On Maximum Values in Certain Applied Stochastic Processes*

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In many stochastic processes, applied in such fields as Queueing, Counter theory, Biology, Traffic flow, etc., the random variable of particular interest is the maximum of the process during a certain time interval. The maximum length of a waiting line is of obvious interest in the design of the waiting room. A left-turn lane, for instance, should be designed so that it rarely exceeds its capacity, lest it interfere with the thru-lanes. In Epidemic theory or counters, again the maximum of the number of infectives or the maximum of the active particles, during a given length of time, are good measures of the virulence of the epidemic or the radiation.

Unfortunately the distribution theory of extreme values is very complicated in all but the simplest cases.

In this paper, we will describe a situation, which occurs frequently and may be put to good use in numerical calculations and in some cases also in theoretical work. In section 1, we formulate the problem. In section 2, we indicate how a general computational method can be developed to get numerical results on extreme values. In section 3, we study the $GI|M|1$ queue and the Type II counter, to indicate how certain simple features of particular problems can be used to get the extreme value distributions more directly.

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I. Formulation of the Problem.

It is known that many processes, arising in applications can be studied in continuous time, in terms of an imbedded semi-Markov process.

As a typical example, we cite the $M|G|1$ queue and its generalizations [ ]. If ξ_n denotes the queue-length after the n-th departure and X_n denotes the length of time between the $(n-1)$st and the n-th departure, then the process $\{(\xi_n, X_n), n \geq 0, X_o = 0\}$ and ξ_o equal to the queuelength at $t = 0+$, is a semi-Markov process. The process $\{\xi(t), t \geq 0\}$ of the queuelength at time t can be studied very simply in terms of this imbedded semi-Markov process. The queuelength $\{\xi(t), t \geq 0\}$ has an additional feature, which enables us to study its maximum over $[o,t]$ directly in terms of the imbedded semi-Markov process. Between successive departure-points the process $\xi(t)$ is non-decreasing, so that the only variables we need to consider in studying the maximum are the $\xi(t_n-)$, where $t_n = X_1 + \ldots + X_n$, $n \geq 1$. But the queuelength before a departure and immediately after a departure are simply related to each other by the service policy. In the $M|G|1$ case $\xi(t_n-) = 1 + \xi(t_n+)$.

Therefore the maximum of $\xi(t)$ can be related very simply to the maximum of a semi-Markov process.

II. The Maximum of a semi-Markov Process.

Let the bivariate process $\{(\xi_n, X_n), n \geq 0\}$ be a semi-Markov process on the non-negative integers, which is regular for all initial distributions. We refer to Pyke [ ] for formal definitions.

Let its transition matrix $Q(.)$ be given by:

(1) $$Q_{ij}(x) = P\{X_n \leq x, \xi_n = j \mid \xi_{n-1} = i\}, \quad i,j \geq 0 .$$

Let $N(t)$ denote the number of transitions in $(o,t]$ and let $J(t) = \xi_{N(t)}$.
We note that:

(2) $$\theta(t) = \max_{o < u \leq t} J(u) = \max \{\xi_1, \ldots, \xi_{N(t)}\}$$

so; setting $\theta(o) = \xi_o$, we get:

(3) $$P\{\theta(t) \leq k,\ J(t) = j \mid \theta(o) = \xi_o = i\}$$

$$= P\{\xi_o \leq k,\ \ldots,\ \xi_{N(t)} \leq k,\ \xi_{N(t)} = j \mid \xi_o = i\}$$

$$= \sum_{n=o}^{\infty} P\{\xi_o \leq k, \ldots, \xi_{N(t)} \leq k,\ \xi_{N(t)} = j,\ N(t) = n \mid \xi_o = i\}$$

and the latter probabilities are of course simple taboo probabilities. To avoid trivialities, we assume $i \leq k$, $j \leq k$.

Let $R_{ij}(n,k;x)$ be the probability that in time x, exactly n transitions occur between states i and j, without a visit to the set $\{k + 1, k + 2, \ldots\}$, then

(4) $$R_{ij}(o,k;x) = \delta_{ij}\,[1 - H_i(x)], \qquad H_i(x) = \sum_{j=o}^{\infty} Q_{ij}(x)$$

and

$$R_{ij}(n,k;x) =$$

$$\sum_{0 \leq i_1, i_2, \ldots, i_{n-1} \leq k} Q_{i\,i_1} * Q_{i_1 i_2} * \ldots Q_{i_{n-1} j} * [1 - H_j]\,(x) .$$

Let ${}_kQ(\cdot)$ denote the matrix obtained from Q by truncating it after the (k+1)st row and column, and let H( ) be a diagonal matrix with diagonal elements $H_o(x), \dots H_k(x)$, then:

(5) $$R_{ij}(n,k;x) = \left[{}_kQ^{(n)} * [I - H](x) \right]_{ij}$$

where the matrix-multiplications are performed, using convolution multiplication. (see Pyke [ ]) .

From (3), we obtain:

(6) $$P\{\theta(t) \leq k, J(t) = j \mid \xi_o = i\} =$$

$$\sum_{n=o}^{\infty} \left[{}_kQ^{(n)} * (I - H)(x)\right]_{ij} =$$

$$\{[I - {}_kQ]^{(-1)} * (I - H)(x)\}_{ij}$$

where the inverse is defined by the previous expression. In some simple cases, we may use (6) to obtain explicit expressions via Laplace-Stieltjes transforms, but in most cases one will have to resort to numerical integration in (4) to obtain approximate values for the probabilities in (6). The amount of computation involved is enormous, since each value of k must be treated separately. In practical problems, where large values of k are highly improbable, it may be worthwhile to set up the computational apparatus required.

In the next section, we will show by two examples, how a more careful study of the imbedded semi-Markov process may lead to simpler recurrence relations.

Baxter [ ] has obtained a decomposition of the transition matrix in the case of Markov chains, which--when known-- leads to simple expressions for the extreme value distributions and it is possible to prove analogous theorems to his for semi-Markov processes, but again these would not lead to simplifications in numerical work.

III. Special treatment in the case of the GI|M|1 queue and a Counter model.

In the GI|M|1 queue, customers arrive according to a renewal process. Let, for simplicity, $t = 0$ be an arrival-epoch and let the distribution of the successive independent inter-arrival times be $F(x)$. The service times are negative exponential variables with parameter μ .

Using very much the same argument, we can also study a type II counter in which particles have independent, identically distributed interarrival times with a distribution $F(x)$ and produce pulses with negative exponentially distributed lengths, of parameter μ .

In the queueing model, the queue-length behaves between successive arrivals like a pure death process with death-rate μ , whereas in the counter process the death-rate is $\mu\, \xi(t)$. This second process is computationally more involved.

It is obvious that the maximum $\alpha(t) = \max_{0 \leq u \leq t} \xi(u)$ is obtained immediately after an arrival epoch.

Let ξ_n be the queuelength immediately after the n-th arrival and let X_n be the time between the (n-1)st and the n-th arrival, then $(\xi_n, X_n, n \geq 0)$, $X_o = 0$ is a semi-Markov process and its transition matrix $Q_{ij}(x)$, $i,j \geq 1$ is given by:

$$Q_{i1}(x) = \int_o^x e^{-\mu u} \sum_{\nu=i}^{\infty} \frac{(\mu u)^{\nu}}{\nu!} \, dF(u), \quad i \geq 1 \tag{7}$$

and

$$Q_{ij}(x) = \int_0^x e^{-\mu u} \frac{(\mu u)^{i-j+1}}{(i-j+1)!} \quad dF(u), \quad j > 1, \ i \geq j-1 .$$

$$Q_{ij}(x) = 0, \quad \text{elsewhere} .$$

In the counter model, the transition matrix $Q^o(\cdot)$ is given by:

(8)
$$Q^o_{i1}(x) = \int_0^x [1-F(u)] \ d\ R^i_o(u), \qquad i \geq 1 .$$

$$Q^o_{ij}(x) = \int_0^x e^{-(j-1)\mu u} \ dF(u) \ \int_0^x e^{+(j-1)\mu v} \ dR^i_{j-1}(v) ,$$

$$j > 1, \ i \geq j-1 .$$

$$Q^o_{ij}(x) = 0, \quad \text{elsewhere}$$

where $R^k_\nu(u)$ is the distribution-function, whose L-S-transform is given by: $(k \geq \nu)$:

(9)
$$\int_0^\infty e^{-sx} \ dR^k_\nu(x) = \prod_{\alpha=0}^{k-\nu-1} \left[\frac{\mu(k-\alpha)}{s+\mu(k-\alpha)} \right] = \frac{B[k+1, \frac{s}{\mu} + \nu + 1]}{B[\nu+1, \frac{s}{\mu} + k + 1]} .$$

This distribution has a density, which is a known, but complicated polynomial in $e^{-\mu x}$.

Let us denote the L.S. transforms of the $Q_{ij}(x)$ - or the $Q^o_{ij}(x)$ - by $\varphi_{ij}(s)$. The remaining discussion depends only on the fact that $\varphi_{ij}(s) = 0$ for $j > 1, \ i < j-1$.

Derivation of the extremum distribution.

Let us assume that $\xi_o = 1$. The first visit to state 2 will occur at some arrival-epoch ξ_2, the first visit to state 3 at some arrival epoch ξ_3, etc.

Let ξ_k be the first entrance time into state k, with $\xi_1 = 0$.

It is obvious that:

$$(10) \qquad P\{\alpha(t) \leq k \mid \xi_o = 1\} = P\{\xi_k \geq t \mid \xi_o = 1\} .$$

Now $\xi_k = Z_1 + Z_2 + \ldots + Z_{k-1}$, where Z_i is the length of time for which $\alpha(t) = i$. The variables Z_k, $k = 1,2,\ldots$ are independent, by the semi-Markov property, so it suffices to derive their distribution, to know the distribution of $\alpha(t)$. Let $g_k(s)$ be the Laplace-Stieltjes transform of the probability distribution of Z_k, $k = 1,\ldots,$ then:

$$(11) \qquad \int_o^\infty e^{-st} P[\alpha(t) \leq k \mid \xi_o = j] \, dt = s^{-1} \left\{1 = \prod_{\nu=j}^{k} g_\nu(s)\right\}, \; j \leq k .$$

We will now prove a simple recursion relation between the functions $g_k(s)$, $k \geq 1$.

<u>Theorem</u>:

$$(12) \qquad g_1(s) = f(s + \mu) \, [1-f(s) + f(s+\mu)]^{-1}$$

and for $k > 1$

$$(13) \qquad g_k(s) = \varphi_{k,k+1}(s) \left\{1 - \varphi_{kk}(s) - \sum_{j=1}^{k-1} \varphi_{kj}(s) \, g_j(s) \ldots g_{k-1}(s)\right\}^{-1} .$$

<u>Proof</u>:

We first prove (13):

Consider the possible ways of going from state k to state $k + 1$ for the first time in less than a time x.

We can either perform a number of transitions from state k to state k and then form k to $k + 1$, without going below k or we can stay in state k for a number of transitions and then make a transition to some state j, $1 \leq j < k$. In order to reach state $k + 1$, we must then again go from j to $j + 1$, from $j + 1$ to $j + 2$, etc. to state k. Once we reach state k again, we are back in the same situation as initially. If we express all probabilities associated with these possible paths, we obtain in terms of L.S. transforms:

(14) $$g_k(s) =$$

$$\frac{\varphi_{k,k+1}(s)}{1-\varphi_{kk}(s)} + \frac{1}{1-\varphi_{kk}(s)} \sum_{j=1}^{k-1} \varphi_{kj}(s)\, g_j(s) \ldots g_{k-1}(s)\, g_k(s)$$

which is equivalent to (13) .

If $k = 1$, then we first make a number of transitions from state 1 to itself, followed by a transition from 1 to 2 . It follows that:

(15) $$g_1(s) = \varphi_{12}(s)\, [1 - \varphi_{11}(s)]^{-1}$$

but

$$\varphi_{12}(s) = f(s + \mu)$$

$$\varphi_{11}(s) = \int_0^\infty e^{-sx} \sum_{\nu=1}^{\infty} e^{-\mu x} \frac{(\mu x)^\nu}{\nu!}\, dF(x) = f(s) - f(s + \mu) \quad .$$

If we set:

(16) $$A_o = 1, \quad A_k(s) = \frac{1}{g_1(s) \ldots g_k(s)}, \quad k \geq 1$$

then

$$(17) \qquad A_k(s) = \frac{1}{f(\mu + s)} \sum_{j=1}^{k} (\varphi_{kj} - \varphi_{kj}(s)) \; A_{j-1}(s)$$

is an alternative version of (14)

In some applications one is interested in the maxima of the process $\xi(t)$, between successive visits to the state 0. We will hence also consider the probability that the maximum changes from k to $k+1$ without an intermediate visit to 0.

Let $G_k(x)$ be the probability that the process $\xi(t)$ goes from state k to state $K + 1$, without an intermediate visit to 0 or $k + 1$, in a length of time $\leq x$. Let $g_k^*(s)$ be the L.S. transform of $G_k(\cdot)$.

The same argument as before leads to the recurrence relation:

$$(18) \qquad g_1^*(s) = f(s + \mu)$$

$$g_2^*(s) = \frac{f^2(s + \mu)}{1-f(s) + f(s+\mu)}$$

and

$$g_k^*(s) = (1-\varphi_{kk}(s))^{-1} \varphi_{k,k+1}(s) + \sum_{j=2}^{k-1} (1-\varphi_{kk}(s))^{-1} \varphi_{kj}(s) \; g_j^* \; g_{j+1}^* \cdots g_k^*(s)$$

for $k \geq 3$.

Let us set:

(19)
$$B_1(s) = 1$$
$$B_k(s) = (g_2^* \ldots g_k^*)^{-1}$$

then the latter formula leads to a linear recursion formula for the $B_k(s)$.

In the case of the $GI|M|1$ queue, we can easily obtain a generating function for the $B_k(s)$ as follows.

Formulae (18) and (19) lead to

(20)
$$B_1(s) = 1$$
$$\theta_o(s)\, B_k(s) = B_{k-1}(s) - \sum_{j=1}^{k-1} \theta_{k-j}(s)\, B_j(s), \quad k \geq 2 .$$

in which

(21)
$$\theta_\nu(s) = \int_o^\infty e^{-(\mu+s)x} \frac{(\mu x)^\nu}{\nu!}\, dF(x), \quad \nu \geq 0 .$$

set:

(22)
$$W(z,s) = \sum_{k=1}^{\infty} B_k(s) z^{k-1} , \quad \text{Re } s \geq 0, \quad |z| \leq 1$$

then (20) and (21) lead to:

(23)
$$W(z,s) = \frac{f(s+\mu)}{1 - (1+z)\, f(s+\mu) + f(s+\mu-\mu z)} ,$$

where $f(\cdot)$ as before denotes the L.S. transform of $F(\cdot)$.

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13. ABSTRACT

We consider the problem of finding the maximum observation in $[0,t]$ for a class of processes having an imbedded semi-Markov process. In a particular subclass which includes the $GI|M|1$ queue and some type II counters, we get more explicit results.

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