THE INTEGRAL OF A STEP FUNCTION DEFINED ON A SEMI-MARKOV PROCESS

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Abstract

Consider an m-state, irreducible, recurrent Semi-Markov process (S.M.P.) and a step function $f(\cdot)$ which takes on the value $v_i (i = 1, \ldots, m)$ when the S.M.P. is in state i. We study the integral of $f(\cdot)$ between 0 and t.

The Laplace transform of the characteristic function of the integral is obtained in a general form by use of matrix notation. In the case of a stationary Semi-Markov process the transform of the expected value of the integral is inverted in closed form. Asymptotic properties of the expected value of the integral are derived by applying "Smith's Key Renewal Theorem".

1. Notation and Assumptions

In this paper we construct a function defined on a Semi-Markov process (S.M.P.). The notation utilized will be the same as that set forth by Pyke (1961a). We consider a double sequence of random variables, $\{(J_n, X_n), n = 0, 1, \ldots\}$, defined on a complete probability space such that $X_o = 0$ a.s., $P\{J_o = j\} = a_j$, and $P\{J_n = k, X_n \leq x \mid J_{n-1} = j\} = Q_{jk}(x)$ $(n = 1, 2, \ldots$ and $j, k = 1, \ldots, m < \infty$ where m is the number of states.) The $Q_{jk}(x)$ are non-decreasing and right continuous mass functions and satisfy $Q_{jk}(x) = 0$ for $x \leq 0$, $Q_{jk}(+\infty) = p_{jk}$, and $\sum_{k=1}^{m} p_{jk} = 1$. Consider a step function taking

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on the value v_j if and only if the S.M.P. is in state j. The definite integral, $I(t)$, of this step function over the interval $[0, t]$ is studied here.

A general Semi-Markov process (G.S.M.P.) is as that defined above except that there will be another set of mass functions $\tilde{Q}_{jk}(x) = P\{J_1 = k, X_1 \leq x \mid J_o = j\}$ for the first transition. Pyke (1961b) has shown that the process is stationary if $a_j = \eta_j \mu_{jj}^{-1}$ and $\tilde{Q}_{jk}(x) = p_{jk}\eta_j^{-1} \int_o^x (1-F_{jk}(y))dy$ where $F_{jk}(x) = p_{jk}^{-1}Q_{jk}(x)$,

$H_j(x) = \sum_{k=1}^{m} Q_{jk}(x)$, and $G_{jk}(x) = P\{N_k(t) > 0 \mid J_o = j\}$ with first and second moments b_{jk}, b_{jk}'', η_j, η_j'', μ_{jk}, and μ_{jk}'' respectively. Also, $N_k(t)$ represents the number of transitions to state k in the interval $(0, t]$ and

$N(t) = \sum_{k=1}^{m} N_k(t)$. We let

$$S_{jk}(n) = \{(\alpha_o, \alpha_1, \ldots, \alpha_n) \mid \alpha_o = j, \alpha_n = k\} \; (1 \leq \alpha_i \leq m, \; i = 0, \ldots, n)$$

be the set of all path functions with n transitions where $J_o = j$ and $J_n = k$.

Henceforth, a small Latin letter will represent the Laplace-Stieltjes transform of the function represented by the same capital letter, i.e.,

$q_{jk}(s) = \int_o^\infty e^{-st} dQ_{jk}(t)$. Whenever the range of a subscript is omitted it will be 1 to m inclusive. Square matrices of double subscripted terms will be denoted by dropping the subscripts, eg., $Q(t) = (Q_{jk}(t))$ with the argument being omitted whenever it is obvious. We will also have use for the notation $[Q(t)]_{jk} = Q_{jk}(t)$.

We will assume throughout that the Markov Renewal process (M.R.P.) $\{(N_1(t), \ldots, N_m(t)), t \geq 0\}$ is irreducible, i.e., there is only one communicating class of states, and that $\eta_j < \infty$ or equivalently that the M.R.P. is positive recurrent since m is finite. It follows from the definition of a positive recurrent M.R.P. that μ_{jj}, the mean time between visits to state j, is finite. Again, for ease in notation we will assume that each $G_{jj}(t)$ is non-lattice. This latter assumption would only be required in the expression for $\tilde{Q}_{jk}(t)$ in the stationary case and in the application of Smith's (1954) renewal theorems.

2. Distribution of the Integral

In order to determine the distribution, $R(y:t)$, of $I(t)$ for a G.S.M.P. we consider a path function of $S_{jk}(n)$. The transitions times $\tau_1, \tau_2, \ldots,$ determined by the given path function satisfy $0 < \tau_1 < \ldots < \tau_n \leq t$. We have

$$(2.1) \qquad I_{jk}(n:t) = v_j\tau_1 + v_{\alpha_1}(\tau_2 - \tau_1) + \ldots + v_{\alpha_{n-1}}(\tau_n - \tau_{n-1}) + v_k(t - \tau_n)$$

and the corresponding mass function

$$R_{jk}(y,n:t) = P\{J_{N(t)} = k,\ I_{jk}(n:t) \leq y \mid J_o = j\} \quad .$$

The various number of transitions in the interval [0, t] are mutually exclusive events, hence the joint distribution of $I_{jk}(t)$, the value of the integral when $J_o = j$ and $J_{N(t)} = k$, is

$$R_{jk}(y:t) = \sum_{n=0}^{\infty} R_{jk}(y,n:t)$$

from which one obtains the marginal distribution

$$R_j(y:t) = \sum_{k=1}^{m} R_{jk}(y:t) \quad .$$

The desired distribution is then given by

$$R(y:t) = \sum_{j=1}^{m} a_j R_j(y:t) \ .$$

We evaluate the Laplace transform of the Fourier transform of $I_{jk}(n:t)$. For $n > 0$ we let

$$(2.2) \qquad \Psi_{jk}(w,n:t) = E(e^{iwI_{jk}(n:t)})$$

$$= \sum_{S_{jk}(n)} \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} e^{iwI_{jk}(n:t)} dP_j(\alpha_1,\tau_1,\ldots,\alpha_n,\tau_n)$$

where $dP_j(\alpha_1, \tau_1, \ldots, \alpha_n, \tau_n)$ is the probability that the transitions α_{i-1} to α_i occur in the interval $(\tau_i, \tau_i + d\tau_i]$, $i = 0, 1, \ldots, n$ and $\alpha_n = k$ given $\alpha_o = j$. By the conditional independence of the G.S.M.P., given the random variables J_i, $i = 0, 1, \ldots, n$, we have that

$$(2.3) \qquad dP_j(\alpha_1,\tau_1,\ldots,\alpha_n,\tau_n) = d\widetilde{Q}_{j\alpha_1}(\tau_1)\ldots dQ_{\alpha_{n-1}k}(\tau_n - \tau_{n-1})(1-H_k(t-\tau_n)) \ .$$

From (2.1) and (2.2) we obtain, upon taking the Laplace transform of (2.2),

$$(2.4) \qquad \Lambda_{jk}^{(n)}(s,w) = \int_0^{\infty} e^{-st}\Psi_{jk}(w,n:t)dt$$

$$= \sum_{S_{jk}(n)} \widetilde{q}_{j\alpha_1}(s-iwv_j)q_{\alpha_1\alpha_2}(s-iwv_{\alpha_1}) \ \ldots$$

$$\cdot \ q_{\alpha_{n-1}\alpha_n}(s-iwv_{\alpha_{n-1}})(s-iwv_k)^{-1}(1-h_k(s-iwv_k))$$

for $n \geq 1$. In the case $n = 0$ we have

$$\Lambda_{jk}^{(o)} = (s-iwv_j)^{-1}(1-\tilde{h}_k(s-iwv_k)) . \tag{2.5}$$

By letting

$$\Gamma_{jk}(s,w) = \Gamma_{jk}(s-iwv_k) = \delta_{jk}(s-iwv_k)^{-1}(1-h_k(s-iwv_k)) \tag{2.6}$$

and

$$q_{jk}(s,w) = q_{jk}(s-iwv_j) \tag{2.7}$$

with similar expressions for $\tilde{\Gamma}_{jk}$ and $\tilde{q}_{jk}$, we obtain in matrix notation

$$\begin{aligned} \Lambda^{(n)}(s,w) &= \tilde{q}(s,w)q^{n-1}(s,w)\Gamma(s,w) & (n \geq 1) \\ &= \tilde{\Gamma}(s,w) & (n = 0) \end{aligned} \tag{2.8}$$

where δ_{jk} is the Kronecker delta and $q^o(s,w)$ is the identity matrix. Since the spectral radius of $q(s,w)$ is less than one we have that $\sum_{n=1}^{\infty} q^{n-1} = (I-q)^{-1}$. The above derivation yields:

<u>Theorem 2.1</u>: Given a G.S.M.P. and the integral $I_{jk}(t)$ defined on this process, then

$$\Lambda = \tilde{\Gamma} + \tilde{q}(I - q)^{-1}\,\Gamma \tag{2.9}$$

where Λ_{jk} represents the Laplace transform of the Fourier transform of $I_{jk}(t)$.

Corollary 2.1: For an ordinary Semi-Markov process (O.S.M.P.) where $\tilde{q} = q$, and the integral $I_{jk}(t)$ defined on this process, then,

$$\Lambda = (I-q)^{-1}\,\Gamma \quad . \tag{2.10}$$

Corollary 2.2: For a stationary Semi-Markov process (S.S.M.P.) and the integral $I_{jk}(t)$ defined on this process

$$\Lambda = W\eta^{-1}W(p-I)(I-q)^{-1}\Gamma \tag{2.11}$$

where $W = (\delta_{jk}(s-iwv_j)^{-1})$ and $\eta = (\delta_{jk}\eta_j)$.

The proof of Corollary 1 is trivial and the proof of Corollary 2 follows from

$$\tilde{q} = \eta^{-1}W(p-q), \quad \tilde{h} = \eta^{-1}W(I-h),$$

and

$$\tilde{\Gamma} = W-\eta^{-1}W^2(I-h)$$

where $h = (\delta_{jk}h_j)$.

The inverse Laplace and Fourier transform of Λ_{jk} gives the desired distribution function $R_{jk}(y:t)$ and, as mentioned previously,

$$R(y:t) = \sum_{j=1}^{m} \sum_{k=1}^{m} a_j R_{jk}(y:t) \quad .$$

3. Joint Distributions.

The determination of the joint distribution

(3.1) $$R_{jkr}(y_1, y_2: t_1, t_2) = P\{J_{N(t_1+t_2)} = r,\ J_{N(t_1)} = k, I(t_1+t_2) - I(t_1) < y_2,\ I(t_1) < y_1 \mid J_0 = j,\ I(0) = 0\}$$

resulting from the two intervals $(0, t_1]$ and $(t_1, t_1 + t_2]$ is a straightforward generalization of the technique utilized in the previous section. In this case a double Laplace transform is required.

A typical path function belongs to the set

$$S_{jkr}(f,g) = \{\alpha_0, \alpha_1, \ldots, \alpha_{f+g} \mid \alpha_0 = j,\ \alpha_f = k,\ \alpha_{f+g} = r\}$$

where f and g are the number of transitions in each respective interval. The transitions for this path function occur at $\tau_1, \ldots, \tau_{f+g}$ where $0 < \tau_1 < \ldots < \tau_f \leq t_1 < \tau_{f+1} < \ldots < \tau_{f+g} \leq t_1 + t_2$.
We now set

(3.2) $$I_{jk}(f:t_1) = v_j \tau_1 + v_{\alpha_1}(\tau_2 - \tau_1) + \ldots + v_{\alpha_f}(t_1 - \tau_f)$$

(3.3) $$I_{kr}(g:t_2) = v_{\alpha_f}(\tau_{f+1} - t_1) + v_{\alpha_{f+1}}(\tau_{f+2} - \tau_{f+1}) + \ldots + v_r(t_1 + t_2 - \tau_{f+g})$$

and form the Fourier transform

$$(3.4)\quad \Psi_{jkr}(w_1,w_2,f,g:t_1,t_2) = \sum_{S_{jkr}(f,g)} \int_{t_1}^{t_1+t_2}\int_{t_1}^{\tau_{f+g}}\cdots\int_{o}^{t_1}\int_{o}^{\tau_f}\cdots\int_{o}^{\tau_2}$$

$$e^{iw_1 I_{jk}(f:t_1)+iw_2 I_{kr}(g:t_2)}\, dP_j(\alpha_1,\ldots,\alpha_{f+g},\tau_1,\ldots,\tau_{f+g}) \quad .$$

The double Laplace transform

$$(3.5)\qquad \Lambda_{jkr}^{(f,g)} = \int_o^\infty e^{-s_1t_1}dt_1 \int_o^\infty e^{-s_2t_2}\, \Psi_{jkr}(w_1,w_2,f,g:t_1,t_2)dt_2$$

may be evaluated, using the substitution $u_n = \tau_{f+n}-t_1$, $n = 1, \ldots, q$ to yield

$$(3.6)\qquad \Lambda_{jkr}^{(f,g)} = [\tilde{q}_1 q^{f-1}]_{jk}(s_{1k}- s_{2k})^{-1}[(q_2- q_1)q_2^{g-1}\Gamma_2]_{kr} \quad (f,g \geq 1)$$

where
$$q_j = q(s_j,\, w_j), \qquad \tilde{q}_j = \tilde{q}(s_j,\, w_j)$$
$$\Gamma_j = \Gamma(s_j,\, w_j), \qquad \tilde{\Gamma}_j = \tilde{\Gamma}(s_j,\, w_j)$$
$$s_{jk} = s_j - iw_jv_k \qquad \text{for } j = 1,2; \quad k = 1,\ldots, m.$$

It remains to evaluate $\Lambda_{jkr}^{(f,g)}$ for the three cases $f = 0,\ g \geq 1$; $f \geq 1,\ g = 0$; and $f = 0,\ g = 0$. These terms are

$$(3.7)\qquad \Lambda_{jkr}(0,g) = \delta_{jk}(s_{1k}- s_{2k})^{-1}[(\tilde{q}_2- \tilde{q}_1)q_2^{g-1}\Gamma_2]_{kr}$$

(3.8) $$\Lambda_{jkr}(f,0) = \delta_{kr}[\tilde{q}_1 q^{f-1}]_{jk}(s_{1k} - s_{2k})^{-1}[\Gamma_2 - \Gamma_1]_{kk}$$

(3.9) $$\Lambda_{jkr}(0,0) = \delta_{jk}\delta_{kr}(s_{1k} - s_{2k})^{-1}[\tilde{\Gamma}_2 - \tilde{\Gamma}_1]_{kk} \quad .$$

By summing the terms of Λ_{jkr} over all values of f and g we have the desired result.

All through the above results one can observe coupling terms resulting from the interval in which the point t_1 is located. The three different types of coupling intervals result from the three various types of intervals, i.e., those associated with $\tilde{Q}_{jk}(t)$, $Q_{jk}(t)$, or $1-H_j(t_1 + t_2 - t)$. The remaining part of the coupling terms, $(s_{1k} - s_{2k})^{-1}$, is determined by the state k which the process is in at t_1 .

In general, the expression for Λ_{jkr} will not factor into a product of two terms, one being a function of s_1 and w_1 and the other of s_2 and w_2, since the contributions to the I-process from each interval will in general not be independent. Conditional independence holds in the case where the elements of the transition matrix are of the form

(3.10) $$Q_{jk}(t) = \tilde{Q}_{jk}(t) = p_{jk}(1-e^{\lambda_j t})$$

where

$$\lambda_j > 0, \quad \sum_{k=1}^{m} p_{jk} = 1 \quad (j = 1, \ldots, m) \quad .$$

The derivation of the n-dimensional joint distribution is a straightforward generalization of the 2-dimensional case. The heavy notation encountered prohibits further derivation, however, the general form can be discussed. In this case there will be n adjacent intervals with $n - 2$ interior end points, i.e. $t_1, t_1 + t_2, \ldots, t_1 + \ldots + t_{n+1}$. There would be 2^n individual terms contributing to the desired Laplace transform with each term made up of a product of elements of matrices. There would be first order coupling terms appearing wherever only one interior end point lies in a transition interval, second order coupling terms wherever two interior end points lie in the same transition interval, and etc. The exact form of these higher order coupling terms can be derived in an analogous way.

4. Moments of the Integral

The Laplace transform of the first two moments of the integral are determined by differentiating Λ with respect to w and simplifying where possible. In the case of a S.S.M.P. the mean of $I(t)$ is determined explicitly by inverting the transform. Throughout, the single and double dot notation will be used for the first and second derivative with respect to w.

Before proceeding further we give the following lemmas.

Lemma 4.1: Given a matrix $q(s,w)$ as defined in (2.7), then

$$(4.1) \qquad (I\dot{-}q)^{-1} = (I-q)^{-1}\,\dot{q}\,(I-q)^{-1} .$$

Proof: The $q_{jk}(s,w)$ are all analytic functions for Real $(s) \geq 0$. In the domain of analyticity of $w \sum_{n=1}^{\infty} q^{n-1}$ converges uniformly and hence can be be differentiated term by term. Thus

$$(I\dot{-}q)^{-1} = \sum_{n=1}^{\infty} \dot{q}^{\,n-1} \quad ,$$

expanding the $\dot{q}^{n-1}$ terms and changing the order of summation gives the desired result.

Lemma 4.2: Given a matrix $\Gamma(s) = s^{-1}(\delta_{ik}(1 - \sum_{j=1}^{m} q_{kj}(s))$ where $(I-q(s))$ is a nonsingular matrix and if we let e represent a $m \times 1$ column vector with each element equal to 1, then

$$(4.2) \qquad (I-q(s))^{-1}\Gamma(s)e = s^{-1}e \quad .$$

Proof: Let $(I-q)^{*}_{jk}$ denote the jkth cofactor of the $(I-q)$ matrix. The ith element of $(I-q)^{-1}\Gamma e$ can then be written as

$$\sum_{k=1}^{m} (I-q)^{-1}_{ik}\Gamma_{kk} = s^{-1}(\det(I-q))^{-1}\sum_{j=1}^{m}\sum_{k=1}^{m}(I-q)^{*}_{ki}(\delta_{kj}-q_{kj})$$
$$= s^{-1}$$

which holds for all i. Thus we have the desired result.

Lemma 4.3: Given matrices $\Gamma(s,w) = (\delta_{jk}(s-iwv_j)^{-1}(1-h_j(s-iwv_j)))$ and $q(s,w) = (q_{jk}(s-iwv_j))$ where $(I-q(s,w))$ is nonsingular, then

$$(4.3) \qquad (\dot{q}(s,w)(I-q(s,w))^{-1}\Gamma(s,w) + \dot{\Gamma}(s,w))e\big|_{w=0} = i\, s^{-1}\Gamma(s)ve$$

where $\qquad v = (\delta_{jk}v_j) \quad .$

Proof: Lemma 4.3 will allow us to write the left-hand side of the above equation as $(s^{-1}\dot{q}(s,0) + \dot{\Gamma}(s,0))e$ where we have set $w = 0$ after the differentiation. The jth element of this vector becomes

$$s^{-1}\sum_{k=1}^{m}\dot{q}_{jk}(s,0) + iv_j s^{-2}(1-\sum_{k=1}^{m} q_{jk}(s)) - s^{-1}\sum_{k=1}^{m}\dot{q}_{jk}(s,0) = iv_j s^{-1}\Gamma_{jj}(s) .$$

Since this holds for all j we have the desired result.

Lemma 4.1 can now be used in the differentiation of equation (2.9) to give

$$\dot{\Lambda} = \dot{\tilde{\Gamma}} + \dot{\tilde{q}}(I-q)^{-1}\Gamma + \tilde{q}(I-q)^{-1}\dot{q}(I-q)^{-1}\Gamma + \tilde{q}(I-q)^{-1}\dot{\Gamma} . \tag{4.4}$$

Utilizing Lemmas 4.2 and 4.3 in post multiplying (4.4) by the column vector e and then setting $w = 0$ gives

$$\dot{\Lambda} e \,\big|_{w=0} = i\, s^{-1}(\tilde{\Gamma} + \tilde{q}(I-q)^{-1})v\, e \,\big|_{w=0} \tag{4.5}$$

since $$(\dot{\tilde{\Gamma}} + s^{-1}\dot{\tilde{q}})e \,\big|_{w=0} = i\, s^{-1}\Gamma\, v\, e \,\big|_{w=0} .$$

Equation (4.5) implies that the vector of Laplace transforms of expected values conditioned on the initial state is

$$s^{-1}[\tilde{\Gamma}(s) + \tilde{q}(s)(I-q(s))^{-1}\Gamma(s)]v\, e . \tag{4.6}$$

Under the hypothesis of Corollary 3.1 or 3.2 for a O.S.M.P. or S.S.M.P. we obtain respectively:

$$s^{-1}[(I-q(s))^{-1}\Gamma(s)]v\, e \tag{4.7}$$

and

$$s^{-2}[I + \eta^{-1}(p-I)(I-q(s))^{-1}\Gamma(s)]v\, e . \tag{4.8}$$

Pre-multiplication of the above expressions by the vector, a, of initial probabilities gives the Laplace transform of the unconditioned mean of $I(t)$. In the case of a S.S.M.P.

$$a = \mu^{-1}\eta \quad \text{where} \quad \mu^{-1} = (\mu_{11}^{-1}, \ldots, \mu_{mm}^{-1}), \quad \text{thus}$$

$$(4.9) \qquad \int_0^\infty e^{-st}E(I(t))dt = s^{-2}\mu^{-1}\eta\, v\, e \quad .$$

The validity of this statement can be seen by observing that we have a positive recurrent class for which

$$\mu_{jj} = \beta_{jj} \sum_{k=1}^{m} \eta_k\, \beta_{kk}^{-1}$$

where β_{kk} is the mean time between visits to the state k in the corresponding Markov Chain (c.M.C.)(Pyke 1964). The c.M.C. is also positive recurrent (Pyke 1961a) so the vectors

$$\beta(p - I) = 0$$

and

$$\mu^{-1}(p-I) = (\sum_{k=1}^{m} \eta_k\beta_{kk}^{-1}\beta(p-I) = 0$$

From (4.9) we obtain

$$(4.10) \qquad E(I(t)) = (\mu^{-1}\eta\, v\, e)t = t \sum_{j=1}^{m} \mu_{jj}^{-1}\eta_j v_j \quad .$$

The techniques used above can be also applied in deriving the second moment. In the case of a O.S.M.P. we would have

$$(4.11) \qquad \ddot{\Lambda}\, e \,\big|_{w=0} = (i^2 s^{-2}(I-q)^{-1}\Gamma\, v^2 e + 2\, i\, s^{-1}(I-q)^{-1}(\dot{q}(I-q)^{-1}\Gamma - s^{-1}\dot{h})ve\big|_{w=0}$$

The first term on the right-hand side of this equation fits into the pattern developed in the derivation of the first moment. The second term, however, defies further simplification. Similar situations arise in G.S.M.P. and S.S.M.P.

Although it is not obvious, it appears that these results could be in agreement with the complicated type variances that Pyke (1964) and Jewell (1964) present for the asymptotic distribution of general functions defined on a S.M.P.

5. Asymptotic Properties of the Integral

The asymptotic distribution of functions, in general, defined on a Semi-Markov sequence has been resolved by Pyke (1964). Application of these results gives that $t^{-\frac{1}{2}}(I(t) - tA)$ converges in law to a normal random variable with zero mean and variance σ^2 where

$$A = \sum_{j=1}^{m} \mu_{jj}^{-1}\eta_j v_j$$

$$\sigma^2 = \mu_{kk} \sum_{j=1}^{m} (v_j - A)\mu_{jj}^{-1}\eta_j'' + 2\mu_{kk} \sum_{j=1}^{m} \sum_{s\neq k} \sum_{r\neq k} (v_j - A)p_{js}b_{js}(v_r - A) \cdot \eta_r \mu_{jj}^{-1}\beta_{rr}^{-1}(\beta_{sk} + \beta_{kr} - \beta_{sr})$$

with σ^2 being independent of the arbitrary state k.

The remainder of this section will be devoted to an alternative derivation of the asymptotic mean value of the integral. In this derivation, some of Pyke's (1961b) results and the renewal theorems published by Smith (1954) will be used. The case of a O.S.M.P. will be the only one considered since the contribution to the integral from the visit to the first state is negligible.

If we let

$$r_j(t) = \int_0^\infty y \, dR_j(y:t)$$

then (4.7) and (2.6) yield the vector

$$\left(\int_0^\infty e^{-st} r_j(t) \, dt \right) = s^{-2}(I-q(s))^{-1}(I-h(s))v \; e$$

$$= s^{-2}(m(s) + I)(I-h(s))v \; e$$

where $m_{jk}(s)$ is the Laplace-Stieltjes transform of

$$M_{jk}(t) = E(N_k(t) \mid J_o = j) \quad .$$

Integration by parts gives

$$\left(\int_0^\infty e^{-st} dr_j(t) \right) = s^{-1}(m(s) + I)(I-h(s))v \; e$$

for which the inverse transform is

(5.1) $$(r_j(t)) = (I + M(t)) * \left(\int_0^t (I-H(x))\eta^{-1} dx \right) \eta \; v \; e$$

where the convolution of two matrices A and B is defined as

$$A * B = \left(\sum_{j=1}^{m} A_{ij} * B_{jk} \right) \quad .$$

We now determine the asymptotic properties of the elements of the matrix $(I + M(t)) * B(t)$ where $B(t)$ is a diagonal matrix with $B_j(t) = \eta^{-1} \int_0^t (1-H_j(x))dx$ which is a distribution function of some non-negative random variable. Assuming that the moments η_j and η_j'' are both finite we define

$$(5.2) \qquad k_j(t) = 2\eta_j(\eta_j'')^{-1}(U(t) - B_j(t))$$

where $U(t)$ is the unit step function with jump at zero. This function will serve as the kernel in the application of Smith's renewal theorem.

Considering only visits to state j we see that $M_{jj}(t)$ is a Renewal function where the time between transitions to state j has the distribution $G_{jj}(t)$, thus

$$(5.3) \qquad B_j(t) * (M_{jj}(t) + 1) = \frac{t}{\mu_{jj}} + \frac{\mu_{jj}''}{2\mu_{jj}^2} - \frac{\eta_j''}{2\eta_j\mu_{jj}} + o(1)$$

for large values of t.

In order to determine the asymptotic value of $B_j(t) * M_{ij}(t)$ we observe that

$$k_j(t) * M_{ij}(t) = k_j * G_{ij}(t) * (M_{jj}(t) + U(t))$$

and hence

$$(5.4) \qquad \lim_{t \to \infty} k_j(t) * M_{ij}(t) = G_{ij}(+\infty)\, \mu_{jj}^{-1} = \mu_{jj}^{-1}$$

since i and j communicate. Using (5.2) in (5.4) gives

$$(5.5) \qquad B_j(t) * M_{ij}(t) = M_{ij}(t) - \eta_j''(2\eta_j\mu_{jj})^{-1} + o(1)$$

for large values of t. Now since

$$M_{ij}(t) = G_{ij}(t) * M_{jj}(t) + G_{ij}(t)$$

we have that

$$(5.6) \qquad M_{ij}(t) = \frac{t}{\mu_{jj}} + \frac{\mu_{jj}''}{2\mu_{jj}^2} - \frac{\mu_{ij}}{\mu_{jj}} + o(1)$$

for large values of t. Substitution of (5.5) into (5.4) and (5.3) into (5.1) yields

$$(5.7) \qquad E(I(t)) = a(r_j(t))$$

$$= t\sum_{j=1}^{m}\mu_{jj}^{-1}\eta_j v_j + \sum_{j=1}^{m}\mu_{jj}^{-1}(\mu_{jj}''(2\mu_{jj})^{-1} - \eta_j''(2\eta_j)^{-1})\eta_j v_j \; -$$

$$- \sum_{\substack{i=1 \\ i\neq j}}^{m}\sum_{j=1}^{m} a_i\mu_{jj}^{-1}\mu_{ij}\eta_j v_j + o(1)$$

for large values of t.

We note that the first term on the right-hand side of (5.7) is the expected value for the stationary case.

6. The Case of Jumps at the Transitions

The model which was described in section 1 may be altered to allow for a real valued jump, $r_{jk} < \infty$, to take place at the point of transition from state j to state k. Modification of the previous notation with a pre-subscript r to indicate the alteration gives

(6.1) $$ {}_rI_{jk}(n:t) = v_j\tau_1 + r_{j\alpha_1} + \dots + r_{\alpha_{n-1}\alpha_n} + v_{\alpha_n}(t - \tau_n) \quad . $$

The evaluation of the distribution of the integral ${}_rI(t)$ is similar to that used in section 2. Hence, only the necessary notation and the final results will be shown here.

Let

$$ {}_r\tilde{q}_{jk} = e^{iwr_{jk}}\tilde{q}_{jk} \quad \text{and} \quad {}_rq_{jk} = e^{iwr_{jk}}q_{jk} $$

then

(6.2) $$ {}_r\Lambda^{(n)} = \int_0^\infty e^{-st}E(e^{iw_rI(n:t)})dt $$

$$ = {}_r\tilde{q}\,{}_rq^{n-1}\Gamma \qquad (n \geq 1) $$

$$ = \tilde{\Gamma} \qquad (n = 0) \quad . $$

It is easily shown that $\rho({}_rq) < 1$, hence

(6.3) $$ {}_r\Lambda = \tilde{\Gamma} + {}_rq(I - {}_rq)^{-1}\Gamma \quad . $$

In deriving the expected value of ${}_rI(t)$ we have that

$$(6.4) \qquad {}_r\dot{\Lambda}\, e \,\Big|_{w=0} = (i\, s^{-1}(\tilde{\Gamma}+\tilde{q}(I-q)^{-1}\Gamma)v\, e + i\, s^{-1}(r_{jk}\tilde{q}_{jk})e +$$

$$+ i\, s^{-1}\tilde{q}(I-q)^{-1}(r_{jk}q_{jk})e)\,\Big|_{w=0} \quad .$$

In the case of a S.S.M.P. we obtain

$$(6.5) \qquad E({}_rI(t)) = t\sum_{j=1}^{m} \mu_{jj}^{-1}\eta_j v_j + t\sum_{j=1}^{m}\sum_{k=1}^{m} \mu_{jj}^{-1}p_{jk}r_{jk} \quad .$$

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