On the non-central distributions of the largest roots of two matrices and some percentage points

by

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1. Introduction and Summary.

Tests of three multivariate hypotheses based on Roy's largest root criterion (Roy, 1945) have been shown to have monotonicity property of power with respect to individual population roots, namely, (i) that of equality of covariance matrices of two p-variate normal populations, (ii) that of equality of p-dimensional mean vectors of ℓ p-variate normal populations having a common covariance matrix, and (iii) that of independence between a p-set and a q-set of variates in a (p + q) - variate normal population (Anderson and DasGupta, 1964a, 1964b, DasGupta, Anderson and Mudholkar, 1964, Roy and Mikhail, 1961). Pillai (1965a) has studied the cdf of the larger of two roots in the linear case, i.e., under a single non-zero population root, in connection with test (iii) and has extended the study to the case of the cdf of the larger of two roots when both population roots are non-zero and that of the largest of three roots in the linear case (1965c). In addition, similar non-central cdf's were obtained for test (ii) for the case of two roots with two non-zero population roots, and three roots in the linear case. Tabulations of the power functions have also been made for both tests for various values of the parameters (1965c). In this paper, approximations to the cdf at the upper end are suggested in the non-central case for the exact cdf obtained previously (1965c) and also an approximation to the non-central cdf of the largest of four roots

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in the linear case.

Further, Pillai's general expression (1965b) approximating at the upper end, the cdf of the largest root under the null hypotheses, has been used to compute the upper 5 and 1 per cent points of the largest of s roots, for values of s ranging from 13 to 20. Such percentage points for $s = 11$ and 12 are available in Pillai (1965c), $s = 8$ to 10 in Pillai (1965b); $s = 7$ in Pillai (1964a) and $s = 2$ to 6 in Pillai (1960).

2. Non-central cdf of the largest root for test (iii) in the linear case.

When there is only a single non-zero population root, ρ^2, the joint density function of the characteristic roots, $r_1^2 \leq r_2^2 \leq \ldots \leq r_p^2$, for test (iii), can be expressed in the form (Pillai, 1965c)

$$(2.1) \qquad K(p) \sum_{i=0}^{\infty} A_i^{(p)} Z_i(R^2) \quad ,$$

where $Z_i(R^2)$ are ith degree zonal polynomials defined in James (1964), R^2 being a diagonal matrix with diagonal elements $r_1^2, \ldots, r_p^2$, $K(p) = (1-\rho^2)^{\frac{1}{2}\nu} C(p,m,n)$ $m = (f_2-p-1)/2$, $n = (f_1-p-1)/2$, $\nu = f_1+f_2$, where f_1 and f_2 are the degrees of freedom,

$$(2.2) \qquad C(p,m,n) = \frac{\pi^{\frac{1}{2}p} \prod_{i=1}^{p} \Gamma\frac{1}{2}(2m+2n+p+i+2)}{\prod_{i=1}^{p} \Gamma\frac{1}{2}(2m+i+1)\Gamma\frac{1}{2}(2n+i+1)\Gamma\frac{1}{2}i} \quad ,$$

and

$$(2.3) \qquad A_i^{(p)} = \frac{[\nu(\nu+2)\ldots(\nu+2(i-1))]^2 \rho^{2i}}{2^i f_2(f_2+2)\ldots(f_2+2(i-1))\, p(p+2)\ldots(p+2(i-1))i!} \quad .$$

Now define

(2.4) $$V(x;q_p,\ldots,q_1) = \begin{vmatrix} \int_0^x \theta_p^{m+q_p}(1-\theta_p)^n d\theta_p & \cdots & \int_0^x \theta_p^{m+q_1}(1-\theta_p)^n d\theta_p \\ \cdot & \cdot & \cdot \\ \int_0^{\theta_2} \theta_1^{m+q_p}(1-\theta_1)^n d\theta_1 & \cdots & \int_0^{\theta_2} \theta_1^{m+q_1}(1-\theta_1)^n d\theta_1 \end{vmatrix} .$$

Then the non-central cdf of the largest root, r_p^2, can be written in the form

(2.5) $$\Pr(r_p^2 \leq x) = K(p) \sum_{i=0}^{\infty} A_i^{(p)} F_i$$

where

(2.6) $$F_0 = V(x;\ s-1,\ \ldots,\ 1,\ 0)\ ,$$

(2.7) $$F_1 = V(x;\ s,\ s-2,\ \ldots\ 1,0)\ ,$$

(2.8) $$F_2 = 3V(x;\ s+1,s-2,\ldots,\ 1,0) - V(x;\ s,s-1,s-3,\ldots,1,0)\ ,$$

(2.9) $$F_3 = 15V(x;\ s+2,s-2,\ldots,1,0) - 6V(x;\ s+1,s-1,s-3,\ldots,1,0) + 3V(x;\ s,s-1,s-2,s-4,\ldots,1,0)\ ,$$

(2.10) $$F_4 = 105V(x;\ s+3,s-2,\ldots,1,0) - 45V(x;\ s+2,s-1,s-3,\ldots,1,0) - 6V(x;\ s+1,s,s-3,\ldots,1,0) + 27V(x;\ s+1,s-1,s-2,s-4,\ldots,1,0) - 15V(x;s,s-1,s-2,s-3,s-5,\ldots,1,0)\ ,$$

(2.11) $$F_5 = 945V(x;s+4,s-2,\ldots,1,0) - 420V(x;s+3,s-1,s-3,\ldots,1,0) - 75V(x;s+2,s,s-3,\ldots,1,0) + 270V(x;s+2,s-1,s-2,s-4,\ldots,1,0) + 45V(x;s+1,s,s-2,s-4,\ldots,1,0) - 180V(x;s+1,s-1,s-2,s-3,s-5,\ldots,1,0) + 105V(x;s,s-1,s-2,s-3,s-4,s-6,\ldots,\ 1,0)\ ,$$

$$(2.12) \quad F_6 = 10395V(x;s+5,s-2,\ldots,1,0) - 4725V(x;s+4,s-1,s-3,\ldots,1,0)$$
$$- 945V(x;s+3,s,s-3,\ldots,1,0) + 3150V(x;s+3,s-1,s-2,s-4,\ldots,1,0)$$
$$- 225V(x;s+2,s+1,s-3,\ldots,1,0) + 720V(x;s+2,s,s-2,s-4,\ldots,1,0)$$
$$- 2250V(x;s+2,s-1,s-2,s-3,s-5,\ldots,1,0) - 45V(x;s+1,s,s-1,s-4,\ldots,1,0)$$
$$-405V(x;s+1,s,s-2,s-3,s-5,\ldots,1,0)+1515V(x;s+1,s-1,s-2,s-3,s-4,s-6,\ldots,1,0)$$
$$-945V(x;s,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0) \quad .$$

(2.6) - (2.12) have been obtained by using the values of $Z_i(R^2)$ (James (1964) and Pillai's lemma (1964b) for multiplying a Vandermode type determinant by powers of elementary symmetric **functions**.

3. Approximation to the non-central cdf of r_2^2, r_3^2 and r_4^2 in the linear case.

The exact cdf of r_3^2 has been obtained in the linear case using zonal polynomials up to the sixth degree (Pillai 1965c). However, it involves too many terms and the terms themselves contain in many cases products of two incomplete beta functions. Thus the computations become extremely heavy and the complexity of the cdf increases many folds as p increases. Thus, an approximation to the non-central cdf at the upper tail end, is attempted for small values of ρ^2, and $p = 2,3$ and 4.

The method of approximation employed is that of Pillai (1954, 1956a, 1965b) neglecting terms involving $(1-x)^{2n+1}$ and higher powers of $1-x$ in the development of the cdf obtained by repeated application of Pillai's reduction formula (1954, 1956, 1965b). Using F_0, F_1 and F_2, the approximation to the cdf of r_3^2 is as follows: --

(3.1) $$\Pr(r_3^2 \le x) = B(x;m,n) + K(3)\Big\{\frac{A_2^{(3)}}{m+n+4}\Big[I_0(x;m+3)V(1;2,0) - \frac{2I_0(x;m+2)}{m+m+3}\, I(2m+3)\Big] - \Big[\frac{3A_2^{(3)}I_0(x;m+4)}{m+n+5} + \frac{B_1^{(3)}I_0(x;m+3)}{m+n+4} + \frac{B_0^{(3)}I_0(x;m+2)}{(m+n+3)}\Big] V(1;1,0) + \Big[\frac{6A_2^{(3)}I(2m+4)}{m+n+5} + \frac{2B_1^{(3)}I(2m+3)}{m+n+4} + \frac{2B_0^{(3)}I(2m+2)}{m+n+3} - \frac{2A_2^{(3)}(m+2)I(2m+3)}{(m+n+4)(m+n+3)}\Big] \frac{I_0(x;m+1)}{(m+n+2)}\Big\} ,$$

where

$$B(x;m,n) = \int_0^x y^m(1-y)^n dy \;/\; \beta(m+1,n+1) ,$$

$$I_0(x;q) = x^q(1-x)^{n+1} ,$$

$$I(jm+k) = \beta(jm+k+1,\; jn+j) ,$$

$B_1^{(3)} = \{3A_2^{(3)}(m+4)/(m+n+5)\}+A_1^{(3)}$, and $B_0^{(3)} = \{B_1^{(3)}(m+3) \;/\; (m+n+4)\} + 1$.

(3.2) $$\Pr(r_4^2 \le x) =$$

$$1 + K(4)\Big\{\frac{2A_2^{(4)}(m+3)I_0(x;m+2)}{(m+n+5)(m+n+4)(m+n+3)} [I(2m+4)I(m+1) - I(2m+5)I(m)] + \frac{2I(m)I_0(x;m+1)}{(m+n+2)}\Big[\frac{3A_2^{(4)}}{(m+n+6)} \Big\{\frac{(m+2)I(2m+6)}{(m+n+3)} - I(2m+7)\Big\} + \frac{B_1^{(4)}}{(m+n+5)}\Big\{\frac{(m+2)I(2m+5)}{(m+n+3)} - I(2m+6)\Big\} + \frac{B_0^{(4)}}{(m+n+4)} \Big\{\frac{(m+2)I(2m+4)}{(m+n+3)} - I(2m+5)\Big\} + \frac{A_2^{(4)}}{(m+n+5)} \Big\{I(2m+7) - \frac{(m+2)(m+3)I(2m+5)}{(m+n+4)(m+n+3)}\Big\}\Big] - V(1;2,1,0)\Big[\frac{3A_2^{(4)}I_0(x;m+5)}{(m+n+6)} + \frac{B_1^{(4)}I_0(x;m+4)}{(m+n+5)} + \frac{B_0^{(4)}I_0(x;m+3)}{(m+n+4)} + \frac{B_0^{(4)}(m+3)I_0(x;m+2)}{(m+n+4)(m+n+3)} \Big]$$

$$+V(1;3,1,0)\left[\left\{B_0^{(4)} - \frac{B_1^{(4)}(m+4)}{(m+n+5)}\right\} \frac{I_0(x;m+2)}{(m+n+3)} + \frac{A_2^{(4)}}{(m+n+5)} \left\{ I_0(x;m+4) + \frac{(m+4)I_0(x;m+3)}{(m+n+4)}\right\}\right] + V(1;4,1,0)\left[\left\{B_1^{(4)} - \frac{3A_2^{(4)}(m+5)}{(m+n+6)}\right\} \frac{I_0(x;m+2)}{(m+n+3)} - \frac{A_2^{(4)}I_0(x;m+3)}{(m+n+4)}\right] + \frac{3A_2^{(4)}V(1;5,1,0)I_0(x;m+2)}{(m+n+3)} \Bigg\} ,$$

where

$$B_1^{(4)} = \{3A_2^{(4)}(m+5)/(m+n+6)\} + A_1^{(4)}, \quad \text{and} \quad B_0^{(4)} = \{B_1^{(4)}(m+4) \;/\; (m+n+5)\} + 1$$

The values of the V determinants in (3.1) and (3.2) can be deduced from the general results available in Pillai and Mijares (1959). In contrast to the approximations to the cdf's for $p = 3$ and 4 given above, that for $p = 2$ is simple:

$$\text{(3.3)} \qquad \Pr(r_2^2 \leq x) = 1 + K(2) \left\{ \frac{A_2^{(2)}I_0(x;m+2)I(m+1)}{(m+n+3)} - \left[\frac{3A_2^{(2)}I_0(x;m+3)}{(m+n+4)} + \frac{B_1^{(2)}I_0(x;m+2)}{(m+n+3)} + \frac{B_0^{(2)}I_0(x;m+1)}{(m+n+2)} \right] I(m) \right\}$$

where

$$B_1^{(2)} = \{3A_2^{(2)}(m+3) \;/\; (m+n+4)\} + A_1^{(2)}, \quad B_0^{(2)} = \{B_1^{(2)}(m+2) \;/\; (m+n+3)\} + 1 .$$

4. Non-central cdf of the largest root for test (ii) in the linear case and an approximation.

When there is only one non-zero population root, ω, then the joint density function of the characteristic roots, $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_p$, of the equation $|L - \ell I| = 0$, for test (ii), is given by (Pillai, 1965c) .

$$(4.1) \qquad K'(p) \sum_{i=0}^{\infty} A_i'^{(p)} Z_i(L)$$

where

$$K'(p) = e^{-\frac{1}{2}\omega} C(p,m,n)$$

and

$$A_i'^{(p)} = \frac{[\nu(\nu+2)\ldots(\nu+2(i-1))]}{f_2(f_2+2)\ldots(f_2+2(i-1))} \frac{(\frac{1}{2}\omega)^i}{p(p+2)\ldots(p+2(i-1))i!} .$$

Then the cdf of the largest root ℓ_3 is obtained from (3.1) by the following changes:

$$r_3^2 \longrightarrow \ell_3$$

$$K(3) \longrightarrow K'(3)$$

$$\text{and} \qquad A_i^{(3)} \longrightarrow A_i'^{(3)} .$$

In a similar manner the cdf of ℓ_4 can be obtained from (3.2), and ℓ_2 from (3.3).

5. Errors of approximation.

Error of approximation of the non-central cdf for test (ii) and that for test (iii) may be considered separately. Test (iii) is discussed first.

Test (iii). Some numerical comparison may be made from Table 1 between the exact and approximate powers for $p = 3$ for test of the hypothesis $\rho = 0$ against different simple alternative hypotheses. The exact cdf and powers are available in Pillai (1965c) and the approximate powers are computed using (3.1). Table 1 shows that the approximate powers are generally closer to the exact (using zonal polynomials up to the sixth degree) than the exact (using only those up to the second degree) although the approximation was derived from the latter.

Table 1. Powers (exact and approximate) of r_3^2 test for testing $\rho = 0$ against different simple alternative hypotheses, $\alpha = .05$

	n = 15			n = 40		
ρ	Approx.	Exact (2nd degree polynomial)	Exact (sixth degree polynomial)	Approx.	Exact (2nd degree polynomial)	Exact (sixth degree polynomial)
			m = 0			
.01	.050070	.050070	.050070	.050184	.050185	.050184
.05	.05176	.05179	.05178	.05471	.05494	.05478
.10	.05716	.05836	.05746	.06826	.07931	.07126
			m = 1			
.01	.050053	.050054	.050054	.050141	.050142	.050142
.05	.05135	.05139	.05137	.05360	.05385	.05366
.10	.05544	.05683	.05572	.06371	.07543	.06622
			m = 2			
.01	.050044	.050045	.050045	.050117	.050118	.050118
.05	.05112	.05117	.05115	.05299	.05325	.05305
.10	.05449	.05608	.05478	.06117	.07359	.06343
			m = 5			
.01	.050031	.050033	.050033	.050083	.050084	.050084
.05	.05079	.05087	.05083	.05210	.05243	.05217
.10	.05311	.05542	.05347	.05747	.07210	.05943

Table 2. Powers (exact and approximate) of r_2^2 test for testing $\rho = 0$ against different simple alternative hypotheses, $\alpha = .05$

ρ	n = 15 Approximate	n = 15 Exact (sixth degree poly.)	n = 40 Approximate	n = 40 Exact (sixth degree poly.)
		m = 0		
.01	.050112	.050112	.050294	.050294
.05	.05283	.05284	.05754	.05763
.10	.06157	.06192	.07964	.08377
		m = 1		
.01	.050081	.050081	.050212	.050212
.05	.05205	.05206	.05543	.05550
.10	.05832	.05862	.07106	.07436
		m = 2		
.01	.050066	.050066	.050172	.050172
.05	.05167	.05168	.05440	.05446
.10	.05675	.05703	.06682	.06968
		m = 5		
.01	.050047	.050047	.050120	.050120
.05	.05119	.05119	.05303	.05308
.10	.05470	.05497	.06112	.06346

It may be seen that the approximation which has been obtained by using zonal polynomials only up to the second degree maintains its accuracy equally well for $p = 3$ as for $p = 2$. Hence, the same order of error of approximation is expected for the approximate powers computed for $p = 4$ and presented in Table 3.

Table 3. Approximate powers of r_4^2 test (using (3.2)) for testing $\rho = 0$ against different simple alternative hypotheses, $\alpha = .05$

ρ	m = 0, n = 15	m = 0, n = 40	m = 1, n = 15	m = 1, n = 40	m = 2, n = 15	m = 2, n = 40	m = 5, n = 15	m = 5, n = 40
.01	.050049	.050130	.050040	.050104	.050034	.050089	.050025	.050065
.05	.05125	.05332	.05100	.05265	.05086	.05225	.05064	.05164
.10	.0550	.0626	.0540	.0598	.0534	.0582	.0524	.0555

Test (ii). Again, numerical comparison may be made from Table 4 between the exact powers (Pillai, 1965c) and approximate powers (see Section 4) for $p = 3$ for test of the hypothesis $\omega = 0$ against different simple alternative hypotheses.

Table 4. Powers (exact and approximate) of ℓ_3 test for testing $\omega = 0$ against different simple alternative hypotheses $\alpha = .05$

ω	n = 15 Approx.	n = 15 Exact (2nd degree polynomial)	n = 15 Exact (sixth degree polynomial)	n = 40 Approx.	n = 40 Exact (2nd degree polynomial)	n = 40 Exact (sixth degree polynomial)
			m = 0			
.01	.050135	.050184	.050184	.050209	.050210	.050210
..05	.05068	.05093	.05093	.05105	.05106	.05105
.10	.05136	.05188	.05187	.05212	.05214	.05213
.50	.0567	.0615	.0598	.0607	.0629	.0613
1.00	.0675	.0819	.0709	.0703	.0847	.0742
			m = 1			
.01	.050133	.050135	.050135	.050157	.050157	.050157
.05	.05067	.05068	.05068	.05079	.05079	.05079
.10	.05135	.05138	.05136	.05158	.05161	.05159
.50	.0567	.0589	.0571	.0580	.0601	.0584
1.00	.0624	.0769	.0651	.0649	.0793	.0680
			m = 2			
.01	.050106	.050108	.050108	.050128	.050128	.050128
.05	.05053	.05054	.05054	.05064	.05065	.05065
.10	.05107	.05110	.05109	.05129	.05132	.05130
.50	.0553	.0575	.0557	.0564	.0586	.0568
1.00	.0597	.0742	.0619	.0619	.0764	.0645
			m = 5			
.01	.050066	.050069	.050069	.050085	.050086	.050086
.05	.05033	.05035	.05034	.05043	.05043	.05043
.10	.05066	.05071	.05069	.05085	.05089	.05087
.50	.0532	.0555	.0536	.0542	.0564	.0545
1.00	.0557	.0703	.0574	.0576	.0721	.0595

Table 5. Powers (exact and approximate) of ℓ_2 test for testing $\omega = 0$ against different simple alternative hypotheses $\alpha = .05$

ω	$n = 15$ Approximate	$n = 15$ Exact (sixth degree polynomial)	$n = 40$ Approximate	$n = 40$ Exact (sixth degree polynomial)
		$m = 0$		
.01	.050311	.050311	.050342	.050342
.05	.051563	.051565	.051720	.051722
.10	.053144	.053153	.053462	.053473
.50	.065887	.066657	.067587	.068436
1.00	.080273	.085408	.083689	.089434
		$m = 1$		
.01	.050213	.050213	.050241	.050241
.05	.051071	.051073	.051212	.051214
.10	.052153	.052160	.052438	.052446
.50	.060813	.061368	.062327	.062965
1.00	.070386	.074137	.073417	.077734
		$m = 2$		
.01	.050166	.050166	.050191	.050192
.05	.050831	.050832	.050962	.050963
.10	.051669	.051674	.051934	.051940
.50	.058326	.058773	.059721	.060243
1.00	.065530	.068546	.068307	.071840
		$m = 3$		
.01	.050102	.050102	.050125	.050125
.05	.050513	.050514	.050626	.050627
.10	.051029	.051033	.051256	.051261
.50	.055052	.055354	.056227	.056589
1.00	.059153	.061185	.061463	.063901

Table 6. Approximate powers of ℓ_4 test for testing $\omega = 0$ against different simple alternative hypotheses, $\alpha = .05$

	m = 0		m = 1		m = 2		m = 5	
ω	n = 15	n = 40	n = 15	n = 40	n = 15	n = 40	n = 15	n = 40
.01	.050124	.050145	.050094	.050113	.050077	.050095	.050051	.050065
.05	.05062	.05073	.05047	.05057	.05039	.05047	.05025	.05033
.10	.05125	.05146	.05095	.05114	.05078	.05095	.05051	.05065
.50	.0562	.0573	.0547	.0557	.0538	.0547	.0524	.0532
1.00	.0614	.0637	.0585	.0605	.0567	.0585	.0541	.0556

For test of hypothesis (ii) also, the approximation using zonal polynomials only up to the second degree maintains the same accuracy for $p = 3$ as for $p = 2$. Hence the approximate powers for $p = 4$ given in Table 6 are also expected to have about the same accuracy.

6. The cdf of the largest root under the three null hypotheses.

It has been shown (Pillai 1954, 1956a,b) that when each of the three null hypotheses is true, the cdf of the largest of s non-null characteristic roots of a matrix is given by

$$C(s,m,n)V(x;s-1,\ldots,0) . \tag{6.1}$$

A general expression approximating (6.1) for computing the upper percentage points has been obtained by Pillai(1965b) for odd values of s and another for even values of s. Using these expressions, upper 5 and 1 per cent points were computed for values of $m = 0(1)\ 5,7,10,15$ and $n = 5(5)\ 30,40,60,80,100,130,160,200,300,500,1000$ and $s = 8,9,$ and $10,$ and were presented in (1965b). Further, such tables have also been prepared for $s = 11$ and 12 (Pillai 1965c). In this paper, tables of upper 5 and 1 per cent points are presented for the same arguments of m and n as above and for $s = 13(1)\ 20$.

7. Some remarks.

All the values given in Tables 7-22 are believed to be accurate within a unit in the last decimal quoted. A detailed examination of the error of approximation for earlier studies has been made by Pillai and Bantegui (1959). In the more recent study for $s = 7$ (Pillai, 1964a) the error of approximation has been further dealt with but in a different manner than in the previous studies, for example, by comparing the frequency of differences between the trial values fed into the IBM 7094 and the final values. Since percentage points were obtained for consecutive values of s, and points for each value of s were computed only after completion of tables of those for the previous value of s, it was possible to extrapolate trial values with reasonable precision. Thus, by comparison, the trial values in general were even closer to the final values in the present computations than before and improved techniques of programming enabled to avoid the usual extrapolations made in the bottom right corner of the previous tables.

A brief account of the tabulations of upper percentage points of the largest root is given earlier (Pillai, 1965b). In addition to the work described in the last paragraph of section 1 above, it may be mentioned that Foster and Rees (1957) have tabulated the upper percentage points (80,85,90,95 and 99) of the largest root for $s = 2$, $m = -0.5,0(1)9$ and $n = 1(1)19(5)49, 59,79$. Foster (1957 ,1958) has further extended these tables for values of $s = 3$ and 4. The argument they have used for tabulation are $\nu_1 = 2n+s+1 = f_1$, and $\nu_2 = 2m+s+1 = f_2$. Heck (1960) has given some charts of upper 5, 2.5 and 1% points for $s = 2(1)5$, $m = -\frac{1}{2},0(1)10$ and $n \geq 5$.

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Table 7. Upper 5% points of the largest root for $s = 13$

n \ m	0	1	2	3	4	5	7	10	15
5	.93352	.93989	.94512	.94951	.95324	.95645	.96171	.96756	.97414
10	.83337	.84627	.85725	.86671	.87496	.88222	.89444	.90859	.92521
15	.74141	.75821	.77282	.78566	.79705	.80724	.82473	.84557	.87095
20	.66370	.68251	.69912	.71393	.72724	.73929	.76029	.78588	.81794
25	.59898	.61867	.63626	.65212	.66651	.67965	.70283	.73160	.76846
30	.54490	.56481	.58276	.59907	.61400	.62772	.65217	.68295	.72315
40	.46049	.47986	.49756	.51385	.52892	.54292	.56825	.60084	.64469
60	.35037	.36755	.38349	.39837	.41234	.42550	.44973	.48179	.52665
80	.28232	.29734	.31140	.32464	.33717	.34906	.37122	.40104	.44379
100	.23628	.24949	.26194	.27373	.28495	.29566	.31575	.34311	.38298
130	.18975	.20089	.21144	.22149	.23110	.24032	.25775	.28177	.31737
160	.15850	.16809	.17721	.18593	.19430	.20236	.21767	.23893	.27080
200	.12994	.13802	.14573	.15312	.16024	.16712	.18024	.19859	.22640
300	.089569	.095344	.10088	.10622	.11138	.11639	.12600	.13958	.16049
500	.055231	.058900	.062432	.065849	.069166	.072396	.078630	.087516	.094040
1000	.028198	.030114	.031965	.033760	.035509	.037216	.040524	.045272	.052751

Table 8. Upper 1% points of the largest root for s = 13

n \ m	0	1	2	3	4	5	7	10	15
5	.95306	.95759	.96131	.96442	.96707	.96934	.97306	.97720	.98184
10	.86553	.87608	.88504	.89274	.89946	.90536	.91527	.92672	.94014
15	.77918	.79377	.80643	.81754	.82738	.83617	.85123	.86913	.89085
20	.70344	.72035	.73526	.74853	.76044	.77120	.78991	.81266	.84104
25	.63886	.65698	.67314	.68768	.70085	.71286	.73401	.76017	.79356
30	.58402	.60265	.61941	.63462	.64851	.66127	.68394	.71240	.74943
40	.49703	.51557	.53247	.54799	.56233	.57565	.59967	.63050	.67180
60	.38135	.39818	.41377	.42832	.44194	.45476	.47832	.50942	.55276
80	.30875	.32365	.33759	.35070	.36309	.37484	.39668	.42601	.46789
100	.25919	.27242	.28486	.29663	.30782	.31848	.33846	.36560	.40502
130	.20878	.22001	.23064	.24075	.25042	.25968	.27715	.30118	.33669
160	.17474	.18446	.19370	.20252	.21097	.21911	.23454	.25592	.28790
200	.14351	.15173	.15957	.16709	.17431	.18129	.19458	.21313	.24117
300	.099162	.10508	.11074	.11620	.12147	.12658	.13638	.15021	.17143
500	.061270	.065045	.068678	.072189	.075595	.078910	.085300	.094394	.099251
1000	.031331	.033310	.035220	.037072	.038874	.040633	.044039	.048919	.056593

Table 9. Upper 5% points of the largest root for s = 14

n \ m	0	1	2	3	4	5	7	10	15
5	.94023	.94569	.95022	.95406	.95733	.96017	.96483	.97008	.97603
10	.84691	.85832	.86809	.87656	.88398	.89054	.90162	.91454	.92982
15	.75899	.77416	.78741	.79911	.80953	.81888	.83498	.85427	.87788
20	.68335	.70058	.71586	.72954	.74186	.75304	.77259	.79652	.82661
25	.61952	.63776	.65411	.66890	.68235	.69466	.71643	.74353	.77839
30	.56565	.58425	.60108	.61641	.63046	.64341	.66653	.69572	.73396
40	.48065	.49898	.51578	.53126	.54561	.55897	.58317	.61438	.65646
60	.36823	.38472	.40006	.41442	.42790	.44062	.46408	.49517	.53871
80	.29792	.31246	.32611	.33898	.35118	.36277	.38439	.41352	.45532
100	.25000	.26287	.27502	.28655	.29753	.30802	.32772	.35458	.39376
130	.20131	.21221	.22257	.23245	.24191	.25099	.26817	.29187	.32703
160	.16845	.17788	.18686	.19547	.20373	.21170	.22684	.24789	.27947
200	.13832	.14629	.15391	.16122	.16828	.17510	.18812	.20635	.23399
300	.095557	.10128	.10678	.11208	.11122	.12221	.13179	.14534	.16621
500	.059034	.062683	.066203	.069613	.072927	.076156	.082393	.091292	.10517
1000	.030184	.032095	.033945	.035742	.037494	.039205	.042525	.047295	.054812

Table 10. Upper 1% points of the largest root for $s = 14$

n \ m	0	1	2	3	4	5	7	10	15
5	.95784	.96171	.96493	.96964	.96996	.97197	.97527	.97897	.98317
10	.87660	.88590	.89386	.90075	.90678	.91210	.92107	.93152	.94385
15	.79444	.80757	.81903	.82914	.83812	.84617	.86001	.87655	.89674
20	.72109	.73655	.75023	.76245	.77346	.78342	.80082	.82205	.84866
25	.65775	.67449	.68948	.70300	.71529	.72651	.74634	.77095	.80248
30	.60342	.62078	.63645	.65071	.66376	.67577	.69718	.72412	.75929
40	.51631	.53380	.54979	.56452	.57815	.59082	.61373	.64320	.68278
60	.39883	.41495	.42993	.44392	.45705	.46941	.49218	.52229	.56430
80	.32422	.33862	.35212	.36484	.37687	.38830	.40957	.43817	.47908
100	.27291	.28576	.29788	.30937	.32030	.33073	.35028	.37689	.41558
130	.22042	.23140	.24181	.25173	.26122	.27033	.28753	.31121	.34623
160	.18481	.19435	.20343	.21212	.22045	.22848	.24372	.26486	.29651
200	.15203	.16012	.16786	.17528	.18243	.18933	.20250	.22090	.24874
300	.10529	.11114	.11675	.12217	.12741	.13249	.14224	.15602	.17718
500	.065179	.068928	.072541	.076040	.079437	.082746	.089131	.098227	.11239
1000	.033379	.035350	.037256	.039107	.040910	.042671	.046084	.050981	.058685

Table 11. Upper 5% points of the largest root for s = 15

n \ m	0	1	2	3	4	5	7	10	15
5	.94597	.95069	.95465	.95801	.96090	.96341	.96758	.97230	.97771
10	.85885	.86900	.87774	.88535	.89205	.89799	.90807	.91990	.93399
15	.77482	.78856	.80062	.81131	.82086	.82946	.84432	.86220	.88421
20	.70130	.71713	.73122	.74387	.75530	.76570	.78393	.80632	.83461
25	.63850	.65542	.67065	.68446	.69705	.70860	.72906	.75463	.78762
30	.58499	.60239	.61818	.63261	.64586	.65809	.67997	.70767	.74407
40	.49969	.51704	.53299	.54772	.56139	.57414	.59727	.62717	.66757
60	.38533	.40118	.41595	.42979	.44282	.45511	.47781	.50795	.55023
80	.31299	.32708	.34033	.35285	.36472	.37601	.39709	.42554	.46641
100	.26333	.27587	.28774	.29901	.30975	.32003	.33934	.36570	.40419
130	.21260	.22328	.23345	.24315	.25246	.26140	.27833	.30171	.33642
160	.17821	.18748	.19633	.20481	.21297	.22084	.23581	.25664	.28793
200	.14656	.15442	.16195	.16919	.17618	.18293	.19585	.21394	.24141
300	.10148	.10715	.11260	.11788	.12298	.12795	.13749	.15101	.17183
500	.062807	.066437	.069943	.073345	.076654	.079880	.086117	.095024	.10893
1000	.032160	.034067	.035915	.037713	.039467	.041182	.044512	.049299	.056851

Table 12. Upper 1% points of the largest root for $s = 15$

n \ m	0	1	2	3	4	5	7	10	15
5	.96191	.96526	.96806	.97044	.97248	.97426	.97720	.98054	.98435
10	.88633	.89459	.90169	.90787	.91331	.91812	.92628	.93583	.94719
15	.80814	.82001	.83042	.83964	.84786	.85525	.86800	.88332	.90212
20	.73718	.75135	.76394	.77522	.78541	.79466	.81085	.83070	.85570
25	.67516	.69066	.70458	.71718	.72866	.73917	.75778	.78096	.81077
30	.62145	.63765	.65233	.66572	.67800	.68932	.70955	.73508	.76852
40	.53445	.55097	.56613	.58011	.59307	.60514	.62700	.65519	.69315
60	.41553	.43099	.44538	.45884	.47150	.48343	.50544	.53458	.57532
80	.33913	.35305	.36613	.37848	.39017	.40128	.42200	.44989	.48983
100	.28621	.29871	.31052	.32173	.33240	.34260	.36174	.38782	.42578
130	.23178	.24251	.25271	.26244	.27176	.28071	.29763	.32096	.35549
160	.19468	.20403	.21296	.22151	.22973	.23765	.25269	.27359	.30490
200	.16040	.16837	.17599	.18333	.19039	.19723	.21027	.22851	.25614
300	.11133	.11712	.12268	.12806	.13326	.13831	.14802	.16173	.18283
500	.069052	.072774	.076369	.079853	.083241	.086542	.092919	.10201	.11618
1000	.035415	.037378	.039280	.041129	.042932	.044694	.048113	.053023	.060755

Table 13. Upper 5% points of the largest root for s = 16

n \ m	0	1	2	3	4	5	7	10	15
5	.95092	.95503	.95850	.96146	.96403	.96627	.97001	.97428	.97920
10	.86944	.87850	.88635	.89321	.89928	.90468	.91388	.92474	.93777
15	.78912	.80161	.81262	.82242	.83120	.83912	.85286	.86947	.89002
20	.71775	.73232	.74535	.75707	.76769	.77738	.79440	.81539	.84203
25	.65607	.67181	.68602	.69893	.71073	.72157	.74083	.76497	.79623
30	.60304	.61935	.63419	.64778	.66028	.67184	.69257	.71887	.75355
40	.51767	.53412	.54927	.56329	.57633	.58851	.61063	.63928	.67808
60	.40173	.41696	.43119	.44454	.45712	.46901	.49099	.52020	.56126
80	.32756	.34121	.35408	.36625	.37780	.38881	.40937	.43715	.47710
100	.27629	.28852	.30010	.31111	.32162	.33169	.35061	.37648	.41428
130	.22364	.23411	.24408	.25362	.26277	.27157	.28825	.31130	.34554
160	.18778	.19690	.20561	.21398	.22203	.22980	.24460	.26520	.29618
200	.15468	.16243	.16987	.17703	.18394	.19064	.20344	.22140	.24867
300	.10733	.11295	.11836	.12360	.12868	.13362	.14312	.15659	.17736
500	.066553	.070162	.073655	.077046	.080348	.083571	.089805	.098713	.11263
1000	.034127	.036029	.037875	.039673	.041428	.043146	.046484	.051287	.058869

Table 14. Upper 1% points of the largest root for s = 16

n \ m	0	1	2	3	4	5	7	10	15
5	.96542	.96833	.97078	.97288	.97469	.97628	.97892	.98193	.98540
10	.89495	.90231	.90868	.91425	.91916	.92353	.93097	.93973	.95022
15	.82050	.83127	.84075	.84918	.85672	.86352	.87530	.88951	.90705
20	.75190	.76491	.77652	.78696	.79641	.80501	.82011	.83869	.86220
25	.69124	.70562	.71858	.73034	.74108	.75094	.76842	.79028	.81850
30	.63825	.65340	.66716	.67975	.69132	.70200	.72113	.74534	.77716
40	.55156	.56719	.58155	.59484	.60718	.61868	.63956	.66653	.70294
60	.43151	.44634	.46017	.47314	.48534	.49686	.51813	.54634	.58585
80	.35351	.36698	.37966	.39164	.40301	.41382	.43398	.46118	.50018
100	.29912	.31128	.32278	.33372	.34415	.35412	.37285	.39840	.43564
130	.24286	.25335	.26335	.27289	.28204	29084	.30749	.33045	.36449
160	.20433	.21352	.22230	.23071	.23881	.24662	.26147	.28212	.31308
200	.16862	.17647	.18399	.19123	.19822	.20498	.21789	.23597	.26338
300	.11730	.12302	.12854	.13387	.13903	.14405	.15371	.16736	.18838
500	.072890	.076586	.080162	.083631	.087008	.090300	.096667	.10575	.11992
1000	.037439	.039394	.041291	.043138	.044941	.046704	.050127	.055047	.062803

Table 15. Upper 5% points of the largest root for s = 17

n \ m	0	1	2	3	4	5	7	10	15
5	.95522	.95882	.96188	.96451	.96679	.96880	.97217	.97604	.98054
10	.87888	.88700	.89407	.90029	.90580	.91072	.91915	.92914	.94121
15	.80210	.81348	.82356	.83255	.84064	.84796	.86069	.87615	.89538
20	.73287	.74632	.75837	.76926	.77915	.78818	.80410	.82381	.84891
25	.67238	.68704	.70031	.71240	.72347	.73366	.75182	.77463	.80428
30	.61993	.63523	.64919	.66200	.67381	.68475	.70440	.72940	.76246
40	.53469	.55030	.56467	.57806	.59050	.60213	.62329	.65075	.68804
60	.41746	.43211	.44582	.45870	.47085	.48235	.50362	.53195	.57182
80	.34165	.35489	.36738	.37922	.39047	.40119	.42123	.44835	.48740
100	.28890	.30082	.31212	.32289	.33317	.34302	.36157	.38694	.42406
130	.23443	.24469	.25448	.26385	.27284	.28150	.29792	.32064	.35443
160	.19718	.20614	.21472	.22297	.23091	.23858	.25320	.27358	.30423
200	.16267	.17031	.17766	.18474	.19158	.19821	.21090	.22871	.25578
300	.11312	.11868	.12405	.12926	.13431	.13922	.14868	.16209	.18279
500	.070270	.073860	.077338	.080719	.084013	.087230	.093458	.10236	.11629
1000	.036086	.037982	.039826	.041623	.043380	.045100	.048444	.053260	.060868

Table 16. Upper 1% points of the largest root for s = 17

n \ m	0	1	2	3	4	5	7	10	15
5	.96846	.97101	.97317	.97503	.97665	.97807	.98044	.98317	.98634
10	.90262	.90921	.91494	.91997	.92443	.92841	.93521	.94327	.95298
15	.83169	.84148	.85015	.85788	.86482	.87109	.88199	.89520	.91160
20	.76539	.77737	.78810	.79778	.80657	.81458	.82868	.84610	.86825
25	.70613	.71950	.73159	.74258	.75265	.76190	.77835	.79898	.82572
30	.65393	.66811	.68104	.69289	.70380	.71389	.73199	.75498	.78528
40	.56771	.58251	.59614	.60878	.62053	.63150	.65144	.67727	.71222
60	.44682	.46105	.47435	.48684	.49861	.50973	.53029	.55761	.59594
80	.36741	.38044	.39273	.40436	.41541	.42592	.44556	.47208	.51016
100	.31165	.32348	.33470	.34537	.35555	.36530	.38363	.40867	.44519
130	.25367	.26394	.27373	.28310	.29208	.30072	.31709	.33971	.37325
160	.21380	.22281	.23144	.23973	.24771	.25541	.27006	.29046	.32107
200	.17670	.18443	.19185	.19900	.20591	.21259	.22537	.24329	.27046
300	.12319	.12885	.13432	.13960	.14473	.14972	.15932	.17291	.19383
500	.076694	.080366	.083921	.087376	.090740	.094024	.10038	.10945	.12362
1000	.039451	.041399	.043291	.045135	.046937	.048700	.052126	.057055	.064831

Table 17. Upper 5% points of the largest root for s = 18

n \ m	0	1	2	3	4	5	7	10	15
5	.95898	.96215	.96485	.96720	.96925	.97105	.97409	.97762	.98186
10	.88733	.89464	.90103	.90668	.91170	.91620	.92393	.93315	.94435
15	.81391	.82431	.83356	.84184	.84930	.85607	.86789	.88230	.90032
20	.74680	.75923	.77041	.78054	.78976	.79819	.81311	.83163	.85532
25	.68755	.70122	.71364	.72497	.73538	.74497	.76210	.78367	.81183
30	.63575	.65013	.66328	.67537	.68653	.69689	.71553	.73931	.77085
40	.55081	.56563	.57933	.59206	.60394	.61505	.63531	.66165	.69750
60	.43257	.44666	.45987	.47231	.48405	.49517	.51576	.54323	.58195
80	.35529	.36813	.38026	.39177	.40272	.41316	.43271	.45919	.49735
100	.30117	.31279	.32382	.33434	.34440	.35404	.37222	.39710	.43354
130	.24499	.25504	.26465	.27385	.28269	.29121	.30738	.32977	.36310
160	.20640	.21521	.22366	.23179	.23962	.24720	.26164	.28179	.31211
200	.17054	.17808	.18533	.19233	.19910	.20566	.21823	.23590	.26276
300	.11884	.12436	.12968	.13485	.13987	.14475	.15416	.16752	.18815
500	.073962	.077531	.080994	.084363	.087649	.090859	.097078	.10598	.11990
1000	.038036	.039928	.041768	.043565	.045322	.047043	.050392	.055219	.062850

Table 18. Upper 1% points of the largest root for s = 18

n \ m	0	1	2	3	4	5	7	10	15
5	.97112	.97336	.97528	.97693	.97838	.97965	.98180	.98428	.98723
10	.90947	.91539	.92057	.92513	.92919	.93283	.93906	.94649	.95550
15	.84185	.85079	.85873	.86584	.87223	.87803	.88814	.90045	.91580
20	.77780	.78886	.79880	.80778	.81596	.82344	.83663	.85299	.87387
25	.71996	.73240	.74369	.75399	.76343	.77212	.78762	.80712	.83249
30	.66859	.68189	.69405	.70521	.71551	.72505	.74221	.76405	.79293
40	.58298	.59701	.60996	.62198	.63318	.64366	.66272	.68746	.72103
60	.46148	.47515	.48796	.49991	.51135	.52209	.54196	.56842	.60560
80	.38083	.39345	.40537	.41666	.42740	.43763	.45675	.48261	.51979
100	.32383	.33534	.34628	.35669	.36664	.37617	.39411	.41863	.45446
130	.26424	.27429	.28388	.29306	.30188	.31038	.32647	.34873	.38179
160	.22307	.23193	.24041	.24857	.25643	.26401	.27847	.29862	.32887
200	.18465	.19226	.19959	.20664	.21347	.22008	.23273	.25047	.27741
300	.12901	.13462	.14003	.14527	.15036	.15531	.16485	.17837	.19921
500	.080467	.084114	.087650	.091089	.094440	.097713	.10405	.11311	.12727
1000	.041452	.043392	.045280	.047121	.048921	.050684	.054112	.059048	.066841

Table 19. Upper 5% points of the largest root for s = 19

n \ m	0	1	2	3	4	5	7	10	15
5	.96228	.96508	.96750	.96959	.97144	.97307	.97582	.97904	.98284
10	.89492	.90152	.90732	.91246	.91705	.92118	.92829	.93681	.94723
15	.82468	.83422	.84272	.85036	.85727	.86354	.87453	.88799	.90490
20	.75965	.77117	.78156	.79100	.79961	.80750	.82148	.83891	.86131
25	.70167	.71445	.72608	.73672	.74651	.75555	.77172	.79215	.81891
30	.65059	.66412	.67651	.68794	.69851	.70832	.72602	.74866	.77877
40	.56609	.58017	.59322	.60537	.61671	.62734	.64674	.67201	.70649
60	.44708	.46065	.47339	.48539	.49673	.50749	.52744	.55407	.59167
80	.36850	.38096	.39274	.40393	.41459	.42476	.44382	.46967	.50697
100	.31312	.32444	.33521	.34550	.35534	.36477	.38257	.40698	.44275
130	.25533	.26518	.27460	.28364	.29233	.30071	.31662	.33868	.37154
160	.21546	.22412	.23244	.24045	.24818	.25565	.26991	.28982	.31982
200	.17828	.18572	.19289	.19981	.20650	.21300	.22545	.24296	.26960
300	.12451	.12997	.13525	.14038	.14537	.15022	.15958	.17288	.19343
500	.077627	.081176	.084624	.087981	.091256	.094459	.10067	.10956	.12348
1000	.039978	.041864	.043702	.045497	.047254	.048976	.052329	.057166	.064816

Table 20. Upper 1% points of the largest root for s = 19

n \ m	0	1	2	3	4	5	7	10	15
5	.97346	.97544	.97714	.97862	.97992	.98107	.98302	.98528	.98796
10	.91561	.92096	.92565	.92980	.93351	.93684	.94257	.94944	.95782
15	.85111	.85929	.86658	.87313	.87904	.88441	.89380	.90529	.91968
20	.78923	.79946	.80869	.81705	.82467	.83166	.84402	.85940	.87911
25	.73281	.74442	.75498	.76463	.77350	.78168	.79630	.81475	.83883
30	.68231	.69481	.70625	.71678	.72652	.73555	.75182	.77259	.80014
40	.59744	.61074	.62306	.63451	.64519	.65519	.67343	.69714	.72940
60	.47555	.48869	.50102	.51262	.52358	.53395	.55318	.57880	.61488
80	.39381	.40604	.41760	.42856	.43900	.44895	.46757	.49279	.52910
100	.33567	.34688	.35753	.36770	.37742	.38673	.40428	.42831	.46344
130	.27456	.28439	.29379	.30280	.31146	.31981	.33563	.35754	.39010
160	.23217	.24086	.24921	.25723	.26497	.27246	.28672	.30661	.33651
200	.19247	.19997	.20719	.21416	.22090	.22744	.23995	.25753	.28423
300	.13477	.14031	.14567	.15087	.15592	.16084	.17032	.18376	.20450
500	.084209	.087831	.091348	.094770	.098109	.10137	.10769	.11674	.13087
1000	.043443	.045376	.047258	.049096	.050895	.052657	.056086	.061027	.068834

Table 21. Upper 5% points of the largest root for s = 20

n \ m	0	1	2	3	4	5	7	10	15
5	.96520	.96769	.96985	.97173	.97340	.97487	.97738	.98032	.98394
10	.90176	.90774	.91302	.91772	.92193	.92572	.93228	.94018	.94989
15	.83454	.84330	.85114	.85820	.86461	.87044	.88068	.89327	.90916
20	.77155	.78224	.79191	.80072	.80877	.81616	.82930	.84572	.86690
25	.71486	.72681	.73772	.74773	.75694	.76547	.78075	.80012	.82557
30	.66453	.67727	.68898	.69978	.70979	.71910	.73592	.75749	.78625
40	.58059	.59399	.60643	.61802	.62886	.63902	.65761	.68188	.71505
60	.46103	.47410	.48638	.49798	.50894	.51935	.53867	.56450	.60103
80	.38130	.39339	.40484	.41572	.42609	.43600	.45459	.47982	.51628
100	.32475	.33579	.34631	.35636	.36598	.37522	.39266	.41658	.45170
130	.26544	.27509	.28434	.29322	.30176	.31000	.32566	.34739	.37979
160	.22435	.23287	.24106	.24895	.25657	.26395	.27803	.29770	.32737
200	.18592	.19326	.20033	.20717	.21379	.22022	.23255	.24990	.27632
300	.13011	.13552	.14076	.14585	.15080	.15563	.16494	.17817	.19863
500	.081267	.084796	.088228	.091572	.094837	.098031	.10423	.11311	.12704
1000	.041912	.043793	.045628	.047421	.049177	.050900	.054256	.059100	.066780

Table 22. Upper 1% points of the largest root for s = 20

n \ m	0	1	2	3	4	5	7	10	15
5	.97552	.97728	.97880	.98013	.98130	.98234	.98411	.98618	.98870
10	.92115	.92599	.93026	.93405	.93744	.94050	.94578	.95214	.95995
15	.83454	.86708	.87379	.87984	.88531	.89030	.89904	.90977	.92329
20	.79978	.80928	.81785	.82565	.83277	.83931	.85090	.86538	.88402
25	.74479	.75564	.76553	.77459	.78293	.79064	.80444	.82191	.84480
30	.69519	.70694	.71773	.72768	.73689	.74544	.76089	.78065	.80695
40	.61113	.62377	.63549	.64640	.65660	.66616	.68361	.70635	.73736
60	.48906	.50169	.51356	.52475	.53533	.54536	.56396	.58878	.62380
80	.40637	.41821	.42943	.44008	.45023	.45991	.47805	.50264	.53810
100	.34717	.35809	.36848	.37840	.38790	.39701	.41418	.43771	.47217
130	.28465	.29428	.30349	.31233	.32083	.32902	.34459	.36614	.39822
160	.24109	.24963	.25783	.26573	.27336	.28073	.29480	.31443	.34398
200	.20016	.20755	.21468	.22156	.22822	.23468	.24706	.26446	.29092
300	.14045	.14594	.15125	.15640	.16142	.16630	.17571	.18908	.20972
500	.087921	.091519	.095016	.098422	.10175	.10500	.11130	.12033	.13446
1000	.045423	.047349	.049227	.051061	.052857	.054618	.058048	.062993	.070818

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