On Some Spectral Properties of a Transition Matrix I

by

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Errata Sheet for

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page 2 the definition of a sub regular measure should be:

$\{\alpha_j\}$ is a <u>sub regular measure</u> if

$$\alpha_j \leq \sum_{k=1}^{\infty} \alpha_k p_{kj}$$

for every j.

page 15 the proof given for lemma 5 is incorrect. The following is a correct proof:

Suppose $u_j < 0$. Then $u_j^+ = 0 \leq \sum_k p_{jk} u_k^+$

Suppose $u_j > 0$. Then $u_j^+ = u_j \leq \sum_k p_{jk} u_k \leq \sum_k p_{jk} u_k^+$. QED

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Let $P = (p_{ij})$ be the transition matrix of an infinite, irreducible, discrete parameter Markov chain with state space $I = \{1,2,\ldots\}$. In this paper we will consider P as a transformation acting on different function spaces (here sequence spaces since the Markov chain is in discrete time) and study some of the associated spectral properties.

Definition: ℓ^∞ is the space of all infinite vectors $x = (x_1, x_2, \ldots)$ of complex numbers satisfying

$$||x||_\infty = \sup_n |x_n| < \infty.$$

We define a transformation T on ℓ^∞ by

$$(Tx)_j = \sum_{k=1}^{\infty} p_{jk}\, x_k \qquad j = 1,2,\ldots\,.$$

Lemma 1: T is a bounded linear transformation on ℓ^∞ into ℓ^∞.

Proof: It is clear that T is linear.

$$||Tx||_\infty = \sup_n |(Tx)_n| = \sup_n |\sum_{k=1}^\infty p_{nk} x_k| \le$$

$$\sup_n \sum_{k=1}^\infty p_{nk}|x_k| \le \sup_n \sum_{k=1}^\infty p_{nk} \sup_\ell |x_\ell|$$

$$= \sup_n \sum_{k=1}^\infty p_{nk} ||x||_\infty = ||x||_\infty < \infty .$$

Thus $Tx \in \ell^\infty$. Also, $||Tx||_\infty \le ||x||_\infty$ implies T is bounded (and therefore continuous). QED

Corollary: $||T|| \le 1$.

Proof: We know that $||Tx||_\infty \le ||x||_\infty$ for every $x \in \ell^\infty$. Therefore, (see Taylor [8], p. 86)

$$||T|| = \sup_{||x||_\infty \neq 0} \frac{||Tx||_\infty}{||x||_\infty} \le 1. \quad \text{QED}$$

Definition: A non-negative, not identically zero, real valued function $\{\alpha_j\}$ on I is called a super regular measure if

$$\alpha_j \ge \sum_{k=1}^\infty \alpha_k p_{kj} \quad \text{for } j = 1,2,\ldots$$

$\{\alpha_j\}$ is on a regular measure if equality holds for every j, and $\{\alpha_j\}$ is a sub regular measure if its negative is a super regular measure. It is known that every irreducible Markov chain possesses at least one super regular measure (see Kendall [6], p. 140). If the chain is recurrent, then every

Since the existence of a super regular measure is assured for an irreducible chain, we will assume that a particular super regular measure $\{\alpha_j\}$ has been chosen, and it will remain fixed throughout the remainder of the paper.

Lemma 2: If $\{\alpha_j\}$ is a super regular measure in a non recurrent chain, then $\sum_{j=1}^{\infty} \alpha_j = \infty$.

Proof: Suppose $\sum_{j=1}^{\infty} \alpha_j < \infty$. By super regularity we have

$$* \qquad \sum_{j=1}^{\infty} \alpha_j \geq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \alpha_k p_{kj} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \alpha_k p_{kj} = \sum_{k=1}^{\infty} \alpha_k .$$

If $\alpha_j > \sum_{k=1}^{\infty} \alpha_k p_{kj}$ for some j, we would obtain strict inequality in * which would be a contradiction. Therefore, under our assumption, the super regular measure $\{\alpha_j\}$ is regular. But

$$\alpha_j = \sum_{k=1}^{\infty} \alpha_k p_{kj} = \sum_{k=1}^{\infty} \alpha_k p_{kj}^{(n)}$$

for every n. Thus, using the dominated convergence theorem,

$$\alpha_j = \lim_{n \to \infty} \sum_{k=1}^{\infty} \alpha_k p_{kj}^{(n)} = \sum_{k=1}^{\infty} \alpha_k \lim_{n \to \infty} p_{kj}^{(n)} = 0$$

since the chain is non recurrent. But this implies that $\alpha_j = 0$ for $j = 1,2,\ldots$ which violates the definition of a super regular measure.

Hence $\sum_{j=1}^{\infty} \alpha_j = \infty$. QED

Definition: For $1 \leq p < \infty$, $\ell^p(\alpha)$ is the space of all infinite vectors $y = (y_1, y_2, \ldots)$ of complex numbers satisfying

$$||y||_p = \left(\sum_{j=1}^{\infty} |y_j|^p \alpha_j\right)^{1/p} < \infty.$$

We define a transformation T^p on $\ell^p(\alpha)$ by

$$(T^p_y)_j = \sum_{k=1}^{\infty} p_{jk} y_k \quad j = 1,2,\ldots .$$

We will let $\ell^\infty(\alpha) = \ell^\infty$, and $T^\infty = T$. The usual proof (see Taylor [8], p. 100) of the completeness of the spaces ℓ^p , $p \geq 1$, can be applied here without revision to the spaces $\ell^p(\alpha)$, $p \geq 1$. It follows that our spaces $\ell^p(\alpha)$, $p \geq 1$, are Banach spaces. In particular, $\ell^2(\alpha)$ is a Hilbert space. It would be of interest to study in detail the properties of the sequences which are elements of $\ell^p(\alpha)$. This will not be done in this paper. We do note, however, that in non recurrent and null recurrent chains (where super regular measures are infinite) constant sequences are not in $\ell^p(\alpha)$. Indeed, by the very definition of $\ell^p(\alpha)$, we must have that $|u_j|^p$ converges to zero at a rate which is rapid enough to insure that

$\sum_{j=1}^{\infty} |u_j|^p \alpha_j < \infty$.

The space $\ell^1(\alpha)$ has been used in the study of potential theory for discrete Markov chains (see Kemeny and Snell [5], p. 200). Transformations T_p (on the usual ℓ^p spaces) defined by

$$(T_p\ x)_j = \sum_k x_k\left(\frac{\alpha_k}{\alpha_j}\right)^{1/q} p_{jk},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, have been studied by Kendall [6] for the case $p = 2$, and for $p > 1$ by Vere-Jones [11].

Lemma 3: T^p is a bounded linear transformation of $\ell^p(\alpha)$ into $\ell^p(\alpha)$. Moreover $||T^p|| \leq 1$ for every $p \in [1, \infty]$.

Proof: T^p is clearly linear. Suppose $x \in \ell^1(\alpha)$. Then

$$||Tx||_1 = \sum_{k=1}^{\infty} |(Tx)_k|\alpha_k = \sum_{k=1}^{\infty} |\sum_{i=1}^{\infty} p_{kj}\ x_j|\alpha_i \leq \sum_{k=1}^{\infty}\ \sum_{j=1}^{\infty} p_{kj}|x_j|\alpha_k$$

$$= \sum_{j=1}^{\infty} |x_j|\left(\sum_{k=1}^{\infty} \alpha_k p_{kj} \right) \leq \sum_{j=1}^{\infty} |x_j|\alpha_j = ||x||_1 < \infty .$$

Therefore, T maps $\ell^1(\alpha)$ into $\ell^1(\alpha)$ and is bounded. We have already shown that the same result is true for T operating on $\ell^\infty(\alpha)$. It follows from the M. Riesz Convexity Theorem (see Taylor [8] pp. 221-224, especially problem 1 p. 224) that T^p is a bounded linear transformation of $\ell^p(\alpha)$ into $\ell^p(\alpha)$ for every p, $1 \leq p \leq \infty$, and that $||T^p|| \leq 1$. QED

Example 1: Suppose the Markov Chain under consideration is positive recurrent. Let u be a real valued, non negative, super regular function

$(u_j \geq \sum_{k=1}^{\infty} p_{jk} u_k$ for every $k)$. Then u is regular and constant, say

$u_j = c$ for every j, and

$$||u||_p = \left(\sum_{j=1}^{\infty} |u_j|^p \alpha_j \right)^{1/p} < \infty.$$

Hence $u \in \ell_p(\alpha)$, $1 \leq p < \infty$ and we have $||T^p|| = 1$. The same example shows $||T^\infty|| = 1$. Thus, for a positive recurrent chain $||T^p|| = 1$, $1 \leq p \leq \infty$.

Example 2: Consider a Markov Chain with state space $I = \{1,2,3,\ldots\}$ and transition probabilities given by

$$\begin{aligned} p_{j,1} &= p_j > 0 \\ & \qquad\qquad\qquad j = 1,2,\ldots . \\ p_{j,j+1} &= 1-p_j > 0 \end{aligned}$$

Let $\alpha_1 = 1$, $\alpha_j = \prod_{k=1}^{j} (1-p_j)$ for $j = 2,3,\ldots$, and $\alpha_\infty = \lim_{j \to \infty} \alpha_j$.

Furthermore, let $\sigma = \lim_{k \to \infty} \sum_{j=1}^{k} \alpha_j$. Now, as is well known (see for example Kemeny and Snell [5], p. 249-250), this Markov chain will be non recurrent if and only if $\alpha_\infty > 0$. If this is the case then $\{\alpha_j\}$ is a super regular measure, with

$$\alpha_1 > \sum_{j=1}^{\infty} \alpha_j p_{j1} = \alpha_1 - \alpha_\infty, \text{ and } \alpha_k = \sum_{j=1}^{\infty} \alpha_j p_{jk}, \text{ for } k = 2,3,\ldots .$$

If $\alpha_\infty = 0$, the chain is recurrent and $\{\alpha_j\}$ is its (essentially) unique regular measure. If $\sigma = +\infty$, the chain is null recurrent and if $\sigma < \infty$, the chain is positive recurrent with $\{\alpha_i/\sigma\}$ as its stationary absolute distribution.

We will consider first the non recurrent case. Here $\{\alpha_j\}$ is a super regular measure (there are no regular measures), but there is no guarantee that this super regular measure is unique. Let us search for another; call it $\{\beta_j\}$. We must have

$$* \qquad \sum_{k=1}^{\infty} \beta_k \, p_{kj} = \beta_{j-1}(1-p_{j-1}) \leq \beta_j \qquad \text{for } j = 2,3,\ldots$$

and

$$** \qquad \sum_{k=1}^{\infty} \beta_k \, p_{k1} = \sum_{k=1}^{\infty} \beta_k \, p_k \leq \beta_1.$$

Any non decreasing sequence of non negative numbers will satisfy equations *. If order to obtain a sequence which will satisfy equation ** as well, we can choose $p_k = p^k$ where $0 < p < 1$, and $\beta_k = \beta^k$ where β is such that $\beta p < 1-p$.

For this choice of p_k we have

$$\alpha_\infty = \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \prod_{m=1}^{n} (1-p^m).$$

Now

$$\log \alpha_\infty = \lim_{n \to \infty} \sum_{m=1}^{n} \log(1-p^m) = \sum_{m=1}^{\infty} \log(1-p^m)$$

$$= - \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{(p^m)^k}{k} = - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=1}^{\infty} (p^k)^m$$

$$= - \sum_{k=1}^{\infty} \frac{1}{k} \frac{p^k}{1-p^k} \text{, and}$$

$$\sum_{k=1}^{\infty} \frac{1}{k} \frac{p^k}{1-p^k} \leq \frac{1}{1-p} \sum_{k=1}^{\infty} \frac{p^k}{k} = \frac{1}{1-p} \log (1-p).$$

Thus the series converges when $0 < p < 1$. It follows that $\alpha_\infty > 0$ for $0 < p < 1$, and that the chain is non recurrent for $0 < p < 1$.

Equations * are satisfied, and equation ** becomes

$$\sum_{k=1}^{\infty} \beta_k p_k = \sum_{k=1}^{\infty} (\beta p)^k = \frac{\beta p}{1-\beta p} < \beta.$$

We see that $\beta_k = \beta^k$, $0 < \beta < \frac{1-p}{p}$, yields a super regular measure for this Markov chain.

Now let $u \in \ell^q(\beta)$, $1 < q < \infty$. We have

$$(||T^q u||_q)^q = \sum_{k=1}^{\infty} |(T^q u)_q|^q \beta_k = \sum_{k=1}^{\infty} | \sum_{j=1}^{\infty} p_{kj} u_j |^q \beta_k$$

$$\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} p_{kj} |u_j|^q \beta_k = \sum_{k=1}^{\infty} [|u_1|^q p^k + |u_{k+1}|^q (1-p)^k] \beta^k$$

$$\leq |u_1|^q \frac{\beta p}{1-\beta p} + \frac{1}{\beta} [(||u||_q)^q - |u_1|^q \beta]$$

$$= \frac{1}{\beta} (||u||_q)^q + |u_1|^q \frac{2p\beta-1}{1-p\beta} .$$

Choose β and p so that $1 < \beta < \min(\frac{1-p}{p}, \frac{2}{3p})$, for example $\beta = 2$ and $p = \frac{1}{4}$ will work. Now $\frac{2p\beta-1}{1-p\beta} < 1$, and we have

$$(||Tu||_q)^q \leq \frac{1}{\beta}(||u||_q)^q, \text{ and } ||T^q|| \leq (\frac{1}{\beta})^{1/q} < 1.$$

If we choose $\beta = 1$, we obtain $||T^q|| = 1$. Thus we have examples of non recurrent chains for which $||T^q|| < 1$ and $||T^q|| = 1$.

Consider the bounded linear operator T^p acting on the complex sequence space $\ell^p(\alpha)$, $1 \leq p \leq \infty$. The <u>point spectrum</u> of T^p is

$$\sigma(T^p) = \{\text{complex numbers } \lambda\text{: } T^p x = \lambda x \text{ for some non-zero } x \in \ell^p(\alpha)\}.$$

The element x of $\ell^p(\alpha)$ involved in this definition is called an <u>eigenvector</u> of T^p and the scalar λ is the <u>eigenvalue</u> of which x belongs. Eigenvalues which lie on the unit circle have been of particular interest. If the Markov chain under consideration is finite with period d, we have the following classical results which follow from the Perron-Frobenius theorems (see Gantmacher [3], Chapter 13; or Rosenblatt [9], pp. 44-51):

For the stochastic matrix P, one is always an eigenvalue of maximum absolute value. There are d eigenvalues of absolute value one, and they are exactly the dth roots of unity. Moreover, a non negative matrix is stochastic if and only if it has the eigenvector $(1,1,\ldots,1)$ for the eigenvalue 1. We will show that some similar results hold in the infinite case when the chain is positive recurrent (every finite irreducible Markov chain is positive recurrent), but that for non recurrent or null recurrent chains that part of the point spectrum on the unit circle may well be empty. Following the notation of Chung [1], p. 12, let $C_1, C_2, \ldots, C_d$ be the cyclic subclasses of I; so that

$$\bigcup_{r=1}^{d} C_r = I, \quad C_r \cap C_s = \phi \quad \text{if} \quad r \neq s,$$

and, if $i \in C_r$ and $p_{ij}^{(n)} > 0$, then $j \in C_s$ where $r + n \equiv s \pmod{d}$. d is the period of the Markov chain. Suppose the chain is positive recurrent. Let u be the sequence defined by

$$u_k = \lambda_r \quad \text{if} \quad k \in C_r$$

where $\lambda_1, \ldots, \lambda_d$ are complex constants each having finite absolute value.

$$(||u||_p)^p = \sum_{k=1}^{\infty} |u_k|^p \alpha_k = \sum_{r=1}^{d} |\lambda_r|^p \sum_{k\in C_r} \alpha_k$$

$$\leq d \max_{r=1,\ldots,d} |\lambda_r|^p \sum_{k=1}^{\infty} \alpha_i < \infty .$$

Hence $u \in \ell^p(\alpha)$ for each $p \in [1,\infty)$. Also, $u \in \ell^\infty(\alpha)$, since

$$||u||_\infty = \sup_k |u_k^{(k)}| < \infty.$$

Suppose $k\in C_r$. Then

$$(T\overset{p}{u})_k = \sum_{j=1}^{\infty} p_{kj} u_j = \sum_{j\in C_{r+1}} p_{kj}\, \lambda_{r+1} = \lambda_{r+1}$$

Thus T^p here acts as a kind of rotation or as a shift operator since the value of u_k is λ_r, and that of $(T^p u)_k$ is λ_{r+1}.

Consequently, let $\lambda_1,\ldots,\lambda_d$ be the dth roots of unity, i.e.,

$$\lambda_r = e^{2\pi i \frac{r}{d}} \qquad r = 1,\ldots,d.$$

Now we have, for $k \in C_r$,

$$(T\overset{p}{u})_k = e^{2\pi i \frac{r+1}{d}} \sum_{j\in C_{r+1}} p_{kj} = e^{2\pi i \frac{r}{d}}\, e^{2\pi i \frac{1}{d}} = e^{2\pi i \frac{1}{d}} u_k .$$

Thus we have shown that

$$T\overset{p}{u} = e^{2\pi i \frac{1}{d}} u,$$

and it follows that $e^{2\pi i \frac{1}{d}} \in \sigma(T^p)$.

We can generalize these ideas in a very straightforward fashion. Let V^s be the function on I defined by

$$V_k^s = e^{2\pi i s \frac{r}{d}} \quad \text{if } k \in C_r.$$

Clearly $V^s \in \ell^p(d)$. For $k \in C_r$, we have

$$(T^p V^s)_k = \sum_{j=1}^{\infty} p_{ij} V_j^s = e^{2\pi i s \frac{r+1}{d}} = e^{2\pi i s \frac{r}{d}} e^{2\pi i s \frac{1}{d}} = e^{2\pi i s \frac{1}{d}} V_k^s .$$

Therefore, $e^{2\pi i s \frac{1}{d}} \in \sigma(T^p)$, and we have proved the following theorem.

Theorem 1: In a positive recurrent Markov chain,

$$\{e^{2\pi i s/d}: s = 1,2,\ldots,d\} \subset \sigma(T^p), \quad 1 \leq p \leq \infty.$$

Lemma 4: Let $u \in \ell^p(\alpha)$ $(1 \leq p < \infty)$ be real, non-negative, and sub regular with respect to the operator T^p, i.e.,

$$(T^p u)_k \geq u_k \quad k = 1,2,3,\ldots .$$

Then $(T^p u)_k = u_k$ for $k = 1,2,\ldots$,

Proof: Using the sub regularity of u, we have

$$(||u||_p)^p = \sum_{j=1}^{\infty} (u_j)^p \alpha_j \leq \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{\infty} p_{jk} u_k \Big)^p \alpha_j .$$

Now $\sum_{k=1}^{\infty} p_{jk}u_k$ is the expected value of the discrete random variable u with respect to the probability mass function $\{p_{j1}, p_{j2}, \dots\}$. This expected value is finite for each j since

$$\sum_{k=1}^{\infty} p_{jk}u_k = (T^p u)_j < \infty .$$

The function $g(u) = u^p$ is convex for $p \geq 1$. Therefore, by Jensen's inequality (Loeve [7], p. 159 e)

$$\sum_{k=1}^{\infty} p_{jk}u_k^p \geq \left(\sum_{k=1}^{\infty} p_{jk} u_k \right)^p$$

Hence,

$$(||u||_p)^p \leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} p_{jk}u_k \right)^p \alpha_j \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_{jk} u_k^p \alpha_j$$

$$= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} p_{jk}\alpha_j \right) u_k^p \leq \sum_{k=1}^{\infty} u_k^p \alpha_k = \left(||u||_p \right)^p .$$

If, for any j, $u_j < \sum_{k=1}^{\infty} p_{jk}u_k$, we would obtain strict inequality in this expression which would be a contradiction. Thus

$$u_j = \sum_{k=1}^{\infty} p_{jk} u_k, \quad j = 1,2,\dots . \quad \text{QED}$$

Lemma 5: Let u be a real valued function in $\ell^p(\alpha)$, and let u^+ be the positive part of u. If $u \leq T^p u$, then $u^+ \leq T^p u^+$.

Proof:

$$u_j^+ = \begin{cases} u_j & \text{if } u_j \geq 0 \\ 0 & \text{if } u_j < 0 \end{cases}$$

$$(T^p u^+)_j = \sum_{k=1}^{\infty} p_{jk} u_k^+ \geq \sum_{k=1}^{\infty} p_{jk} u_k = (T^p u)_j = u_j \geq u_j + \text{QED}$$

Lemma 6: If $u \in \ell^p(\alpha)$ and $u = T^p u$, then u is identically equal to a constant.

Proof: Suppose u is real. By lemma 5, we know that $u^+ \leq T^p u^+$ and by lemma 4, it follows that $u^+ = T^p u^+$. Therefore, either $u_j^+ = 0$ for every j, in which case u is non positive; or $u_j^+ > 0$ for every j, in which case u is a strictly positive function. Thus we see that either $u_j \leq 0$ for every j, or $u_j > 0$ for every j. In the same way, by considering the function V defined by $V_j = u_j - a$ (which also satisfies $T^p V = V$) where a is an arbitrary constant, we see that either $u_j \leq a$ for every j, or $u_j > a$ for every j. Now suppose u is not identically equal to a constant. Then there exist two states ℓ and k such that $u_\ell < u_k$. Choose any number a so that $u_\ell < a < u_k$. This contradicts $u_j \leq a$ or $u_j > a$ for every j. Thus u is identically equal to a constant. The result follows by applying what has just been shown to the real and imaginary parts of u. QED

Theorem 2: Suppose the Markov chain is null recurrent or non recurrent. Then there are no eigen values of T^p ($1 \le p < \infty$) on the unit circle (having modulus 1).

Proof: Suppose there exists a complex number λ such that $T^p_u = \lambda u$ for some non zero u in $\ell^p(\alpha)$, and for which $|\lambda| = 1$. For every $j = 1,2,3,\ldots$ we have

$$(T^p|u|)_j = \sum_{k=1}^{\infty} p_{jk} |u_k| \ge |\sum_{k=1}^{\infty} p_{jk} u_k| = |\lambda u_j| = |u_j| \; .$$

It follows from lemmas 4 and 6 that $|u_j|$ is identically equal to a constant, say $|u_j| \equiv c$. But $u \in \ell^p(\alpha)$. Therefore,

$$\infty > (||u||_p)^p = \sum_{j=1}^{\infty} |u_j|^p \alpha_j = c^p \sum_{j=1}^{\infty} \alpha_j \; .$$

But $\sum_{j=1}^{\infty} \alpha_j = \infty$ for a null recurrent chain and for a non recurrent chain.

Hence $c = 0$ and this implies that $u_j = 0$ for every j. This is a contradiction. QED

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