) \qquad \begin{matrix} i = 1,2,\ldots, n_x \\ j = 1,2,\ldots, n_y \end{matrix} \quad .$$

We abbreviate

$$h_{ij} = k(x_o + (i-1)\Delta_x,\ y_o + (j-1)\Delta_y) \quad .$$

The covariance estimates are

$$(12)\qquad r_{ab} = \frac{1}{(n_x-a)(n_y-b)} \sum_{i=1}^{n_x-a} \sum_{j=1}^{n_y-b} h_{i+a,j+b} h_{i,j};\quad 0 \le a \le m_x,\ 0 \le b \le m_y$$

$$= \frac{1}{(n_x-a)(n_y-b)} \sum_{i=1-a}^{n_x} \sum_{j=1}^{n_y-b} h_{i+a,j+b} h_{i,j};\quad -m_x \le a < 0,\ 0 < b \le m_y \quad .$$

The maximum lag numbers are normally less than one half of the numbers of data points in each direction. The definition (12) is extended, for convenience in writing formulas, to the lower half of the grid of all possible pairs by making it symmetric through the origin;

$$(13)\qquad r_{-a,-b} = r_{a,b} \quad .$$

It is evident that the expected values of r_{ab} are $R(a\Delta_x, b\Delta_y)$ (5), so that they are unbiased estimates. Their statistical characteristics will depend on the number of terms averaged. This number varies in each and is largest for the smaller lags. The choice of the maximum lag numbers is made with an eye to keeping the mean lagged products statistically reliable at the larger lags.

It is natural to attempt an estimate of $|s(\lambda,\mu)|^2$ by means of a Fourier series in analogy to (6). Then the estimate of a spectral value would be a linear combination of the covariances:

$$(14)\qquad f = \sum_{a=-m_x}^{m_x} \sum_{b=-m_y}^{m_y} w_{ab}\, r_{ab} \quad .$$

Since the r's are symmetric through the origin, the w's can be taken that way also (by redefining as $\frac{1}{2}(w_{ab} + w_{-a-b})$) so the sum may be cut in half for computational purposes. As with the one dimensional spectral analysis the w's will be modifications of the Fourier coefficients suggested in analogy to (6). The statistical properties of such linear estimates are derived easily as in the one dimensional case.

$$(15) \qquad E\{f\} = \sum_{a=-m_x}^{m_x} \sum_{b=-m_y}^{m_y} w_{ab}\, R(a\Delta_x,\, b\Delta_y)$$

$$= \sum_{a} \sum_{b} w_{ab} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |s(\lambda,\mu)|^2 \exp\{2\pi i(a\Delta_x\lambda + b\Delta_y\mu)\}\, d\lambda\, d\mu$$

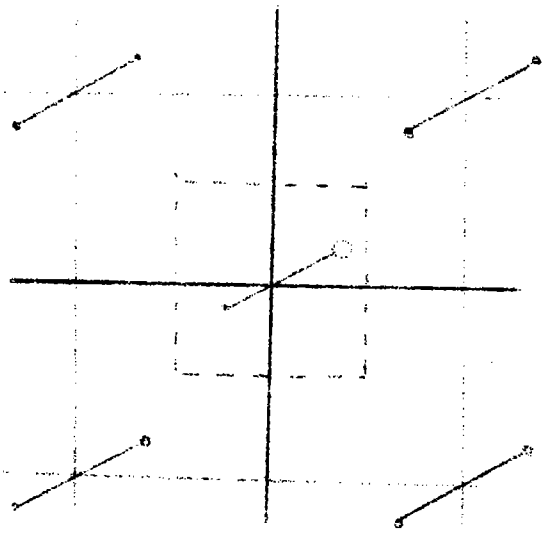
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |s(\lambda,\mu)|^2 \Big[\sum_{a} \sum_{b} w_{ab} \exp\{2\pi i(a\Delta_x\lambda + b\Delta_y\mu)\} \Big] d\lambda\, d\mu \quad .$$

The real valued function

$$(16) \qquad W(\lambda,\mu) = \sum_{a=-m_x}^{m_x} \sum_{b=-m_y}^{m_y} w_{ab} \exp 2\pi i(a\Delta_x\lambda + b\Delta_y\mu) = \sum_{a} \sum_{b} w_{ab} \cos 2\pi(a\Delta_x\lambda + b\Delta_y\mu)$$

is often called the spectral window. If it were a Dirac δ-function centered on (λ_o,μ_o) the result of the integration (15) would be the spectral value $|s(\lambda_o,\mu_o)|^2$. But, of course it is not possible to choose the coefficients w_{ab} to effect this. In fact the window $W(\lambda,\mu)$ is continuous, periodic, and symmetric through the origin for all choices of w_{ab}. Its periodicity and symmetry through the origin give rise to aliasing. Imagine designing the w's so that the window has a peak over (λ_o,μ_o). Then it will also have a peak at $(-\lambda_o,-\mu_o)$ because of its symmetry. This is no trouble since the spectrum is also symmetric. But this pair of peaks is repeated about all the grid points

$(k/\Delta_x, \ell/\Delta_y)$ in the (λ,μ) plane. It is clear that after our points of estimation have covered the rectangle $|\lambda| \leq 1/2\Delta_x$, $|\mu| \leq 1/2\Delta_y$, any further calculations will repeat estimates that already have been obtained. Each estimate (14) has an expected value which is a weighted sum of value of $|s(\lambda,\mu)|^2$ some near the point (λ_o,μ_o) and others quite remote. The remote values are "aliased" (a statistician says "confounded") with the ones inside the rectangle of half a wavelength. We will name this rectangle the "Nyquist rectangle". The experimenter must control aliasing by choosing the spacing of his data points, i.e. the Δ_x and Δ_y so that the spectral values, $|s(\lambda,\mu)|^2$ are small outside the Nyquist rectangle. We shall assume this is so in our discussion.



We wish to attain, as closely as possible, unbiased estimates of the point values $|s(\lambda,\mu)|^2$ for different (λ_o,μ_o). It is clear from the continuity of the function $W(\lambda,\mu)$ that we must average values of the spectrum from other points near (λ_o,μ_o). In fact since the relation (16) is a Fourier series, its coefficients are integrals

$$(17) \qquad w_{ab} = \Delta_x\Delta_y \int_{-\frac{1}{2}\Delta_x}^{\frac{1}{2}\Delta_x} \int_{\frac{1}{2}\Delta_y}^{\frac{1}{2}\Delta_y} W(\lambda,\mu) \cos 2\pi(\lambda\Delta_x a + \mu\Delta_y b)\, d\lambda\, d\mu \quad .$$

These coefficients are zero for $|a| \geq m_x$, $|b| > m_y$. If we hope to make $W(\lambda,\mu)$ small except in the neighborhood of (λ_o,μ_o) then the area in which it is non-negligible must cover both positive and negative parts of the cosines with frequencies higher than m_x,m_y. For this to be so, the area of the non-negligible

values of the window must be about that of the rectangle $|\lambda| \leq 1/2\Delta_x m_x$, $|\mu| \leq 1/2\Delta_y m_y$.

Another remark that can be made on general principles is that the estimates (14) are linear combinations of a finite number of r's ($2m_x m_y + m_x + m_y + 1$, to be exact) and thus an equal number (perhaps less in singular cases) of f's will be a linearly independent set. All other values are functionally dependent on these. We may therefore restrict our computational effort to obtaining a number of spectral estimates equal to the number of covariances. We shall see later that it is advantageous statistically to space the estimates so that the windows do not overlap significantly. Then combined with the observation in the preceeding paragraph we are led to choose our points of estimation to be equally spaced within the Nyquist rectangle. Since the boundaries of the Nyquist rectangle are reflections of each other through the origin, we choose our points as follows:

$$\lambda_r = \frac{r}{(2m_x + 1)\Delta_x} \qquad r = 0, \pm 1, \pm 2, \ldots, \pm m_x \tag{18}$$

$$\mu_s = \frac{s}{(2m_y + 1)\Delta_y} \qquad s = 0, \pm 1, \pm 2, \ldots, \pm m_y \quad .$$

In one sense the closest we can make $W(\lambda,\mu)$ come to a Dirac function centered on (λ_r,μ_s) is to make it the finite part of the Fourier cosine series of the Dirac function, i.e. to use

$$w_{ab} = \Delta_x \Delta_y \cos 2\pi(a\Delta_x\lambda_r + b\Delta_y\mu_s) \quad .$$

The window function is easily calculated using Formula 428 of Jolley [4], or by putting the cosines in exponential form.

$$(19)\qquad W(\lambda,\mu) = \sum_{a=-m_x}^{m_x} \sum_{b=-m_y}^{m_y} \Delta_x\Delta_y \cos 2\pi(a\Delta_x\lambda_r + b\Delta_y\mu_s) \cos 2\pi(a\Delta_x\lambda + b\Delta_y\mu)$$

$$= \frac{\Delta_x\Delta_y}{2}\Big[\sum_a \sum_b \cos 2\pi(a\Delta_x(\lambda_r + \lambda) + b\Delta_y(\mu_s + \mu))$$

$$+ \sum_a \sum_b \cos 2\pi(a\Delta_x(\lambda_r - \lambda) + b\Delta_y(\mu_s - \mu))\Big]$$

$$= \frac{\Delta_x\Delta_y}{2} \frac{\sin 2\pi(m_x + \frac{1}{2})\Delta_x(\lambda_r + \lambda)}{\sin \pi\Delta_x(\lambda_r + \lambda)} \frac{\sin 2\pi(m_y + \frac{1}{2})\Delta_y(\mu_s + \mu)}{\sin \pi\Delta_y(\mu_s + \mu)} +$$

$$+ \frac{\Delta_x\Delta_y}{2} \frac{\sin 2\pi(m_x + \frac{1}{2})\Delta_x(\lambda_r - \lambda)}{\sin \pi\Delta_x(\lambda_r - \lambda)} \frac{\sin 2\pi(m_x + \frac{1}{2})\Delta_y(\mu_s - \mu)}{\sin \pi\Delta_y(\mu_s - \mu)} .$$

In case $(\lambda_r + \lambda)$ or one of the other frequencies is zero, the ratio of sines is to be interpreted as the ratio of the arguments, e.g. as $(2m_x + 1)$. $W(\lambda,\mu)$ has peaks at (λ_r,μ_s) and its reflection through the origin. The shape of the peaks may be inferred from the graphs of the corresponding one dimensional window plotted in Blackman and Tukey [2] or in Hannan [3], for each of the sine ratios is half of the one dimensional window function. The window takes large negative values near the peak which gives it rather poor characteristics. It is common to consider modifications of it in the form

$$(20)\qquad W'_{ab} = \Delta_x\Delta_y\, d_{ab} \cos 2\pi(a\Delta_x\lambda_r + b\Delta_y\mu_s), \qquad d_{-a,-b} = d_{a,b} .$$

From (19) we can see that the cosines in (20) and (16) form an orthogonal set under summation

$$(21)\qquad \sum_{a=-m_x}^{m_x} \sum_{b=-m_y}^{m_y} \cos 2\pi\left(\frac{aj}{2m_x+1} + \frac{bk}{2m_y+1}\right) \cos 2\pi\left(\frac{aj'}{2m_x+1} + \frac{bk'}{2m_y+1}\right)$$

$$= \begin{cases} 0 & \text{if } j \neq \pm j' \text{ or } k \neq \pm k' \\ \frac{1}{2}(2m_x+1)(2m_y+1) & \text{if } j = \pm j' \text{ and } k = \pm k' \\ (2m_x+1)(2m_y+1) & \text{if } j = j' = k = k' = 0 \end{cases}$$

where all four $\pm$ signs are taken the same way.

With these relations we may compute the formula for the window $W'(\lambda,\mu)$ gotten from coefficients (20). It is the convolution of the window $W(\lambda,\mu)$ with the Fourier cosine series of the coefficients d_{ab}. By convolution, we mean, not the integral, but the finite sum over the lattice points (18). If the modified window, $W'(\lambda,\mu)$ is to be zero at points remote from (λ_r,μ_s) the finite Fourier transform of the coefficients d_{ab} cannot be large remote from the origin. Commonly this finite set of values is such that only few of them around the origin are non zero. It is also usual, in the interest of economical computation to take the d_{ab} independent of the (r,s). In this way they may be applied as multipliers to the covariances after which an ordinary cosine transform is computed, thus accomplishing (14), or the computation of (14) may be done as a cosine transform and the resulting f's "smoothed" by convolving them with the cosine transform of the d's. If a little care is taken at the boundaries, both methods are arithmetically equivalent. Because of the form of the elementary window (19) if we use products of the one dimensional spectral smoothing coefficients, their effects in modifying the window can be assessed from the discussion of the one dimensional case. For example the counterpart of the Hamming window is gotten by smoothing the spectrum by running average with

$$g_{ab} = g_a g_b \; ; \qquad g_o = .54, \quad g_1 = g_{-1} = .23 \quad .$$

Statistical reliability of estimates.

To this point we have discussed only the expected value of our estimates based on linear combinations of covariances. The form of the estimate is virtually completely given by requiring certain desired properties of the expected value. Let us now consider the statistical variability of such estimates. The work follows closely that done in Blackman and Tukey [2] for the one dimensional case and thus we shall only give a sketch.

Consider two spectral estimates (or one estimate denoted two ways)

$$(22) \qquad f^{(1)} = \sum_a \sum_b w_{ab}^{(1)} r_{ab}, \qquad f^{(2)} = \sum_a \sum_b w_{ab}^{(2)} r_{ab} \ .$$

$$\operatorname{cov}\{f^{(1)}, f^{(2)}\} = \sum_a \sum_b \sum_c \sum_d w_{ab}^{(1)} w_{cd}^{(2)} \operatorname{cov}\{r_{ab}, r_{cd}\}$$

The covariance of two r's may be found in terms of the spectrum by a tedious calculation based on the assumption that the expected value of the product of four values of $h(x,y)$ is given in terms of the spectrum by the formula that would hold if the stationary process were Gaussian:

$$(23) \qquad \operatorname{cov}\{r_{ab}, r_{cd}\} = 8 \int\int\int\int K(\Delta_x w, n_x - |a|) \cdot K(\Delta_x w, n_x - |c|) \cdot$$

$$\cdot K(\Delta_y z, n_y - |b|)\, K(\Delta_y z, n_y - |d|) \cdot$$

$$\cdot \cos 2\pi(a\Delta_x w' + b\Delta_y z') \cos 2\pi(c\Delta_x w' + d\Delta_y z') \cdot$$

$$\cdot |s(w-w', z-z')|^2 \, |s(w+w', z+z')|^2 \, dw\, dw'\, dz\, dz' \ ,$$

$$\text{where} \qquad K(x,n) = \frac{\sin 2\pi x n}{n \sin 2\pi x} \ .$$

The sine ratio $K(x,n)$ is an even, periodic function with peaks at $x = 0, \pm\frac{1}{2}, \pm 1$, etc. The width of a peak is determined by n. The product of the sine ratios in the integral has the effect of restricting the integration to the region near $w = 0$, $z = 0$, and to other such hyperplanes outside the four dimensional Nyquist rectangle. The degree of concentration depends on the various sizes of $n_x - |a|$, etc. We will take m_x and m_y small enough so that the approximation of $n_x - |a|$ etc. by common values N_x, etc. will be satisfactory. This choice will depend on the rapidity of variation of the spectral function $|s(\lambda,\mu)|^2$. With this approximation we may replace the covariances in (22) with the integrals (23) and take the sums inside the integral signs to get

$$(24) \qquad \operatorname{cov}\{f^{(1)},f^{(2)}\} \approx 8 \int\int\int\int \left(\frac{\sin 2\pi\Delta_x w N_x}{N_x \sin 2\pi\Delta_x w}\right)^2 \left(\frac{\sin 2\pi\Delta_y z\, N_y}{N_y \sin 2\pi\Delta_y z}\right)^2 \cdot$$

$$\cdot\, W^{(1)}(w',z')\, W^{(2)}(w',z') \cdot$$

$$\cdot\, |s(w-w',z-z')|^2\, |s(w+w',z+z')|^2\, dw\, dw'\, dz\, dz' \,.$$

We are assuming that $|s(\lambda,\mu)|^2$ is small outside the Nyquist rectangle, and we assume further that it is not rapidly varying with respect to the width of the peaks of the sine ratios. We may, then, approximately carry out the integration over w and z to get

$$(25) \qquad \operatorname{cov}\{f^{(1)}, f^{(2)}\} \approx 8 \int\int W^{(1)}(w',z')\, W^{(2)}(w',z') |s(w',z')|^4 \frac{1}{2N_x\Delta_x} \frac{1}{2N_y\Delta_y}\, dw'\, dz' \,.$$

The integral of the sine ratio, or Fejer's integral, may be found in Titchmarsh, [9], p.413.

Formula (25) reveals that if two different window functions do not overlap, i.e. if each is small where the other is large, the corresponding spectral estimates are essentially uncorrelated. For the situation where $f^{(1)}$ and $f^{(2)}$ are the same estimate, the formula is still valid and gives the variance of f,

$$\text{(26)} \qquad \text{Var } \{f\} = \frac{2}{\Delta_x \Delta_y N_x N_y} \int\int \left[W(w,z)|s(w,z)|^2\right]^2 dw\, dz \quad .$$

This result is used to indicate the statistical reliability of the procedure in the following way. The spectral estimate

$$\text{(27)} \qquad f = \sum_a \sum_b w_{ab} r_{ab}$$

is a quadratic function of the data and as such has a distribution which is well approximated by a distribution of the gamma type. Chi square tables may be used to calculate confidence intervals on f using the degrees of freedom formula,

$$\text{d.f.} = 2 \frac{(E\{f\})^2}{\text{Var } \{f\}} \quad .$$

A simple approximation to this may be found by taking $W(\lambda,\mu)$ to be a "perfect" rectangular window and letting $m_x/n_x = m_y/n_y = g$.

$$\text{(28)} \qquad \text{d.f.} \approx 2\left(\frac{1-g}{g}\right)^2 \quad .$$

Prewhitening.

With some applications the general shape of the spectrum will be no surprise to the experimenter. For example, roughness spectra of ground profiles are usually very high near zero frequency and fall off rapidly as frequency increases in any direction. In this kind of spectrum the behavior of $|s(\lambda,\mu)|^2$ around

the origin is anything but slowly varying and we may expect difficulty in estimation. A procedure given in Blackman and Tukey [2], called "prewhitening" may be used to avoid these difficulties. For the two dimensional case prewhitening proceeds as follows. Coefficients

$$c_{\alpha,\beta};\quad -d_x \le \alpha \le d_x,\quad -d_y \le \beta \le d_y$$

are specified. We will take them so $c_{-\alpha,-\beta} = c_{\alpha,\beta}$. From the data, h_{ij}, we compute

$$h'_{ij} = \sum_{\alpha=-d_x}^{d_x} \sum_{\beta=-d_y}^{d_y} h_{i+\alpha,j+\beta}\, c_{\alpha,\beta} \tag{29}$$

for $d_x + 1 \le i \le n_x - d_x$, $d_y + 1 \le j \le n_y - d_y$. There are $(n_x - 2d_x)(n_y - 2d_y)$ items of the modified data. This data is analyzed as above, yielding covariances, r'_{ab} and spectral estimates $f'_{\alpha\beta}$. If, as is usually done in explaining this procedure, the random variables h'_{ij} are represented in terms of (1a), the true spectrum $|s'(\lambda,\mu)|^2$ of the h'_{ij} is seen to be a product of $|s(\lambda,\mu)|^2$ by the "spectrum" of the coefficients $c_{\alpha\beta}$. But such an explanation is glib. The prewhitened data and the old data are presented as mathematically equivalent, and the reader is apt to wonder what has been accomplished. A treatment in terms of the arithmetical operations of spectral estimation is more revealing.

$$(30)\qquad r'_{ab} = \frac{1}{(n_x-2d_x-a)(n_y-2d_y-b)} \sum_{i=d_x+1}^{n_x-2d_x-a} \sum_{j=d_y+1}^{n_y-2d_y-b} h'_{i+a,j+b}\, h'_{i,j}$$

$$= \frac{1}{(n'_x-a)(n'_y-b)} \sum_i \sum_j \sum_{\alpha=-d_x}^{d_x} \sum_{\beta=-d_y}^{d_y} h_{i+\alpha+a,j+\beta+b}\, c_{\alpha\beta} \cdot$$

$$\cdot \sum_{\gamma=-d_x}^{d_x} \sum_{\delta=-d_y}^{d_y} h_{i+\gamma,\, j+\delta}\, c_{\gamma\delta}$$

$$= \frac{1}{(n'_x-a)(n'_y-b)} \sum_\alpha \sum_\beta c_{\alpha\beta} \sum_\gamma \sum_\delta c_{\gamma\delta} \sum_i \sum_j h_{i+\alpha+a,j+\beta+b}\, h_{i+\gamma,j+\delta} \,.$$

We have abbreviated $n_x - 2d_x$ and $n_y - 2d_y$ and n'_x and n'_y. The sums

$$\frac{1}{(n'_x-a)(n'_y-b)} \sum_{i=d_x+1}^{n'_x-j} \sum_{j=d_y+1}^{n'_y-b} h_{i+\alpha+a,j+\beta+b}\, h_{i+\gamma,j+\delta}$$

are approximately those of

$$r_{a+\alpha-\gamma,\, b+\beta-\delta}$$

except that some products on either end are omitted. Some compensation is made by dividing by the number of terms actually present. This is the first deviation of the arithmetic from the ideal and it is probably minor in practice.

It makes the notation simpler if we extend the range of definition of the $c_{\alpha\beta}$ to all pairs of integers by making them zero for $|\alpha| > d_x$, $|\beta| > d_y$. Then the limits of summation may be ignored. Let $\alpha-\gamma=\xi$ and $\beta-\delta=\eta$

$$(31) \qquad r'_{ab} \approx \sum_{\xi} \sum_{\eta} \sum_{\gamma} \sum_{\delta} c_{\xi+\gamma,\eta+\delta} \; r_{a+\xi,\, b+\eta}$$

$$\approx \sum_{\xi} \sum_{\eta} p_{\xi\eta} \, r_{a+\xi,\, b+\eta} \quad .$$

Now this is the same sort of running average, or convolution, as (29), but done with the covariances and using coefficients

$$(32) \qquad p_{\xi\eta} = \sum_{\gamma} \sum_{\delta} c_{\xi+\gamma,\, \eta+\delta} \; c_{\gamma\delta} \quad .$$

These are the summed (not averaged) lagged products of the coefficients used in prewhitening. Since all possible pairs are used, they are not truncated as are the covariances which they resemble. The range of subscripts, (ξ,η) which covers the non zero values of the p's is $|\xi| \le 2d_x$, $|\eta| \le 2d_y$.

Let us consider the spectral estimates given by the simple Fourier cosine transform of the r'_{ab} .

$$f'_{\alpha\beta} = \Delta_x \Delta_y \sum_{a=-m_x}^{m_x} \sum_{b=-m_y}^{m_y} \cos 2\pi \left(\frac{a\alpha}{2m_x+1} + \frac{b\beta}{2m_y+1} \right) r'_{ab} \quad .$$

From the convolution formula for finite Fourier series, we have, since r'_{ab} is a convolution of p_{ab} and r_{ab} ,

$$(33) \qquad f'_{\alpha\beta} = t_{\alpha\beta} \, f_{\alpha\beta}$$

where $t_{\alpha\beta}$ is the cosine transform of the p_{ab}'s and $f_{\alpha\beta}$ is the cosine transform of the r_{ab}'s. Therefore all that need be done to find $f_{\alpha\beta}$ from $f'_{\alpha\beta}$ is calculate the cosine transform of the p_{ab} and multiply $f'_{\alpha\beta}$ term by term by

its reciprocals. Actually the computation is usually done with $f'_{\alpha\beta}$ not the cosine transforms of the covariances, but the spectral estimates made with some more suitable window. This is the second deviation from the ideal and probably the important one in practice.

Prewhitening has a real effect on practical results. Far from doing something and then undoing it, some part of the information in the data is removed completely. To indicate this let us consider a mode of prewhitening which is appropriate to data taken from a terrain profile. The areas measured were 200 ft. by 200 ft. (61 m by 61 m) and there were 100 by 100 grid points. The finiteness of scope prevents any effective observation of frequencies smaller than 1/200 cycle per ft. The sizes of ground height variations with wavelengths this long or longer are considerable, in fact they are dominant over any of the wavelengths within the scope of effective measurement. Such height variations appear in the finite record as trends. In our effort to remove these trends we resorted to a prewhitening with coefficients $c_{\alpha\beta}$ which, in effect, subtracted from each point's height the average of all the heights (including itself) in a square surrounding it. It is easy to show that when $h(x,y)$ has a linear trend this removes it. To be exact, if we have two ground profiles which differ only in

$$h^{(1)}(x,y) - h^{(2)}(x,y) = Ax + By + C \quad ,$$

then they will be reduced to the same prewhitened data. The linear trend has been removed and is not brought back by the correction of the spectral estimates. It is simple to construct coefficients that remove any polynomial trend, even though this trend may appear as one or two cycles of a low frequency wave in the finite data.

Errors in the data.

No data is free from the possibility of gross errors and noise. As a first approximation to the effect of errors let us take them to be independent of the randomness in the data itself. We write

$$h_{ij} = h_{ij}^{(1)} + \delta_{ih} \tag{34}$$

where $h_{ij}^{(1)}$ are the grid point values of the random process we would like to observe, and δ_{ij} are the errors made in reading the values at each point. The covariance estimates of our data are

$$r_{ab} = \frac{1}{(n_x-a)(n_y-b)} \sum_a \sum_b \left[h_{i+a,j+b}^{(1)} h_{i,j}^{(1)} + h_{i+a,j+b}^{(1)} \delta_{i,j} + h_{i,j}^{(1)} \delta_{i+a,j+b} + \delta_{i+a,j+b} \delta_{i,j} \right] . \tag{35}$$

The two central terms have expected value zero due to the independence of $h_{ij}^{(1)}$ and δ_{ij}. Their actual values, though perhaps small in usual practice, contribute somewhat to the statistical error in the covariance estimate.

As a first case suppose the δ_{ij} are all zero except for two relatively large values at (i_1,j_1) and (i_2,j_2). From (35) leaving out the central terms, we have

$$r_{ab} = r_{ab}^{(1)} + r_{ab}^{(\delta)} \tag{36}$$

$$(37) \qquad r_{ab}^{(\delta)} = \begin{cases} \dfrac{\delta_{i_1 j_1}^2 + \delta_{i_2 j_2}^2}{n_x\, n_y} & \text{if } (a,b) = (0,0) \\ \dfrac{\delta_{i_1 j_1}\, \delta_{i_2 j_2}}{(n_x - |i_1 - i_2|)(n_y - |j_1 - j_2|)} & \text{if } (a,b) = (i_1 - i_2, j_1 - j_2) \text{ and if } |i_1 - i_2| \le m_x,\ |j_1 - j_2| \le m_y \\ 0 & \text{otherwise .} \end{cases}$$

The unsmoothed spectral estimates of $r_{ab}^{(\delta)}$ are

$$f_{\alpha\beta}^{(\delta)} = \Delta_x \Delta_y \sum_a \sum_b r_{ab}^{(\delta)} \cos 2\pi \left(\frac{a\alpha}{2m_x+1} + \frac{b\beta}{2m_y+1}\right)$$

$$= \Delta_x \Delta_y \left[\frac{\delta_{i_1 j_1}^2 + \delta_{i_2 j_2}^2}{n_x\, n_y} + \frac{\delta_{i_1 j_1}\, \delta_{i_2 j_2}}{(n_x - |i_1 - i_2|)(n_y - |j_1 - j_2|)} \cdot \right.$$

$$\left. \cdot \cos 2\pi \left(\frac{|i_1 - i_2|\alpha}{2m_x+1} + \frac{|j_1 - j_2|\beta}{2m_y+1}\right) \right] .$$

The first term is constant over the entire spectrum and is like the effect of reading error. The second term is present only if the two anomalous points are close enough together to be within the maximum lag rectangle of size m_x by m_y. Prewhitening and smoothing of the spectrum tend to blur these effects somewhat. Some of our one dimensional ground profile data had occasional errors of one foot, caused by blunders in recording the levels while surveying. These errors were about three times the average absolute variation in height but nevertheless caused a noticeable periodic wobble in the spectral graph and pips on the covariance curve. The data was searched and the anomalies corrected with resulting improvement in the appearance of the spectral estimates.

A type of random error that can occur in two dimensional data was reported by Pierson et al. [8]. This is a directional error caused by the method of carrying out the measurements of h_{ij}. In the case of surveying, for example, the surveyor may set his level and measure several complete lines in the y direction. Then he may move the level, record its new height, and measure several more. The errors in measuring level heights form, let us suppose, a stationary random process in the x direction, but they are constant in the y direction. That is to say the δ_{ij} are constant in j. We denote them by δ_i. The covariances $r_{ab}^{(\delta)}$ are also constant in their second subscript and are actually covariances of the one dimensional process, δ_i. Such a covariance gives rise to spectral estimates

$$f_{\alpha\beta}^{(\delta)} = \Delta_x\Delta_y \sum_a \sum_b r_a^{(\delta)} \cos 2\pi \left(\frac{a\alpha}{2m_x+1} + \frac{b\beta}{2m_y+1}\right)$$

$$= \Delta_x\Delta_y \frac{\sin \pi\beta}{\sin \frac{\pi\beta}{2m_y+1}} \sum_a r_a^{(\delta)} \cos 2\pi \frac{a\alpha}{2m_x+1}$$

These are zero for (integral) $\beta \neq 0$. Therefore the estimates are concentrated on the λ axis of the spectral graph. The effect of spectral smoothing or modification of the window is to spread these away from their axis positions. Prewhitening has very little effect since the running average operation on the data preserves the constancy of this error in one direction.

If the δ's are independent random variables, their covariances would be zero except at the origin and the spectral estimates would consist of a ridge on the λ axis of equal height all along. The surveyor's level setting error, on the other hand, gives in the first approximation, independent random variables, $\alpha_1,\alpha_2,\ldots,\alpha_{n/b}$ which are repeated b times each $\alpha_1,\alpha_1,\ldots,\alpha_1,\alpha_2,\ldots$.

The covariances of this sequence are:

$$r_o^{(\delta)} = \frac{1}{n}\, b \sum_{i=1}^{n/b} \alpha_i^2$$

$$r_1^{(\delta)} = \frac{b-1}{n-1} \sum_{i=1}^{n/b} \alpha_i^2 + \frac{1}{n-1} \sum_{i=1}^{n/b-1} \alpha_i\, \alpha_{i+1}$$

$$r_2^{(\delta)} = \frac{b-2}{n-2} \sum_{i=1}^{n/b} \alpha_i^2 + \frac{2}{n-2} \sum_{i=1}^{n/b-2} \alpha_i\, \alpha_{i+2} \quad .$$

From the mutual independence of the α's the latter terms are likely to be small. The first terms are largest for $r_o^{(\delta)}$ then decrease steadily to become zero after $r_{b-1}^{(\delta)}$ (more terms of the second kind appear then also). The spectral estimates of this type of error will be given by a sum of b cosine terms of low frequency beginning with the constant one. They will not be constant as for independent errors, but will still be concentrated on the λ axis.

An error in data acquisition which is the sum of two independent directional effects gives a spectral error composed of a ridge on each axis. The spectral smoothing or modification of the window spreads these ridges but in the plain cosine transform of the covariances, the high values are on the axes of the spectral graph only. If this transform is computed and printed out the detection and correction of this kind of error is easier.

Field Results.

Figures 1 and 2 are two of a series of twenty 2-dimensional spectral estimates made from ground profile data measured with rod and level. The measurements of height were made at 2 ft. by 2 ft. grid points in a 200 ft. by 200 ft.

square. The maximum lag number in both axes was 20 so that the degrees of freedom for each estimate, given approximately by formula (28), are 32. The units on the axes are cycles per foot and the heights are $(ft.)^4$ or square feet per cycles per ft. per cycles per ft. Note that the contours are not at equal height intervals but exponentially increasing.

The spectrum in Figure 1 is from data taken at Ft. Carson, Colorado. It shows two ridges, one on the horizontal axis and the other about 60° to it. The latter ridge, which is mostly in the higher frequencies, is roughly at right angles to the general slope of the ground on which the measurements were taken. This spectral ridge indicates that the ground has, to some extent, ribs of width 14 ft. and less which run up and down the slope. This is a miniature of the appearance of mountain slopes as seen, for example, in aerial photographs and is what one might have expected. The expected correspondence between spectral ridge and ground slope occurs in a large number of our measured spectra. A few spectra have ridges which are not in accord with the ground slope. An example is the axial ridge in Figure 1, which extends into the long wavelengths. Several other spectra have ridges which indicate that the ground is ribbed across the slope. Explanation of these awaits further scrutiny of the data.

The data for Fig. 2 was taken at Ft. McClellan, Alabama. The dominant part of it is symmetric and is in accord with the observation that the ground profile has no definite slope, the square having been laid out in a draw. The extension of the .02 contour of the spectral surface along the horizontal axis is possibly due to the directional measurement error described above. If so the error must arise from a more complex situation than we have discussed since the ridge is wider than would have resulted from smoothing high values on the axis only. The variance of this presumed error is somewhat large as compared to presumed errors in our other data. Because this kind of error was unforseen,

we failed to keep adequate records of the surveying operation and so may have lost our chance to explain this spectrum.

A comparison of the two spectral charts shows that the McClellan spectral surface is everywhere above that of Carson which indicates that the ground was much rougher at McClellan. This is supported by visual observation. The results obtained at our twenty sites are in good agreement with visual observation. However our purpose is to put ground roughness into quantitative terms for engineering uses. Qualitative discussions such as we have been giving serve a purpose, but the real value of the spectra will be shown only by future work.

I would like to express my gratitude to Mr. Ronald A. Liston of the Land Locomotion Laboratory, ATAC, for his support of this project and for many conferences leading to an understanding of the real problem. Also thanks go to my colleagues and co-workers, J.L. Bogdanoff and F. Kozin, whose thoughts on this problem I can no longer separate from mine.

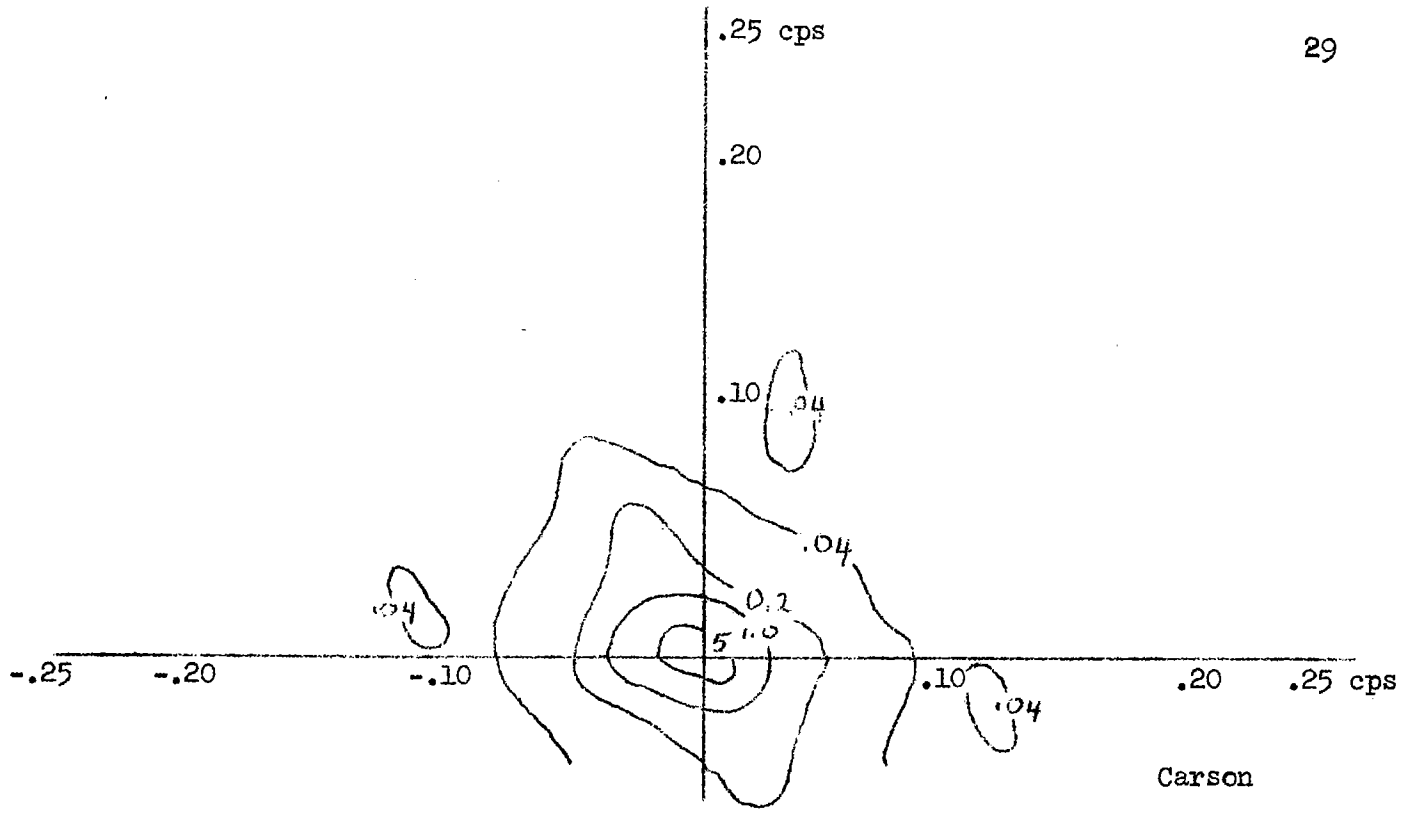

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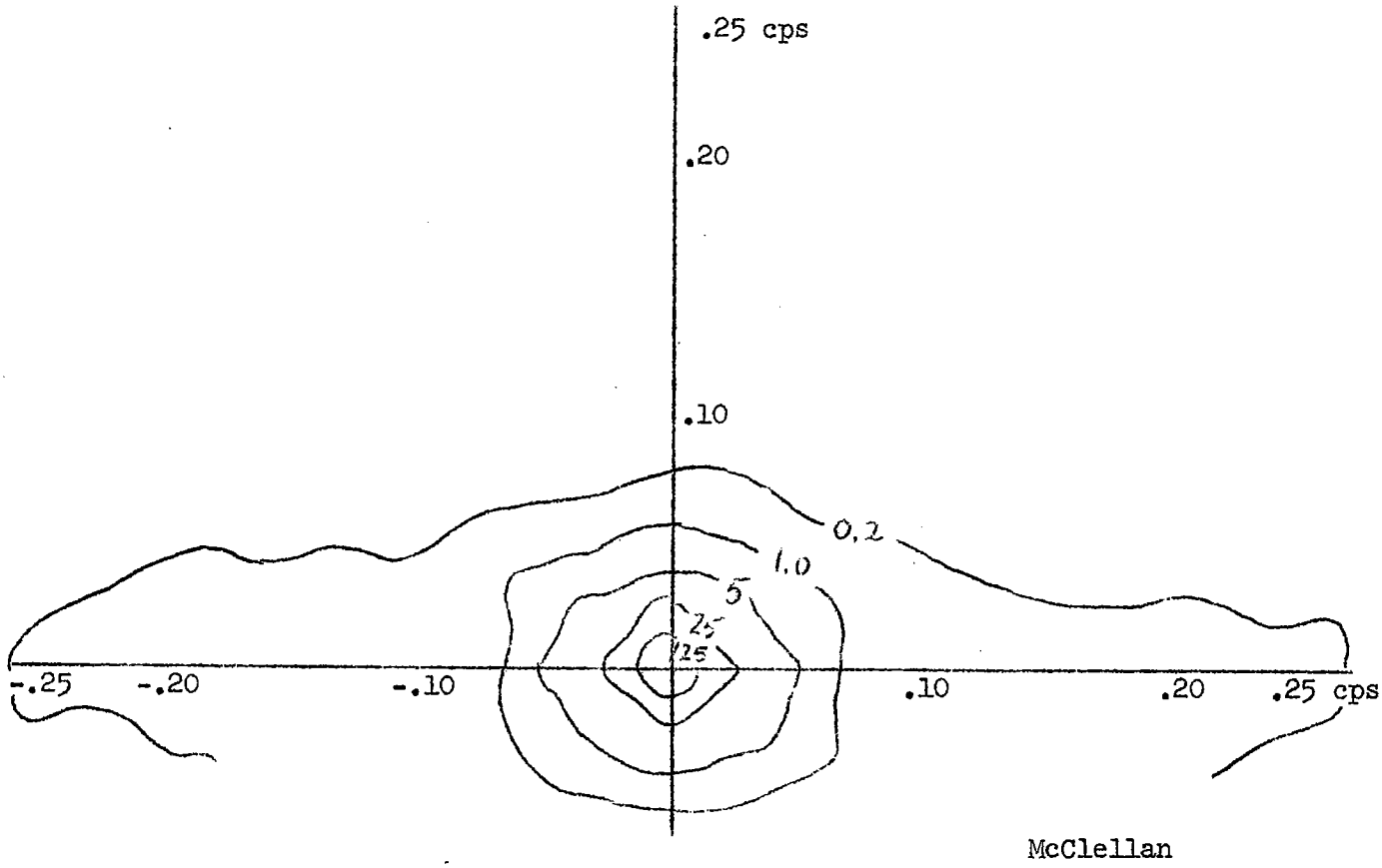
Fig. 1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-.10
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.20
.25 cps
McClellan

Fig. 2

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