On Second Moments of Stopping Rules

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Summary. The current investigation is a natural outgrowth of [2], being concerned with the variance of stopping rules and the effect of non-zero means on the variance of a randomly stopped sum. Some martingale generalizations of applications of [2] also appear.

1. Introduction. A stopping rule or stopping variable of a sequence $\{X_n, n \geq 1\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ is a positive integer-valued random variable t such that for every $n \geq 1$ the event $\{t = n\} \in \mathcal{F}_n$, the Borel field generated by $X_1 \ldots X_n$. In contradistinction, a stopping time[1] (likewise of a sequence $\{X_n\}$) will be defined as a positive integer or $+\infty$ valued function on Ω subject to the same proviso that $\{t = n\} \in \mathcal{F}_n$, $n \geq 1$. Thus, a stopping time t is a stopping variable or stopping rule if and only if $P\{t < \infty\} = 1$. In numerous problems of probability theory and statistics it is necessary to demonstrate that what is obviously a stopping time is further a stopping variable and even to obtain detailed information about the latter.

2. Comparison of Stopping Rules. Let the basic process $\{X_n, n \geq 1\}$ consist of independent random variables with $EX_n = 0$, $EX_n^2 = 1$, $P\{|X_n| \leq a < \infty\} = 1$ for $n \geq 1$. Set $S_n = \sum_{i=1}^{n} X_i$ and define $t_m(c)$ to be the smallest positive index $n \geq m$ $(m = 1, 2, \ldots)$ for which $S_n^2 > c^2 n$ where c is a positive

constant. For the case of coin tossing $(a = 1)$, it was shown in [1] that for all m, $E\, t_m(c)$ is finite or infinite according as $c < 1$ or $c \geq 1$ and this was generalized in [2] to the uniformly bounded case.[2] Apropos of these results it may be noted that for $m = 0$, the lemma of theorem 1 of the next section gives an upper bound for $E\, t_1(c)$ when the latter is finite. It will be proved in section 3 that if $c^2 < 3 - \sqrt{6}$, $E\, t_m^2(c) < \infty$, all $m \geq 1$ while if $c^2 > 3 - \sqrt{6}$ then $E\, t_m^2(c) = \infty$ for all sufficiently large (but not necessarily all) m.

It is clear from a comparison technique that there is a non-increasing sequence of non-negative constants $\{c_k, k \geq 1\}$ such that $E\, t_m^k(c) < \infty$ for $c < c_k$ (if $c_k > 0$) while $E\, t_m^k(c) = \infty$ for all sufficiently large m if $c > c_k$. Such comparisons may be formalized by the following Definition: A stopping time t will be called "more restrictive" than a stopping time s if $\{t = n\} \subset \{s \leq n\}$ for $n = 1, 2, \ldots$ that is, if $t \geq s$.

Clearly, if t is more restrictive than s, and t is a bonafide stopping variable, so is s; moreover, the finiteness of $E\, t^{\alpha}$ implies that of $E\, s^{\alpha}$ for any $\alpha > 0$.

Thus, if $c < c'$, $E\, t_m^k(c) \leq E\, t_m^k(c')$, $(k, m = 1, 2, \ldots)$ corroborating the prior statement about the sequence c_k. It is a natural conjecture that c_k decreases to zero but currently the authors know of no method for attacking this seemingly simple question.

3. Second Moments. When $c^2 < 1$, the situation changes in the coin tossing example $(a = 1)$ alluded to earlier since now $P\{t_m(c) = 1\} = 1$ for $m = 1$. Thus, to allow the second moment to attain an infinite value, it is necessary to dawdle for a while so as to insure that S_n does not prematurely escape

its parabolic bonds. This accounts for the appearance of the phrase "for all sufficiently large m" in

Theorem 1: Let $\{X_n\}$ be independent random variables with $P\{|X_n| \leq a < \infty\} = 1$, $E X_n = 0$, $E X_n^2 = 1$ for $n \geq 1$ and define t_m = smallest integer $n \geq m$ for which $S_n^2 > c^2 n$ $(n = 1, 2, \ldots)$. If $c^2 < 3 - \sqrt{6}$, then $E t_m^2 < \infty$, all $m \geq 1$ while if $c^2 > 3 - \sqrt{6}$, $E t_m^2 = \infty$ for all sufficiently large m.

Proof: In the case $c^2 < 3 - \sqrt{6}$ we write t for t_m. Set $\gamma_n = E X_n^3$, $\beta_n = E X_n^4$ and $t' = \min(t,k)$ where $k > m$. Since $E t' \sum_{j=1}^{t'} \beta_j \leq a^4 E t'^2 < \infty$, by Theorem 3 or [2],

$$(1) \qquad E S_{t'}^4 = 6 E t' S_{t'}^2 - 3 E t' (t'+1) + 4 E S_{t'} \sum_1^{t'} \gamma_j + E \sum_1^{t'} \beta_j$$

whence

$$E(S_{t'}^2 - c^2 t')^2 = (6-2c^2) E t' S_{t'}^2 - (3-c^4) E t'^2 - 3 E t' + 4 E S_{t'} \sum_1^{t'} \gamma_j + E \sum_1^{t'} \beta_j$$

implying

$$(2) \qquad (3-c^4) E t'^2 + (2c^2-6) E t' S_{t'}^2 \leq (a^4-3) E t' + 4a^3 E t' |S_{t'}| \quad .$$

Let $A_k = \{m < t \leq k\}$. From (2), recalling that $E t' \leq E t < \infty$ for $c^2 < 1$ [2] ,

$$(3 - c^4)\left[\int_{[t>k]} k^2 + \int_{A_k} t^2\right] + (2c^2 - 6)\left[\int_{[t>k]} c^2k^2 + \int_{A_k} t(ct^{1/2}+a)^2\right]$$

$$\leq 4a^3\left[\int_{[t>k]} ck^{3/2} + \int_{A_k} t(ct^{1/2}+a)\right] + O(1) .$$

Consequently,

$$(c^4-6c^2+3)\left[k^2P\{t > k\} + \int_{A_k} t^2\right] \leq B\left[k^{3/2}\, P\{t > k\} + \int_{A_k} t^{3/2}\right] + O\ (1)$$

where $B > 0$ is a constant depending only on c and a. Thus, letting $k \longrightarrow \infty$, $E\ t^2 < \infty$ regardless of m.

In the alternative case, we may clearly suppose $3 - \sqrt{6} < c^2 < 1$. Define $u_m(c)$ to be the first index $n \geq 1$ for which $S_n^2 > c^2(n + m)$ where m is an arbitrary non-negative quantity.

Suppose it has been established for every c^2 in $(3 - \sqrt{6}, 1)$ that $E\ u_m^2(c) = \infty$ for all sufficiently large m. Then, for any c^2 in $(3 - \sqrt{6}, 1)$ we may choose c_o^2 likewise in this interval but less than c^2 and be assured of the existence of an integer m_o such that $E\ u_{m_o}^2(c_o) = \infty$. Select the integer m_1 so that $c^2n > c_o^2\ (n + m_o)$ for all $n \geq m_1$. Then by the comparison technique $E\ t_m^2 = \infty$ for $m \geq m_1$.

Thus, it suffices to prove the anxiliary proposition involving $u_m(c)$ and in so doing we denote the latter variable by t.

Lemma: For $0 < c < 1$ and $m \geq 0$,

$$\frac{c^2m}{1-c^2} \leq E\ t \leq [ac(1-c^2)^{-1} + \sqrt{(m-1)(1-c^2)^{-1} + a^2(1-c^2)^{-2}}]^2 - m + 1$$

and thus $E\ t = O(m)$.

Proof: Choose $c < c_1 < 1$ and $m_1 > 0$ such that $c_1^2 n \geq c^2(n+m)$ for all $n \geq m_1$. By the comparison technique and Corollary 2 of [2], $E\ t < \infty$. By Theorem 2 of [2], $E\ t = E\ S_t^2 \geq c^2 E(t + m)$ proving the first inequality. On the other hand,

$$Et = E\ S_t^2 \leq E\ [c(t+m-1)^{1/2} + a]^2 \leq c^2 E(t+m-1) + 2ac\ E^{1/2}(t+m-1) + a^2$$

or

$$(1-c^2)\ E\ (t+m-1) - 2ac\ E^{1/2}(t+m-1) - (a^2+m-1) \leq 0$$

yielding the second.

Suppose now that $E\ t^2 < \infty$ for all m. By Theorem 3 of [2],

$$E\ S_t^4 = 6\ E\ t\ S_t^2 - 3\ E\ t(t+1) + 4\ E\ S_t \sum_{j=1}^{t} \gamma_j + E \sum_{1}^{t} \beta_j$$

$$\geq 6\ c^2\ E\ t(t+m) - 3\ E\ t(t+1) - 4\ a^3\ E\ t\ |S_t|$$

(3) $$\geq (6c^2-3)\ E\ t^2 + (6mc^2-3)\ E\ t - 4\ a^3 c\ E(t+m-1)^{3/2} - 4\ a^4\ E\ t$$

On the other hand,

(4) $$E\ S_t^4 \leq E[c(t+m-1)^{1/2} + a]^4 = c^4\ E(t+m-1)^2 + 4\ a\ c^3\ E(t+m-1)^{3/2}$$
$$+ 6c^2 a^2\ E(t+m-1) + 4\ ca^3\ E(t+m-1)^{1/2} + a^4$$

whence, combining (3) and (4) and recalling that $E\ t = O(m)$

$$(6c^2-3-c^4)\ E\ t^2 \leq m^2c^4 - 2mc^2(3-c^2)\ E\ t + 4\ ac(a^2+c^2)\ E(t+m-1)^{3/2} + O(m).$$

Since $E(t+m-1)^{3/2} \leq 2\ E\ t^{3/2} + 2\ m^{3/2} \leq 2\ E^{3/4}\ t^2 + 2\ m^{3/2}$ and $E\ t > m\ c^2(1-c^2)^{-1}$ (by the lemma),

$$(5) \qquad (6c^2-3-c^4)\ E\ t^2 \leq m^2c^4[1-2(3-c^2)(1-c^2)^{-1}] + 8\ ac(a^2+c^2)(E^{3/4}t^2+m^{3/2}) + O(m$$

Employing the lemma again, we have $E\ t^2 \geq E^2 t \geq m^2c^4(1-c^2)^{-2} \longrightarrow \infty$ and

$$(6) \qquad 6\ c^2 - 3 - c^4 \leq O(E^{-1/4}t^2) + O(m^{-1/2}).$$

Hence $6\ c^2 - 3 - c^4 \leq 0$ which is patently false for c^2 in $(3 - \sqrt{6},\ 1)$. Thus, $E\ t^2 = \infty$ for all sufficiently large m and the theorem is proved.

Theorem 2: Let $\{X_n\}$ be independent random variables with $P\{|X_n| \leq a < \infty\} =$ [illegible] $E\ X_n = 0$, $E\ X_n^2 = 1$ for $n \geq 1$. If t designates the smallest integer $n \geq m$ such that $|S_n| > c\ n^{1/\alpha}$, then $E\ t^2 < \infty$ for all $\alpha > 2$, $c > 0$ and $m \geq 1$.

Proof: For any $c > 0$ and $\alpha > 2$, if m is sufficiently large $c\ n^{1/\alpha} < 4^{-1}\ n^{1/2}$ for $n \geq m$. It follows therefore from the comparison technique and Theorem 1 that $E\ t^2 < \infty$ for all sufficiently large m. Consequently, $E\ t^2 < \infty$ for all $m \geq 1$, $\alpha > 2$, $c > 0$.

4. <u>Non-Zero Means</u>. Let the random variables $\{X_n\}$ of the basic process be independent with $EX_n = \mu_n$, $EX_n^2 = 1 + \mu_n^2$, $n \geq 1$. If $S_n = \sum_{i=1}^{n} X_i$ and t is a stopping variable with $E\ t < \infty$, then

$$E\,(S_t - \sum_{i=1}^{t} \mu_i)^2 = E\,t \tag{7}$$

by Theorem 2 of [2]. If, in addition $\mu_n = 0$, $E\,S_t = 0$ by Wald's theorem and the L.H.S. of (7) is just the variance of S_t, say $\sigma^2_{S_t}$. On the other hand if $\mu_n \neq 0$, this is no longer the case and $\sigma^2_{S_t}$ may even be infinite despite the finiteness of (7).

For example, let $P\{X_n = \mu+1\} = P\{X_n = \mu-1\} = \frac{1}{2}$, $\mu \neq 0$ and define t as the first index $n \geq m$ such that $(S_n - n\mu)^2 > 3n/4$. According to Theorem 1 of the preceding section, $E\,t^2 = \infty$ for all $m \geq m'$ (and it will now be stipulated that $m \geq m'$) while according to (7), $E(S_t - t\mu)^2 < \infty$. In view of the elementary inequality $\mu^2 Et^2 \leq 2E(S_t - t\mu)^2 + 2E\,S_t^2$, it follows that $E\,S_t^2 = \infty$. By Wald's theorem, $E\,S_t = \mu Et < \infty$ and thus $\sigma^2_{S_t} = \infty$.

s Even when both quantities are finite, no general inequality between $E(S_t - \sum_1^t \mu_i)^2$ and $\sigma^2_{S_t}$ obtains. It is not difficult to verify that

$\text{Cov}\,(2S_t - \sum_1^t \mu_i, \sum_1^t \mu_i) \leq 0$ is necessary and sufficient for $\sigma^2_{S_t} \leq E(S_t - \sum_1^t \mu_i)^2$ if $E(\sum_1^t \mu_i)^2 < \infty$, $E\sum_1^t E|X_i| < \infty$. When $EX_n = \mu$, $EX_n^2 = 1 + \mu^2$ and t is a stopping variable with $E\,t^2 < \infty$, the simple condition $\mu\,\text{Cov}\,(t, S_t) \leq 0$ implies $\sigma^2_{S_t} \leq E(S_t - t\mu)^2$. If $P\{X_n = 1\} = p = 1 - P\{X_n = -b\}$, $b > 0$ and t denotes the first $n \geq 1$ for which $X_n = 1$, then $S_t = -b(t-1)+1$. Since t and S_t are negatively correlated and $Et^2 < \infty$, $\sigma^2_{S_t} \leq E(S_t - t\mu)^2$ if $\mu \geq 0$, i.e., if $p \geq b/(b+1)$. Here, this condition is necessary as well.

5. Martingale Generalizations. In the following, the basic process $\{X_n\}$ will be postulated to satisfy $E|X_n| < \infty$, $E\{X_{n+1} | \mathcal{F}_n\} = 0$, $n \geq 1$ so that $S_n = \sum_1^n X_i$ is a martingale.

Theorem 3: Let $\{S_n, n \geq 1\}$ satisfy $E\{X_{n+1} | \mathcal{F}_n\} = 0$, $E \sup X_n^2 < \infty$. If $u_n^2 = E\{X_n^2 | \mathcal{F}_{n-1}\}$, define t as the first integer $n \geq m$ for which $S_n^2 > c^2 \sum_1^n u_j^2$ where $0 < c < 1$ and $m = 1, 2 \ldots$. Then $\int_{[t \leq n]} \sum_1^t u_j^2 = O(1)$ and $\int_{[t>n]} \sum_1^n u_j^2 = O(1)$ as $n \longrightarrow \infty$.

Proof: For any integer $k \geq m$, set $t' = \min(t,k)$ and define $z = \sup |X_n|$, $A_k = \{m < t \leq k\}$. By Theorem 1 of [2]

$$\int_{[t\leq k]} \sum_1^t u_j^2 + \int_{[t>k]} \sum_1^k u_j^2 = E \sum_1^{t'} u_j^2 = ES_{t'}^2 \leq \int_{A_k} [c(\sum_1^t u_j^2)^{\frac{1}{2}} + z]^2 + \int_{[t>k]} c^2 \sum_1^k u_j^2 + O(1)$$

Thus,

$$(1-c^2) [\int_{[t>k]} \sum_1^k u_j^2 + \int_{A_k} \sum_1^t u_j^2] \leq 2c(\int_{A_k} z^2)^{\frac{1}{2}} (\int_{A_k} \sum_1^t u_j^2)^{\frac{1}{2}} + O(1)$$

and the conclusion follows.

Corollary 1: If further, $P\{ \sum_1^\infty u_j^2 = \infty\} = 1$, $P\{t > k\} = o(1)$ and $E \sum_1^t u_j^2 < \infty$.

Corollary 2: If moreover $P\{ u_j^2 > \delta > 0\} = 1$, $j \geq 1$ then $Et < \infty$.

Corollary 3: If $\{X_n\}$ are independent with $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $E(\sup X_n^2) < \infty$, $\sum_1^\infty \sigma_n^2 = \infty$ and $t = 1\text{st } n \geq m$ such that $S_n^2 > c^2 \sum_1^n \sigma_j^2$, $0 < c < 1$, then

$P\{t < \infty\} = 1$ and $E(\sum_1^t \sigma_j^2) < \infty$. If $\sigma_n^2 > \delta > 0$, $Et < \infty$.

Corollary 3 generalizes corollary 2 of Theorem 2 of [2] wherein $\sigma_n^2 = 1$, $n \geq 1$.

Finally, the method of stopping rules will be utilized to generalize a Kolmogoroff inequality and to derive a result of Doob's [3, p.320] which does not follow from this inequality.

Theorem 4: Let $\{X_n, n \geq 1\}$ satisfy $EX_n^2 < \infty$, $E\{X_{n+1} | \mathcal{F}_n\} = 0$ and set $u_n^2 = E\{X_{n+1}^2 | \mathcal{F}_n\}$, $z = \sup|X_n|$. Then, if $\epsilon > 0$ for any positive integer k,

$$\int_{\left[\max_{n<k} S_n^2 \leq \epsilon^2\right]} \sum_1^k u_j^2 \leq E(\epsilon + z)^2$$

Proof: Let t = first $n \geq 1$ such that $S_n^2 > \epsilon^2$. Set $t' = \min(t,k)$. Then

$$E(\epsilon + z)^2 \geq ES_{t'}^2 = E\sum_1^{t'} u_j^2 \geq \int_{[t\geq k]} \sum_1^k u_j^2 = \int_{\left[\max_{n<k} S_n^2 \leq \epsilon^2\right]} \sum_1^k u_j^2$$

Corollary 1: If moreover $Ez^2 < \infty$, S_n diverges a.e. on $A = \left[\sum_1^\infty u_j^2 = \infty\right]$.

Proof: Let $t = 1\text{st } n \geq m$ for which $S_n^2 > \epsilon^2$. Then for $k \geq m$ it follows from the theorem that

$$E(\epsilon + z)^2 \geq \int_{[t \geq k]} \sum_m^k u_j^2 \geq \int_{A[t \geq k]} \sum_m^k u_j^2 \geq \int_{A[t=\infty]} \sum_m^k u_j^2$$

whence $P\{A[t=\infty]\} = 0$, i.e., $\sup_{n \geq m} |S_n - S_{m-1}| > \epsilon$, a.e. in A. Since m is arbitrary S_n diverges a.e. in A.

Corollary 2: If, further $Ez^2 < \infty$ and $P\{\sum_1^\infty u_n^2 = \infty\} = 1$, t is a bonafide stopping variable.

Footnotes

1. In [2] the terms are used synonymously but it is clearly desirable to make such a distinction.

2. For $c \geq 1$, the hypothesis of a uniform bound is superfluous and was not stipulated in [2].

References

1. Blackwell, D. and Friedman, D. (1964). A remark on the coin tossing game. Ann. Math. Stat. 35, 1345-1347.

2. Chow, Y. S., Robbins, H. and Teicher, H. (1965) Moments of randomly stopped sums. Ann. Math. Stat. 36, 789-799.

3. Doob, J. L. (1953) Stochastic Processes, Wiley, New York