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ON VALUES ASSOCIATED WITH A STOCHASTIC SEQUENCE

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1. Introduction

Let $\{z_n\}_1^\infty$ be a sequence of random variables with a known joint distribution. We are allowed to observe the z_n sequentially, stopping anywhere we please; the decision to stop with z_n must be a function of $z_1, \cdots, z_n$ only (and not of $z_{n+1}, \cdots$). If we decide to stop with z_n, we are to receive a reward $x_n = f_n(z_1, \cdots, z_n)$ where f_n is a known function for each n. Let t denote any rule which tells us when to stop and for which $E(x_t)$ exists, and let v denote the supremum of $E(x_t)$ over all such t. How can we find the value of v, and what stopping rule will achieve v or come close to it?

2. Definition of the γ_n sequence

We proceed to give a more precise definition of v and associated concepts. *We assume given always*

(a) a probability space $(\Omega, \mathfrak{F}, P)$ with points ω;
(b) a nondecreasing sequence $\{\mathfrak{F}_n\}_1^\infty$ of sub-Borel fields of $\mathfrak{F}$;
(c) a sequence $\{x_n\}_1^\infty$ of random variables $x_n = x_n(\omega)$ such that for each $n \geq 1$, x_n is measurable $(\mathfrak{F}_n)$ and $E(x_n^-) < \infty$.

(In terms of the intuitive background of the first paragraph, $\mathfrak{F}_n$ is the Borel field $\mathcal{B}(z_1, \cdots, z_n)$ generated by $z_1, \cdots, z_n$. Having served the purpose of defining the $\mathfrak{F}_n$ and x_n, the z_n disappear in the general theory which follows.) Any random variable (r.v.) t with values $1, 2, \cdots$ (not including ∞) such that the event $[t = n]$ (that is, the set of all ω such that $t(\omega) = n$) belongs to $\mathfrak{F}_n$ for each $n \geq 1$, is called a *stopping variable* (s.v.); $x_t = x_{t(\omega)}(\omega)$ is then a r.v. Let C denote the class of all t for which $E(x_t^-) < \infty$. We define the *value* of the stochastic sequence $\{x_n, \mathfrak{F}_n\}_1^\infty$ to be

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$$v = \sup_{t \in C} E(x_t). \tag{1}$$

Similarly, for each $n \geq 1$ we denote by C_n the class of all t in C such that $P[t \geq n] = 1$, and set

$$v_n = \sup_{t \in C_n} E(x_t). \tag{2}$$

Then

$$C = C_1 \supset C_2 \supset \cdots \quad \text{and} \quad v = v_1 \geq v_2 \geq \cdots ; \tag{3}$$

since $t = n \in C_n$, we have $v_n \geq E(x_n) > -\infty$.

For any family $(y_t, t \in T)$ of r.v.'s we define $y = \text{ess sup}_{t \in T}\, y_t$ if (a) y is a r.v. such that $P[y \geq y_t] = 1$ for each t in T, and (b) if z is any r.v. such that $P[z \geq y_t] = 1$ for each t in T, then $P[z \geq y] = 1$. It is known that there always exists a *sequence* $\{t_k\}_1^\infty$ in T such that

$$\sup_k y_{t_k} = \text{ess sup}_{t \in T}\, y_t. \tag{4}$$

We may therefore define for each $n \geq 1$ a r.v. γ_n measurable $(\mathfrak{F}_n)$ by

$$\gamma_n = \text{ess sup}_{t \in C_n} E(x_t | \mathfrak{F}_n); \tag{5}$$

then $\gamma_n \geq x_n$ (equalities and inequalities are understood to hold up to sets of P-measure 0) and $E(\gamma_n^-) \leq E(x_n^-) < \infty$.

It might seem more natural to consider, instead of C_n, the larger class $\tilde{C}_n$ of all s.v.'s t such that $P[t \geq n] = 1$ and $E(x_t)$ exists, that is $E(x_t^-)$ and $E(x_t^+)$ not both infinite. However, this would yield the same v_n and γ_n. For if $t \in \tilde{C}_n$, define

$$t' = \begin{cases} t \text{ if } E(x_t | \mathfrak{F}_n) \geq x_n, \\ n \text{ otherwise.} \end{cases} \tag{6}$$

Then setting $A = [E(x_t | \mathfrak{F}_n) \geq x_n]$, we have

$$E(x_{t'}^-) \leq E(x_n^-) + \int_A x_t^-. \tag{7}$$

But $-\infty < \int_A x_n \leq \int_A x_t$, so $\int_A x_t^- < \infty$. Hence, $E(x_{t'}^-) < \infty$ and $t' \in C_n$. Now $E(x_{t'} | \mathfrak{F}_n) = \max(x_n, E(x_t | \mathfrak{F}_n)) \geq E(x_t | \mathfrak{F}_n)$, and hence $E(x_{t'}) \geq E(x_t)$. *It follows that v_n and γ_n are unchanged if we replace C_n by $\tilde{C}_n$ in their definitions.*

3. Some lemmas

LEMMA 1. *For each $n \geq 1$ there exists a sequence $\{t_k\}_1^\infty$ in C_n such that*

$$x_n \leq E(x_{t_k} | \mathfrak{F}_n) \uparrow \gamma_n \qquad \text{as} \quad k \to \infty. \tag{8}$$

PROOF. Choose $\{t_k\}_1^\infty$ in C_n with $t_1 = n$ such that $\gamma_n = \sup_k E(x_{t_k} | \mathfrak{F}_n)$. By lemmas 2 and 3 below, we can assume that (8) holds.

LEMMA 2. *For any $t \in C_n$, define $t' =$ first $k \geq n$ such that $E(x_t | \mathfrak{F}_k) \leq x_k$. Then*

(a) $t' \leq t$, $t' \in C_n$,

(b) $E(x_{t'}|\mathfrak{F}_n) \geq E(x_t|\mathfrak{F}_n)$,

(c) $t' > j \geq n \Rightarrow E(x_{t'}|\mathfrak{F}_j) > x_j$.

PROOF. If $t = j \geq n$, then $E(x_t|\mathfrak{F}_j) = x_j$, so $t' \leq j$; hence, $t' \leq t$. Now

$$(9) \qquad E(x_{t'}^-) = \sum_{k=n}^{\infty} \int_{[t'=k]} x_k^- \leq \sum_{k=n}^{\infty} \int_{[t'=k]} E^-(x_t|\mathfrak{F}_k) \leq \sum_{k=n}^{\infty} \int_{[t'=k]} E(x_t^-|\mathfrak{F}_k) = E(x_t^-) < \infty,$$

so that $t' \in C_n$. Hence (a) holds. For any $A \in \mathfrak{F}_j$ with $j \geq n$,

$$(10) \qquad \int_{A[t' \geq j]} x_{t'} = \sum_{k=j}^{\infty} \int_{A[t'=k]} x_k \geq \sum_{k=j}^{\infty} \int_{A[t'=k]} E(x_t|\mathfrak{F}_k) = \int_{A[t' \geq j]} x_t.$$

Putting $j = n$ gives (b). For $t' > j$ we obtain $E(x_{t'}|\mathfrak{F}_j) \geq E(x_t|\mathfrak{F}_j) > x_j$, which gives (c).

Any $t' \in C_n$ satisfying (c) of lemma 2 will be called *n-regular.*

LEMMA 3. *Let $\{t_i\}_1^\infty \in C_n$ be n-regular for some fixed $n \geq 1$, and define $\tau_i = \max(t_1, \cdots, t_i)$. Then $\tau_i \in C_n$ is n-regular and*

$$(11) \qquad \max_{1 \leq k \leq i} E(x_{t_k}|\mathfrak{F}_n) \leq E(x_{\tau_i}|\mathfrak{F}_n) \leq E(x_{\tau_{i+1}}|\mathfrak{F}_n).$$

PROOF. That $\tau_i \in C_n$ is clear. For $j \geq n$ and $A \in \mathfrak{F}_j$,

$$(12) \qquad \begin{aligned} \int_{A[\tau_i \geq j]} x_{\tau_i} &= \sum_{k=j}^{\infty} \left(\int_{A[\tau_i = k \geq t_{i+1}]} x_{\tau_{i+1}} + \int_{A[\tau_i = k < t_{i+1}]} x_k \right) \\ &\leq \sum_{k=j}^{\infty} \left(\int_{A[\tau_i = k \geq t_{i+1}]} x_{\tau_{i+1}} + \int_{A[\tau_i = k < t_{i+1}]} x_{t_{i+1}} \right) \\ &= \int_{A[\tau_i \geq j]} x_{\tau_{i+1}}. \end{aligned}$$

For $j = n$, this gives

$$(13) \qquad E(x_{\tau_{i+1}}|\mathfrak{F}_n) \geq E(x_{\tau_i}|\mathfrak{F}_n) \geq \cdots \geq E(x_{\tau_1}|\mathfrak{F}_n) = E(x_{t_1}|\mathfrak{F}_n),$$

and hence, by symmetry,

$$(14) \qquad E(x_{\tau_i}|\mathfrak{F}_n) \geq \max_{1 \leq k \leq i} E(x_{t_k}|\mathfrak{F}_n).$$

To prove that τ_i is n-regular, we observe by the above that

$$(15) \qquad \tau_i \geq j \Rightarrow E(x_{\tau_i}|\mathfrak{F}_j) \leq E(x_{\tau_{i+1}}|\mathfrak{F}_j).$$

Since t_1 is n-regular,

$$(16) \qquad t_1 < j \Rightarrow x_j < E(x_{t_1}|\mathfrak{F}_j) = E(x_{\tau_1}|\mathfrak{F}_j) \leq \cdots \leq E(x_{\tau_i}|\mathfrak{F}_j),$$

and by symmetry,

$$(17) \qquad \tau_i > j \Rightarrow x_j < E(x_{\tau_i}|\mathfrak{F}_j).$$

so that $t' \in C$. The same argument without the $^-$ and with reversed inequality proves the inequality $E(x_t) \leq E(x_{t'})$.

A s.v. $t \in C$ is *optimal* if $v = E(x_t)$. A s.v. t in C is *regular* if it is 1-regular; that is, if for each $n \geq 1$, $t > n \Rightarrow E(x_t|\mathfrak{F}_n) > x_n$.

THEOREM 2. (a) *If $\sigma \in C$ and is regular, then it is optimal.* (b) *If $v < \infty$ and an optimal s.v. exists, then $\sigma \in C$ and is optimal and regular; moreover, σ is the minimal optimal s.v. and*

$$\sigma \geq n \Rightarrow E(x_\sigma|\mathfrak{F}_n) = E(\gamma_\sigma|\mathfrak{F}_n) = \gamma_n \qquad (n \geq 1). \tag{27}$$

PROOF. (a) If $\sigma \in C$ and is regular, then $\sigma > n \Rightarrow E(x_\sigma|\mathfrak{F}_n) > x_n$ for each $n \geq 1$. And for any $t \in C$, $\sigma = n$, $t \geq n \Rightarrow E(x_t|\mathfrak{F}_n) \leq \gamma_n = x_n$ by lemma 4. Hence by lemma 1 of [1], σ is optimal.

(b) Since $v < \infty$, $v_n = E(\gamma_n) < \infty$ for each $n \geq 1$. Let s in C be any optimal s.v., set $A = [s = n < \sigma]$, and suppose $P(A) > 0$. Then

$$\int_A \gamma_n > \int_A x_n + \epsilon \qquad \text{for some} \quad \epsilon > 0. \tag{28}$$

Choose $\{t_k\}_1^\infty$ in C_n by lemma 1; then $\int_A x_{t_k} \uparrow \int_A \gamma_n$, so that we can find k so large that $\int_A x_{t_k} > \int_A \gamma_n - \epsilon$. Set

$$s' = \begin{cases} s & \text{off } A \\ t_k & \text{on } A \end{cases}; \tag{29}$$

then it is easy to see that s' is a s.v. in C. But

$$E(x_{s'}) = \int_{\Omega - A} x_s + \int_A x_{t_k} > \int_{\Omega - A} x_s + \int_A x_n = E(x_s), \tag{30}$$

a contradiction. Hence $P(A) = 0$, and thus $P[\sigma \leq s] = 1$, so σ is a s.v. By lemma 5, $\sigma = \min(s, \sigma)$ is in C and σ is optimal and minimal.

For any $n \geq 1$, let $A = [E(x_\sigma|\mathfrak{F}_n) < \gamma_n, \sigma > n] \in \mathfrak{F}_n$. If $P(A) > 0$, then $\int_A \gamma_n > \int_A x_\sigma$, since $E(\gamma_n) \leq E(\gamma_1) = v < \infty$. By lemma 1, there exists t in C_n such that $\int_A x_t > \int_A x_\sigma$. Define

$$\tau = \begin{cases} t & \text{on } A \\ \sigma & \text{off } A \end{cases}; \tag{31}$$

then it is easy to see that τ is a s.v. in C and $E(x_\tau) > E(x_\sigma) = v$, a contradiction. Hence $P(A) = 0$, and by lemma 4,

$$\sigma > n \Rightarrow E(\gamma_\sigma|\mathfrak{F}_n) = E(x_\sigma|\mathfrak{F}_n) = \gamma_n > x_n, \tag{32}$$

so σ is regular and the last part of (b) holds.

6. Bounded stopping variables

The r.v.'s γ_n and the constants v_n are in general impossible to compute directly. To this end we define for any $N \geq 1$ and $1 \leq n \leq N$ the expressions

$$C_n^N = \text{all } t \in C_n \text{ such that } P[t \leq N] = 1; \; v_n^N = \sup_{t \in C_n^N} E(x_t); \tag{33}$$

$$\gamma_n^N = \operatorname*{ess\,sup}_{t \in C_n^N} E(x_t|\mathfrak{F}_n). \tag{34}$$

Then

(35) $$-\infty < E(x_n) = v_n^n \leq v_n^{n+1} \leq \cdots \leq v_n \text{ and } x_n = \gamma_n^n \leq \gamma_n^{n+1} \leq \cdots \leq \gamma_n,$$

so that we can define

(36) $$v_n' = \lim_{N\to\infty} v_n^N, \qquad \gamma_n' = \lim_{N\to\infty} \gamma_n^N,$$

and we have

(37) $$-\infty < E(x_n) \leq v_n' \leq v_n, \qquad x_n \leq \gamma_n' \leq \gamma_n.$$

By the argument of theorem 1 applied to the *finite* sequence $\{x_n\}_1^N$, we have

(38) $$\begin{aligned} &\gamma_N^N = x_N, \\ &\gamma_n^N = \max(x_n, E(\gamma_{n+1}^N|\mathfrak{F}_n)), \qquad (n = 1, \cdots, N-1), \end{aligned}$$

and $E(\gamma_n^N) = v_n^N$, so that γ_n^N and v_n^N are computable by recursion. By the monotone convergence theorem for expectations and conditional expectations, $E(\gamma_n') = v_n'$, and

(39) $$\gamma_n' = \max(x_n, E(\gamma_{n+1}'|\mathfrak{F}_n)), \qquad (n \geq 1).$$

Hence $\{\gamma_n'\}_1^\infty$ *satisfies the same recursion relation as does* $\{\gamma_n\}_1^\infty$. (In [2], $\gamma_n^N = \beta_n^N$, $\gamma_n' = \beta_n$.)

THEOREM 3. *If the condition* A^-: $E(\sup_n x_n^-) < \infty$ *holds, then*

(40) $$\gamma_n' = \gamma_n \quad \text{and} \quad v_n' = v_n, \qquad (n \geq 1).$$

PROOF. For any $t \in C_n$ and $A \in \mathfrak{F}_n$,

(41) $$\int_{A[t\leq N]} x_t \leq \int_A x_{\min(t,N)} + \int_{A[t>N]} x_N^-.$$

Since $E(x_{\min(t,N)}|\mathfrak{F}_n) \leq \gamma_n^N \leq \gamma_n'$,

(42) $$\int_{A[t\leq N]} x_t \leq \int_A \gamma_n' + \int_{A[t>N]} (\sup_m x_m^-).$$

Letting $N \to \infty$,

(43) $$\int_A x_t \leq \int_A \gamma_n', \qquad E(x_t|\mathfrak{F}_n) \leq \gamma_n', \qquad \gamma_n \leq \gamma_n',$$

so $\gamma_n = \gamma_n'$ and $v_n = v_n'$.

COROLLARY. *If* A^- *holds and* $\{x_n\}_1^\infty$ *is Markovian, and* $\mathfrak{F}_n = \mathfrak{B}(x_1, \cdots, x_n)$, *then* [illegible] $= E(\gamma_n|x_n)$.

PROOF. The Markovian property of $\{x_n\}_1^\infty$ implies (by downward induction on n) $\gamma_n^N = E(\gamma_n^N|x_n)$ which entails $\gamma_n' = E(\gamma_n'|x_n)$, and then $\gamma_n = E(\gamma_n|x_n)$. (The assumption A^- will be dropped in the corollary to theorem 9.)

7. Supermartingales

A sequence $\{y_n\}_1^\infty$ of r.v.'s is a *supermartingale* (or lower semimartingale) if for each $n \geq 1$, y_n is measurable $(\mathfrak{F}_n)$, $E(y_n)$ exists, $-\infty \leq E(y_n) \leq \infty$, and $E(y_{n+1}|\mathfrak{F}_n) \leq y_n$. We shall denote by D the class of all supermartingales $\{y_n\}_1^\infty$ such that $y_n \geq x_n$ for each $n \geq 1$. The sequences $\{\gamma_n\}_1^\infty$ and $\{\gamma_n'\}_1^\infty$ are in D.

THEOREM 4. *The sequence* $\{\gamma_n'\}$ *is the minimal element of* D.

PROOF. For any $\{y_n\}_1^\infty$ in D,

$$y_n \geq x_n = \gamma_n^n,$$

(44) $$y_{n-1} \geq E(y_n|\mathfrak{F}_{n-1}) \geq E(\gamma_n^n|\mathfrak{F}_{n-1}),$$

$$y_{n-1} \geq \max(x_{n-1}, E(\gamma_n^n|\mathfrak{F}_{n-1})) = \gamma_{n-1}^n, \cdots, y_i \geq \gamma_i^n, \cdots$$

so that

(45) $$y_i \geq \lim_{n\to\infty} \gamma_i^n = \gamma_i', \qquad (i \geq 1).$$

We shall define various types of "regularity" for elements of D, according to the class of s.v.'s t for which $E(y_t)$ is assumed to exist and the relation

(46) $$t \geq n \Rightarrow E(y_t|\mathfrak{F}_n) \leq y_n, \qquad (n \geq 1)$$

to hold. An element $\{y_n\}_1^\infty$ of D is said to be

(a) *regular* if for every s.v. t, $E(y_t)$ exists and (46) holds;

(b) *semiregular* if for every s.v. t such that $E(y_t)$ exists, (46) holds;

(c) *C-regular* if for every s.v. $t \in C$ (for which $E(y_t)$ necessarily exists), (46) holds.

Clearly, for elements of D, regular $\Rightarrow$ semiregular $\Rightarrow$ C-regular.

We shall use the notation A^+: $E(\sup_n x_n^+) < \infty$, A^*: $E(x_t)$ exists for every s.v. t. Clearly, $A^+ \Rightarrow A^* \Leftarrow A^-$.

LEMMA 6. *If* A^* *holds, then for any* $\epsilon > 0$ *and* $n \geq 1$, *there exists* $s \in C_n$ *such that*

(47) $$E(x_s|\mathfrak{F}_n) > \gamma_n - \epsilon \qquad \text{on} \quad [\gamma_n < \infty].$$

PROOF. Choose $\{t_k\}_1^\infty$ in C_n by lemma 1. On $[\gamma_n < \infty]$ define α = first $k \geq 1$ such that $E(x_{t_k}|\mathfrak{F}_n) > \gamma_n - \epsilon$, and set

(48) $$s = \begin{cases} t_\alpha \text{ on } [\gamma_n < \infty] \\ n \text{ elsewhere.} \end{cases}$$

Then $E(x_s)$ exists, and on $[\gamma_n < \infty]$, $E(x_s|\mathfrak{F}_n) > \gamma_n - \epsilon$. Hence,

(49) $$E(x_s) \geq \int_{[\gamma_n < \infty]} (\gamma_n - \epsilon) + \int_{[\gamma_n = \infty]} x_n > -\infty,$$

so that $s \in C_n$.

LEMMA 7. (a) *Condition* A^- *implies* $E(\gamma_t^-) = E((\gamma_t')^-) < \infty$ *for every s.v.* t, *and* (b) *condition* A^+ *implies* $E((\gamma_t')^+) \leq E(\gamma_t^+) < \infty$ *for every s.v.* t.

PROOF. (a) Since by theorem 3 $x_n \leq \gamma_n' = \gamma_n$, $\gamma_t^- = (\gamma_t')^- \leq \sup x_n^-$.

(b) Since

(50) $$\gamma_n^+ = \operatorname*{ess\,sup}_{t\in C_n} E^+(x_t|\mathfrak{F}_n) \leq E(\sup_j x_j^+|\mathfrak{F}_n),$$

then

(51) $$E((\gamma_t')^+) \leq E(\gamma_t^+) = \sum_{n=1}^\infty \int_{[t=n]} \gamma_n^+ \leq \sum_{n=1}^\infty \int_{[t=n]} E(\sup_j x_j^+|\mathfrak{F}_n) = E(\sup_j x_j^+).$$

THEOREM 5. (a) *If* $\{y_n\}_1^\infty \in D$ *and is C-regular, then* $y_n \geq \gamma_n$ *for each* $n \geq 1$;
(b) $A^* \Rightarrow \{\gamma_n\}_1^\infty$ *is semiregular;*
(c) A^- *or* $A^+ \Rightarrow \{\gamma_n\}_1^\infty$ *is regular;*
(d) $\{\gamma_n\}_1^\infty$ *is C-regular.*

PROOF. (a) If $\{y_n\}_1^\infty \in D$ and is C-regular, then

$$\gamma_n = \operatorname*{ess\,sup}_{t \in C_n} E(x_t|\mathfrak{F}_n) \leq \operatorname*{ess\,sup}_{t \in C_n} E(y_t|\mathfrak{F}_n) \leq y_n. \tag{52}$$

(b) Let τ be any s.v. such that $P[\tau \geq n] = 1$ and $E(\gamma_\tau)$ exists. For arbitrary $\epsilon > 0$, $k \geq n$, and $m \geq 1$, setting $A_m = [\gamma_n < m]$, we have

$$m \geq \int_{A_m} \gamma_n \geq \int_{A_m} \gamma_{n+1} \geq \cdots \geq \int_{A_m} \gamma_k \geq \cdots, \tag{53}$$

so that $\gamma_k < \infty$ on A_m. Hence, $\gamma_k < \infty$ on $A = [\gamma_n < \infty]$. By lemma 6, we can choose $t_k \in C_k$ such that

$$E(x_{t_k}|\mathfrak{F}_k) > \gamma_k - \epsilon \qquad \text{on } A. \tag{54}$$

Define

$$t = \begin{cases} t_k & \text{on } A[\tau = k], \\ \tau & \text{off } A. \end{cases} \tag{55}$$

Then $E(x_t)$ exists, and on A,

$$\begin{aligned} E(x_t|\mathfrak{F}_n) &= E\left(\sum_{k=n}^{\infty} I_{[\tau=k]} \cdot E(x_{t_k}|\mathfrak{F}_k)|\mathfrak{F}_n\right) \geq E\left(\sum_{k=n}^{\infty} I_{[\tau=k]}(\gamma_k - \epsilon)|\mathfrak{F}_n\right) \\ &= E(\gamma_\tau|\mathfrak{F}_n) - \epsilon; \end{aligned} \tag{56}$$

and therefore on A, by the remark preceding lemma 1,

$$\gamma_n = \operatorname*{ess\,sup}_{t \in \tilde{C}_n} E(x_t|\mathfrak{F}_n) \geq E(\gamma_\tau|\mathfrak{F}_n) - \epsilon \tag{57}$$

(recall that $\tilde{C}_n$ = all s.v.'s $t \geq n$ such that $E(x_t)$ exists). Hence,

$$\gamma_n \geq E(\gamma_\tau|\mathfrak{F}_n) \qquad \text{on } \Omega. \tag{58}$$

Now let t be any s.v. such that $E(\gamma_t)$ exists. Set $\tau = \max(t, n)$. Then if $E(\gamma_t^+) = \infty$, $E(\gamma_t^-) < \infty$, and hence

$$E(\gamma_\tau^-) = \int_{[t>n]} \gamma_t^- + \int_{[t \leq n]} \gamma_n^- < \infty, \tag{59}$$

while if $E(\gamma_t^+) < \infty$, then

$$E(\gamma_\tau^+) = \int_{[t>n]} \gamma_t^+ + \int_{[t \leq n]} \gamma_n^+ < \infty, \tag{60}$$

since

$$\infty > \int_{[t \leq n]} \gamma_t = \sum_{k=1}^{n} \int_{[t=k]} \gamma_k \geq \sum_{k=1}^{n} \int_{[t=k]} \gamma_n = \int_{[t \leq n]} \gamma_n. \tag{61}$$

Hence $E(\gamma_\tau)$ exists. By the previous result, $\gamma_n \geq E(\gamma_\tau|\mathfrak{F}_n)$, and hence,

$$t \geq n \Rightarrow \gamma_n \geq E(\gamma_\tau|\mathfrak{F}_n) = E(\gamma_t|\mathfrak{F}_n). \tag{62}$$

(c) This statement follows from (b) and lemma 7.

(d) For $0 \le b < \infty$, let $x_n(b) = \min(x_n, b)$, and let $\gamma_n^b\ (\le \gamma_n)$ denote γ_n for the sequence $\{x_n(b)\}_1^\infty$. As $b \to \infty$, $-x_n^- \le \gamma_n^b \uparrow \tilde{\gamma}_n$, say, where $\tilde{\gamma}_n \le \gamma_n$, and for any t in C_n, $x_t(b) \ge -x_t^-$, so that $E(x_t(b)|\mathfrak{F}_n) \uparrow E(x_t|\mathfrak{F}_n)$. Since $\tilde{\gamma}_n \ge \gamma_n^b \ge E(x_t(b)|\mathfrak{F}_n)$, $\tilde{\gamma}_n \ge E(x_t|\mathfrak{F}_n)$, and hence $\tilde{\gamma}_n \ge \gamma_n$, $\tilde{\gamma}_n = \gamma_n$. Now if $t \in C$, then by (c), $t \ge n \Rightarrow E(\gamma_t^b|\mathfrak{F}_n) \le \gamma_n^b \le \gamma_n$. As $b \to \infty$, since $\gamma_t^b \ge -x_t^-$ and $E(x_t^-) < \infty$, $t \ge n \Rightarrow E(\gamma_t|\mathfrak{F}_n) \le \gamma_n$, so $\{\gamma_n\}_1^\infty$ is C-regular.

COROLLARY 1. (a) *The sequence* $\{\gamma_n\}_1^\infty$ *is the minimal C-regular element of D.*

(b) *Condition A^* implies that* $\{\gamma_n\}_1^\infty$ *is the minimal semiregular element of D.*

(c) *Either A^- or A^+ implies that* $\{\gamma_n\}_1^\infty$ *is the minimal regular element of D.*

We remark that under A^-, $E(\sup_n \gamma_n^-) \le E(\sup_n x_n^-) < \infty$. Hence, by a well-known theorem, $\{\gamma_n\}_1^\infty$ is regular, and similarly for $\{\gamma_n'\}_1^\infty$. By theorems 4 and 5(a), $\{\gamma_n'\}_1^\infty = \{\gamma_n\}_1^\infty$, which gives an alternative proof of theorem 3.

COROLLARY 2. *If* $\gamma_n^b = \operatorname{ess\,sup}_{t \in C_n} E(\min(x_t, b)|\mathfrak{F}_n)$, *then*

$$\gamma_n = \lim_{b \to \infty} \gamma_n^b. \qquad (n \ge 1). \tag{63}$$

8. Almost optimal stopping variables

LEMMA 8. *If $v < \infty$, then for any $\epsilon > 0$, $P[x_n \ge \gamma_n - \epsilon$, i.o.$] = 1$.*

PROOF. Since $\infty > v = E(\gamma_1) \ge E(\gamma_2) \ge \cdots$, we have $P[\gamma_n < \infty] = 1$ for each $n \ge 1$. Choose any $\epsilon > 0$ and $r > 0$, and define for $n \ge 1$,

$$B_n = \left[E(x_{t_n}|\mathfrak{F}_n) > \gamma_n - \frac{\epsilon}{r}\right], \tag{64}$$

where $\{t_n\}_1^\infty$ is chosen by lemma 1 for each $n \ge 1$ so that $t_n \in C_n$ and $P(B_n) > 1 - 1/r$ (convergence a.e. $\Rightarrow$ convergence in probability). Define

$$B = [x_n < \gamma_n - \epsilon \quad \text{for all} \quad n \ge m] \tag{65}$$

where m is any fixed positive integer. Then

$$x_n \le \gamma_n - \epsilon I_B \qquad \text{for} \quad n \ge m, \tag{66}$$

so on B_n for any $n \ge m$,

$$\begin{aligned} \gamma_n - \frac{\epsilon}{r} < E(x_{t_n}|\mathfrak{F}_n) &\le E(\gamma_{t_n}|\mathfrak{F}_n) - \epsilon P(B|\mathfrak{F}_n) \\ &\le \gamma_n - \epsilon P(B|\mathfrak{F}_n) \qquad \text{by theorem 5(d).} \end{aligned} \tag{67}$$

Hence on B_n, $P(B|\mathfrak{F}_n) \le 1/r$, and therefore $P(BB_n) \le 1/r$. It follows that $P(B) \le P(BB_n) + P(\Omega - B_n) \le (1/r) + (1/r) = (2/r)$. Since r can be arbitrarily large, $P(B) = 0$, and therefore,

$$P[x_n \ge \gamma_n - \epsilon \text{ for some } n \ge m] = 1 \tag{68}$$

and

$$P[x_n \ge \gamma_n - \epsilon, \text{ i.o.}] = \lim_{m \to \infty} 1 = 1. \tag{69}$$

THEOREM 6. *For any $\epsilon \ge 0$, define*

$$s = \text{first } n \ge 1 \text{ such that } x_n \ge \gamma_n - \epsilon \ (s = \infty \text{ if no such } n \text{ exists}). \tag{70}$$

Assume the following: (a) $P[s < \infty] = 1$,

(b) $E(x_s)$ *exists*,

(c) $\liminf_{n\to\infty} \int_{[s>n]} E^+(\gamma_{n+1}|\mathfrak{F}_n) = 0$.

Then $E(x_s) \geq v - \epsilon$.

PROOF. We can assume $E(x_s) < \infty$. Since $\gamma_s \leq x_s + \epsilon$, $E(\gamma_s) < \infty$. Now

$$v = E(\gamma_1) = \int_{[s=1]} \gamma_s + \int_{[s>1]} E(\gamma_2|\mathfrak{F}_1) \tag{71}$$

$$= \int_{[s=1]} \gamma_s + \int_{[s=2]} \gamma_s + \int_{[s>2]} E(\gamma_3|\mathfrak{F}_2) = \cdots$$

$$= \int_{[1\leq s\leq n]} \gamma_s + \int_{[s>n]} E(\gamma_{n+1}|\mathfrak{F}_n) \leq \int_{[1\leq s\leq n]} \gamma_s + \int_{[s>n]} E^+(\gamma_{n+1}|\mathfrak{F}_n).$$

Letting $n \to \infty$, $v \leq E(\gamma_s) \leq E(x_s) + \epsilon$.

COROLLARY. *For any* $\epsilon \geq 0$, *define* s *by* (70). *Then*

(i) *for* $\epsilon > 0$, $A^+ \Rightarrow P[s < \infty] = 1$ *and* $E(x_s) \geq v - \epsilon$;

(ii) *for* $\epsilon = 0$, $\{A^+, P[s < \infty] = 1\} \Rightarrow E(x_s) = v$.

PROOF. Condition A^+ implies $\dot{v} < \infty$, and by lemma 8, this implies that $P[s < \infty] = 1$. Condition A^+ also implies (b) and (c).

THEOREM 7. *Let* $\{\alpha_n\}_1^\infty$ *be any sequence of r.v.'s such that* α_n *is* $(\mathfrak{F}_n)$ *measurable and* $E(\alpha_n)$ *exists for each* $n \geq 1$, *and such that*

(a) $\alpha_n = \max (x_n, E(\alpha_{n+1}|\mathfrak{F}_n))$,

(b) $P[x_n \geq \alpha_n - \epsilon \text{ i.o.}] = 1$ *for every* $\epsilon > 0$,

(c) $\{E^+(\alpha_{n+1}|\mathfrak{F}_n)\}_1^\infty$ *is uniformly integrable*,

(d) *either* $E(\sup_n \alpha_n^-) < \infty$, *or* A^+ *holds*.

Then for each $n \geq 1$, $\alpha_n \leq \gamma_n$.

PROOF. For $m \geq 1$, $A \in \mathfrak{F}_m$, and $\epsilon > 0$, define t = first $n \geq m$ such that $x_n \geq \alpha_n - \epsilon$. Then $P[m \leq t < \infty] = 1$. If the first part of (d) holds, then $E(\alpha_t^-) < \infty$, and since $x_t \geq \alpha_t - \epsilon$, it follows that $E(x_t^-) < \infty$, and hence, by theorem 5(d),

$$\int_A \alpha_t \leq \int_A x_t + \epsilon \leq \int_A \gamma_t + \epsilon \leq \int_A \gamma_m + \epsilon. \tag{72}$$

If A^+ holds, then $E(\alpha_t^+) \leq E(x_t^+) + \epsilon < \infty$, and the same result follows from theorem 5(c). Now

$$\int_A \alpha_m = \int_{A[t=m]} \alpha_t + \int_{A[t>m]} \alpha_{m+1} = \cdots = \int_{A[m\leq t\leq m+k]} \alpha_t \tag{73}$$

$$+ \int_{A[t>m+k]} \alpha_{m+k+1} \leq \int_{A[m\leq t\leq m+k]} \alpha_t + \int_{A[t>m+k]} E^+(\alpha_{m+k+1}|\mathfrak{F}_{m+k}).$$

Letting $k \to \infty$, it follows from (c) that

$$\int_A \alpha_m \leq \int_A \alpha_t \leq \int_A \gamma_m + \epsilon, \tag{74}$$

so since ϵ was arbitrarily small, $\int_A \alpha_m \leq \int_A \gamma_m$, and therefore, $\alpha_m \leq \gamma_m$.

COROLLARY. *Assume that* A^- *holds. If* $\{\alpha_n\}_1^\infty$ *is any sequence such that* α_n *is measurable* $(\mathcal{F}_n)$, $E(\alpha_n)$ *exists for each* $n \geq 1$, *and* (a), (b), *and* (c) *hold, then*

$$\alpha_n = \gamma_n. \tag{75}$$

PROOF. By theorems 7, 3, and 4, since A^- implies (d),

$$\gamma_n' \leq \alpha_n \leq \gamma_n = \gamma_n'. \tag{76}$$

9. A theorem of Dynkin

We next prove a slight generalization of a theorem of Dynkin [3]. Let $\{z_n\}_1^\infty$ be a homogeneous discrete time Markov process with arbitrary state space Z. For any nonnegative measurable function $g(\cdot)$ on Z, define the function $Pg(\cdot)$ by

$$Pg(z) = E(g(z_{n+1})|z_n = z), \tag{77}$$

and set

$$Qg = \max(g, Pg), \qquad Q_g^{k+1} = Q(Q^k g), \qquad (k \geq 0), \quad Q_g^\circ = g. \tag{78}$$

Then $g \leq Qg \leq Q^2 g \leq \cdots$, so

$$h = \lim_{N\to\infty} Q^N g \tag{79}$$

exists. Let $\mathcal{F}_n = \mathcal{B}(z_1, \cdots, z_n)$ and consider the sequence $\{x_n\}_1^\infty$ with $x_n = g(z_n)$.

THEOREM 8. *For the process defined above,* $\sup_t E(g(z_t)) = E(h(z_1))$.

PROOF. By theorem 3,

$$\gamma_1 = \gamma_1' = \lim_{N\to\infty} \gamma_1^N, \tag{80}$$

where

$$\begin{aligned}
\gamma_N^N &= g(z_N),\\
\gamma_{N-1}^N &= \max(g(z_{N-1}), E(g(z_N)|z_{N-1})) = Qg(z_{N-1}),\\
\gamma_{N-2}^N &= \max(g(z_{N-2}), E(Qg(z_{N-1})|z_{N-2})) = \max(g(z_{N-2}), PQg(z_{N-2}))\\
&= \max(g(z_{N-2}), Pg(z_{N-2}), PQg(z_{N-2})) = Q^2 g(z_{N-2}),\\
&\vdots\\
\gamma_1^N &= Q^{N-1} g(z_1) \to h(z_1) \quad \text{as} \quad N \to \infty.
\end{aligned} \tag{81}$$

Hence $\gamma_1 = h(z_1)$ and $v = E(\gamma_1) = E(h(z_1))$.

10. The triple limit theorem

LEMMA 9. *Assume* A^+ *holds, and define*

$$\begin{aligned}
x_n(a) &= \max(x_n, -a), \qquad (0 \leq a < \infty),\\
\gamma_n^a &= \operatorname*{ess\,sup}_{P[t \geq n]=1} E(x_t(a)|\mathcal{F}_n).
\end{aligned} \tag{82}$$

Then

$$\gamma_n = \lim_{a\to\infty} \gamma_n^a. \tag{83}$$

PROOF. Since $\gamma_n^a = \max(x_n(a), E(\gamma_{n+1}^a|\mathfrak{F}_n))$ and $\gamma_n(a) \downarrow \gamma_n^*$, say, as $a \to \infty$, where $\gamma_n^* \geq \gamma_n$, it follows from A^+ that $\gamma_n^* = \max(x_n, E(\gamma_{n+1}^*|\mathfrak{F}_n))$. For any $\epsilon > 0$ and $m \geq 1$, define s = first $n \geq m$ such that $x_n \geq \gamma_n^* - \epsilon$ ($= \infty$ if no such n exists). Then $\{\gamma_{\min(s,n)}^*\}_{n=m}^{\infty}$ is a martingale, since

$$(84)\qquad E(\gamma_{\min(s,n+1)}^*) = I_{[s>n]}E(\gamma_{n+1}^*|\mathfrak{F}_n) + I_{[s\leq n]}E(\gamma_s^*|\mathfrak{F}_n)$$
$$= I_{[s>n]}\cdot\gamma_n^* + I_{[s=m]}\cdot\gamma_m^* + \cdots + I_{[s=n]}\cdot\gamma_n^* = \gamma_{\min(s,n)}^*.$$

Since $E((\gamma_{\min(s,n)}^*)^+) \leq E(\sup_n x_n^+) < \infty$, and since $E((\gamma_m^*)^-) < \infty$, we have by a martingale convergence theorem,

$$(85)\qquad \gamma_{\min(s,n)}^* \to \text{a finite limit} \qquad \text{as} \quad n \to \infty,$$

and hence,

$$(86)\qquad \gamma_n^* \to \text{a finite limit on } [s = \infty] \qquad \text{as} \quad n \to \infty.$$

But on $[s = \infty]$, $\gamma_n^* > x_n + \epsilon$ for $n \geq m$, so

$$(87)\qquad \limsup_n x_n \leq \limsup_n \gamma_n^* - \epsilon \qquad \text{on} \quad [s = \infty].$$

Since $\gamma_n^a \leq E(\sup_{j\geq m} x_j(a)|\mathfrak{F}_n)$ for $n \geq m$,

$$(88)\qquad \limsup_n \gamma_n^* \leq \limsup_n \gamma_n^a \leq \sup_{j\geq m} x_j(a),$$

and hence,

$$(89)\qquad \limsup_n \gamma_n^* \leq \limsup_n x_n(a) = \max(\limsup_n x_n, -a),$$

and

$$(90)\qquad \limsup_n \gamma_n^* \leq \limsup_n x_n,$$

but $\gamma_n^* \geq x_n$. Hence,

$$(91)\qquad \limsup_n \gamma_n^* = \limsup_n x_n,$$

contradicting (87) unless $P[s = \infty] = 0$. Hence,

$$(92)\qquad P[x_n \geq \gamma_n^* - \epsilon, \text{ i.o.}] = 1,$$

and by theorem 7, $\gamma_n^* \leq \gamma_n$. Therefore, $\gamma_n^* = \gamma_n$.

THEOREM 9. *The random variables γ_n are equal to*

$$(93)\qquad \gamma_n = \lim_{b\to\infty} \lim_{a\to-\infty} \lim_{N\to\infty} \gamma_n^N(a, b),$$

where

$$(94)\qquad \gamma_n^N(a, b) = \operatorname*{ess\,sup}_{P[n\leq t\leq N]=1} E(x_t(a, b)|\mathfrak{F}_n)$$

and

$$(95)\qquad x(a, b) = \begin{cases} a & \text{if} \quad x < a, \\ x & \text{if} \quad a \leq x \leq b, \\ b & \text{if} \quad x > b. \end{cases}$$

PROOF. This follows from lemma 9, theorem 3, and corollary 2 of theorem 5.

COROLLARY 1. *The values v_n are equal to*

$$\lim_{b\to\infty} \lim_{a\to-\infty} \lim_{N\to\infty} v_n^N(a, b). \tag{96}$$

COROLLARY 2. *If* $\{x_n\}_1^\infty$ *is Markovian and* $\mathfrak{F}_n = \mathcal{B}(x_1, \cdots, x_n)$, *then*

$$\gamma_n = E(\gamma_n | x_n). \tag{97}$$

If the x_n *are independent, then*

$$E(\gamma_{n+1}|\mathfrak{F}_n) = E(\gamma_{n+1}) = v_{n+1}, \tag{98}$$

and the v_n *satisfy the recursion relation*

$$v_n = E\{\max (x_n, v_{n+1})\}, \qquad (n \geq 1). \tag{99}$$

PROOF. By induction $\gamma_n^N(a, b) = E(\gamma_n^N(a, b)|x_n)$ from $n = N$ down, as in the proof of the corollary of theorem 3. Letting N, a, b become infinite yields (97). Under independence,

$$E(\gamma_{n+1}|\mathfrak{F}_n) = E(E(\gamma_{n+1}|x_{n+1})|\mathfrak{F}_n) = E(\gamma_{n+1}) = v_{n+1}. \tag{100}$$

And from $\gamma_n = \max (x_n, E(\gamma_{n+1}|\mathfrak{F}_n)) = \max (x_n, v_{n+1})$, we obtain (99) on taking expectations.

11. Remarks on the independent case

THEOREM 10. *Let the* $\{x_n\}_1^\infty$ *be independent with* $\mathfrak{F}_n = B(x_1, \cdots, x_n)$. *Set* s = *first* $n \geq 1$ *such that* $x_n \geq \gamma_n - \epsilon$ *for* $\epsilon > 0$ ($= \infty$ *if no such* n *exists*). *Then*

$$v < \infty \Rightarrow P[s < \infty] = 1, \tag{101}$$

and if in addition $E(x_s)$ *exists, then*

$$E(x_s) \geq v - \epsilon. \tag{102}$$

PROOF. By lemma 8 and theorem 6, since by (87)

$$\int_{[s>n]} E^+(\gamma_{n+1}|\mathfrak{F}_n) = \int_{[s>n]} v_{n+1}^+ = v_{n+1}^+ P[s > n] \leq v^+ P[s > n] \to 0. \tag{103}$$

We remark that when $\epsilon = 0$ the conditions $v < \infty$, $P[s < \infty] = 1$, $E(x_s)$ exists, imply $E(x_s) = v$.

THEOREM 11. *Let the* $\{x_n\}_1^\infty$ *be independent with* $\mathfrak{F}_n = \mathcal{B}(x_1, \cdots, x_n)$, *and let* $\{\alpha_n\}_1^\infty$ *be any sequence of r.v.'s such that* α_n *is measurable* ($\mathfrak{F}_n$) *and* $E(\alpha_n)$ *exists,* $n \geq 1$. *If*

(a) $\alpha_n = \max (x_n, E(\alpha_{n+1}|\mathfrak{F}_n))$, $(n \geq 1)$,

(b) $P(x_n \geq \alpha_n - \epsilon \text{ i.o.}) = 1$ *for every* $\epsilon > 0$,

(c) $E(\alpha_{n+1}|\mathfrak{F}_n) = c_n$ = *constant, with* $E(\alpha_1) = c_1 < \infty$,

(d) A^+ *holds, or* $\liminf_n E(x_n) > -\infty$,

then

$$\alpha_n \leq \gamma_n, \qquad (n \geq 1). \tag{104}$$

PROOF. Define A and t as in theorem 7. Since

(105) $$c_n = E\{\max (x_{n+1}, c_{n+1})|\mathfrak{F}_n\} \geq c_{n+1},$$

we have

(106) $$\begin{aligned}\int_A \alpha_m &= \int_{A[m \leq t \leq m+k]} \alpha_t + \int_{A[t > m+k]} \alpha_{m+k+1} \\ &= \int_{A[m \leq t \leq m+k]} \alpha_t + \int_{A[t > m+k]} c_{m+k} \\ &\leq \int_{A[m \leq t \leq m+k]} \alpha_t + c_1 P[t > m+k].\end{aligned}$$

Hence under A^+ (or A^-),

(107) $$\begin{aligned}\int_A \alpha_m &\leq \liminf_{k\to\infty} \int_{A[m \leq t \leq m+k]} \alpha_t \leq \liminf_{k\to\infty} \int_{A[m \leq t \leq m+k]} x_t + \epsilon \\ &\leq \liminf_{k\to\infty} \int_{A[m \leq t \leq m+k]} \gamma_t + \epsilon = \int_A \gamma_t + \epsilon \leq \int_A \gamma_m + \epsilon\end{aligned}$$

by theorem 5(c), so $\alpha_m \leq \gamma_m$. If the second part of (d) holds, then since $c_n \downarrow c$, say, where $c \geq \liminf_n E(x_n) > -\infty$, and $x_t \geq c_t - \epsilon \geq c - \epsilon$, it follows that $E(x_t^-) < \infty$, so theorem 5(d) yields the same conclusion.

REMARKS. 1. Lemmas 2 and 3 are slight extensions of lemmas 1 and 2 of [2].

2. Theorem 1 has been proved independently by G. Haggstrom [4] when $E|x_n| < \infty$ and $E(\sup_n x_n^+) < \infty$, as have theorem 4, corollary 1(c) of theorem 5 under A^+, and the corollary of theorem 6. The latter was also proved by J. L. Snell [5].

3. We are greatly indebted to Mr. D. Siegmund for improvements in the statement and proof of many of our results. In particular, theorem 9 is largely due to him.

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