On the Non-Central Distributions of the Largest Roots of Two Matrices in Multivariate Analysis

by

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1. Introduction and Summary. Tests of three multivariate hypotheses based on Professor Roy's largest characteristic root criterion [18] have been shown to have monotonicity of power, namely, (i) that of equality of covariance matrices of two p-variate normal populations, (ii) that of equality of p-dimensional mean vectors of ℓ p-variate normal populations having a common covariance matrix, and (iii) that of independence between a p-set and a q-set of variates in a (p+q)-variate normal population [1], [2], [3], [19]. To facilitate these tests, the cdf of the largest characteristic root criterion under the null hypothesis (i)-(iii) has been studied by Pillai [6], [8], [10], [11], [12], [15] with a view to obtaining an approximation to the cdf useful for computing the upper percentage points. Further, for test (iii), the cdf of the largest of two characteristic roots under a single non-zero population characteristic root has been obtained by Pillai [14] and the power function studied. In the present paper, this study is extended to the case of the cdf of the largest of two roots when both population roots are non-zero and that of the largest of three roots

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under a single non-zero population root. In addition, similar non-central cdf's are obtained for test (ii) for the case of two roots with two non-null population roots, and that of three roots with single non-zero population root. Tabulations of the power functions have been made for both tests for various values of the parameters.

Further, Pillai's general expression [15] approximating at the upper end, the cdf of the largest root under the null hypotheses, has been used to compute the upper 5 and 1 percent points of the largest of eleven and twelve roots.

2. Non-central cdf of the largest root for test (iii). Let the columns of $\begin{bmatrix} X \\ Y \end{bmatrix}$ be ν independent normal $(p+q)$-variates, ($p \leq q$, $p+q \leq \nu$, $\nu + 1 = n'$, the sample size) with zero means and covariance matrix

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{bmatrix} \begin{matrix} p \\ q \end{matrix} \quad .$$
$$\qquad\qquad \begin{matrix} p & \quad q \end{matrix}$$

Let $R = \text{diag}(r_i)$, where $r_1^2, \ldots, r_p^2$ are the characteristic roots of the equation

$$|XY'(YY')^{-1}YX' - r^2 XX'| = 0 \tag{2.1}$$

and $P = \text{diag}(\rho_i)$, where $\rho_1^2, \ldots, \rho_p^2$ are the characteristic roots of the equation

$$|\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}' - \rho^2 \Sigma_{11}| = 0 \quad . \tag{2.2}$$

Then, the distribution of $r_1^2,\ldots,r_p^2$ is given by Constantine [4], in the following form: [5]

$$(2.3)\qquad |I-P^2|^{\frac{\nu}{2}}\ {}_2F_1(\tfrac{1}{2}\nu,\ \tfrac{1}{2}\nu;\ \tfrac{1}{2}f_2;\ P^2,\ R^2)$$

$$\cdot\ C(p,m,n)|R^2|^m|I-R^2|^n \prod_{i>j}(r_i^2-r_j^2)\prod_{i=1}^{p} dr_i^2$$

$$0 < r_1^2 \leq \ldots \leq r_p^2 < 1\quad ,$$

where $f_2 = q$, $m = \frac{1}{2}(q-p-1)$, $n = \frac{1}{2}(\nu-q-p-1)$, (and if n is defined as $\frac{1}{2}(f_1-p-1)$, $\nu = f_1+f_2$),

$$C(p,m,n)= \Pi^{\frac{1}{2}p} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(2m+2n+p+i+2)]\ /\{\Gamma[\tfrac{1}{2}(2m+i+1)]\Gamma[\tfrac{1}{2}(2n+i+1)]\Gamma(\tfrac{1}{2}i)\}\quad ,$$

and the hypergeometric function of matrix argument is defined by [5]

$$(2.4)\qquad {}_sF_t(a_1,\ldots,a_s;\ b_1,\ldots,b_t;\ S,T) = \sum_{k=o}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_s)_\kappa C_\kappa(S)C_\kappa(T)}{(b_1)_\kappa\cdots(b_t)_\kappa C_\kappa(I_p)k!}\quad ,$$

where $a_1,\ldots,a_s$, $b_1,\ldots,b_t$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is given by

$$(a)_\kappa = \prod_{i=1}^{p}(a-\tfrac{1}{2}(i-1))k_i\quad ,$$

where

$$(a)_k = a(a+1)\ldots(a+k-1)\quad ,$$

partition $\mathcal{K}$ of k is such that

$$\mathcal{K} = (k_1, k_2, \ldots k_p),\ k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0 \quad ,$$

$k_1+\ldots+k_p = k$, and the zonal polynomials, $C_{\mathcal{K}}(S)$, are expressible in terms of elementary symmetric functions (esf) of the characteristic roots of S, [5].

Now define by $V(x;\ q_p,\ldots,q_1;n)$ the determinant

$$(2.5) \qquad \begin{vmatrix} \int_0^x x_p^{q_p}(1-x_p)^n dx_p \ldots & \int_0^x x_p^{q_1}(1-x_p)^n dx_p \\ \int_0^{x_2} x_1^{q_p}(1-x_1)^n dx_1 \ldots & \int_0^{x_2} x_1^{q_1}(1-x_1)^n dx_1 \end{vmatrix} .$$

It may be observed that the cdf of the largest root from (2.3) under the null hypothesis (iii) can be thrown into the form (2.5) multiplied by $C(p,m,n)$, [6], [7], [9]. Further, in view of the fact that the zonal polynomials $C_{\mathcal{K}}(S)$ in (2.4) can be expressed in terms of esf's of characteristic roots of S, by the use of Pillai's lemma on the multiplication of the basic Vandermonde type determinant by powers of esf's, [13], it is easy to see that the non-central distribution of the cdf of r_p^2 in (2.3) can be expressed as a series whose terms are linear compounds of determinants of the type (2.5). Further, it has been shown that [6], [7]

$$(2.6) \qquad V(x;\ q_s,\ q_{s-1},\ldots,q_1;\ n) = (q_s+n+1)^{-1}(A^{(s)}+B^{(s)}+q_sC^{(s)})$$

where

$$A^{(s)} = -I_0(x;q_s,\ n+1)\ V(x;\ q_{s-1},\dots,\ q_1;\ n)\ ,$$

$$B^{(s)} = 2\sum_{j=s-1}^{1}(-1)^{s-j-1}I(x;q_s+q_j,2n+1)V(x;q_{s-1},\dots,q_{j+1},q_{j-1},\dots,q_1;n)$$

$$C^{(s)} = V(x;\ q_s-1,\ q_{s-1},\dots,q_1;\ n)\ ,$$

$$I_0(x;\ q_s,\ n+1) = x^{q_s}\ (1-x)^{n+1}\quad ,$$

and

$$I(x;\ q,\ r) = \int_0^x x_1^q\ (1-x_1)^r\ dx_1\quad .$$

It may be noted that $C^{(s)}$ vanishes if $q_s = q_{s-1}+1$. Using (2.6) in each of the determinants of the linear compounds involved in the series obtainable from (2.3), after the necessary number of reductions, the cdf of the largest canonical correlation coefficient, r_p^2, can be ultimately reduced in terms of simple incomplete beta functions.

3. Non-central cdf of r_2^2. Now putting $p = 2$ in (2.3) and using the method outlined in the preceding section the cdf of the largest of two canonical correlation coefficients is obtained in the following form:

$$\text{(3.1)}\quad \Pr\{r_2^2 \leq x\} = K\Big\{-I_0(x;m+1,n+1)\Big[\Big(\sum_{i=0}^{6}B_ix^i\Big)I(x;m,n)+\Big(\sum_{i=2}^{6}C_ix^{i-1}\Big)I(x;m+1,n)$$
$$+\Big(\sum_{i=4}^{6}D_ix^{i-2}\Big)I(x;m+2,n)+E_6x^3I(x;m+3,n)\Big]$$
$$+2\Big[(B_6+C_6+D_6+E_6)I(x;2m+7,2n+1)+(B_5+C_5+D_5)I(x;2m+6,2n+1)$$
$$+(B_4+C_4+D_4)I(x;2m+5,2n+1)+(B_3+C_3)I(2m+4,2n+1)$$
$$+(B_2+C_2)I(x;2m+3,2n+1)+B_1I(x;2m+2,2n+1)+B_0I(x;2m+1,2n+1)\Big]\Big\}\ ,$$

where

$$K = \left[(1-\rho_1^2)(1-\rho_2^2)\right]^{\frac{1}{2}\nu} C(2,m,n)$$

$$B_6 = \frac{a_{61}}{m+n+8}, \quad B_5 = \frac{a_{51}+(m+7)B_6}{m+n+7}, \quad B_4 = \frac{a_{41}+(m+6)B_5}{m+n+6}$$

$$B_3 = \frac{a_{31}+(m+5)B_4}{m+n+5}, \quad B_2 = \frac{a_{21}+(m+4)B_3}{m+n+4}, \quad B_1 = \frac{a_{11}+(m+3)B_2}{m+n+3}$$

$$B_0 = \frac{1+(m+2)B_1}{m+n+2}; \quad C_6 = \frac{a_{62}}{m+n+7}, \quad C_5 = \frac{a_{52}+(m+6)C_6}{m+n+6},$$

$$C_4 = \frac{a_{42}+(m+5)C_5}{m+n+5}, \quad C_3 = \frac{a_{32}+(m+4)C_4}{m+n+4}, \quad C_2 = \frac{a_{22}+(m+3)C_3}{m+n+3}$$

$$D_6 = \frac{a_{63}}{m+n+6}, \quad D_5 = \frac{a_{53}+(m+5)D_6}{m+n+5}, \quad D_4 = \frac{a_{43}+(m+4)D_5}{m+n+4}$$

$$E_6 = \frac{a_{64}}{m+n+5}, \quad a_{61} = 231\, A_{61}, \quad a_{62} = 35(A_{62}-3A_{61})$$

$$a_{63} = 3(A_{63}-5A_{62}-7A_{61}), \quad a_{64} = A_{64}-A_{63}-2A_{62}-5A_{61},$$

$$a_{51} = 63A_{51}, \quad a_{52} = 5A_{52}-28A_{51}, \quad a_{53} = A_{53}-2A_{52}-5A_{51},$$

$$a_{41} = 35A_{41}, \quad a_{42} = 3(A_{42} - 5A_{41}), \quad a_{43} = A_{43} - A_{42} - 2A_{41}$$

$$a_{31} = 5A_{31}, \quad a_{32} = A_{32} - 2A_{31}, \quad a_{21} = 3A_{21}, \quad a_{22} = A_{22} - A_{21},$$

$$a_{11} = A_{11} = \frac{\nu^2}{4f_2} b_1,$$

$$A_{61} = \frac{[\nu(\nu+2)\ldots(\nu+10)]^2}{2^6 f_2(f_2+2)\ldots(f_2+10)} \; \frac{[231b_1^6-1260b_1^4b_2+1680b_1^2b_2^2-320b_2^3]}{8!4!176},$$

$$A_{62} = \frac{[\nu(\nu+2)\ldots(\nu+8)(\nu-1)]^2}{2^6 f_2(f_2+2)\ldots(f_2+8)(f_2-1)8!44} \; \frac{3}{} [35b_1^4 b_2-120b_1^2b_2^2+48b_2^3] ,$$

$$A_{63} = \frac{[\nu\ldots(\nu+6)(\nu-1)(\nu+1)]^2}{2^6 f_2\ldots(f_2+6)(f_2-1)(f_2+1)} \; \frac{10}{7!3} \; [3b_1^2b_2^2-4b_2^3] ,$$

$$A_{64} = \frac{[\nu(\nu+2)(\nu+4)(\nu-1)(\nu+1)(\nu+3)]^2}{2^6 f_2(f_2+2)(f_2+4)(f_2-1)(f_2+1)(f_2+3)} \; \frac{4}{5.7.9} \; b_2^3 ,$$

$$A_{51} = \frac{[\nu(\nu+2)\ldots(\nu+8)]^2}{2^5 f_2(f_2+2)\ldots(f_2+8)} \; \frac{1}{8!48} [63b_1^5-280b_1^3b_2+240b_1b_2^2]$$

$$A_{52} = \frac{[\nu(\nu+2)(\nu+4)(\nu+6)(\nu-1)]^2}{2^5 f_2\ldots(f_2+6)(f_2-1)} \; \frac{1}{6!3} \; [5b_1^3b_2-12b_1b_2^2] ,$$

$$A_{53} = \frac{[\nu(\nu+2)(\nu+4)(\nu-1)(\nu+1)]^2}{2^5 f_2(f_2+2)(f_2+4)(f_2-1)(f_2+1)} \; \frac{1}{7.5} \; b_1b_2^2 ,$$

$$A_{41} = \frac{[\nu(\nu+2)\ldots(\nu+6)]^2}{2^4 f_2(f_2+2)\ldots(f_2+6)} \; \frac{3}{8!8} [35b_1^4-120b_1^2b_2+48b_2^2] ,$$

$$A_{42} = \frac{[\nu(\nu+2)(\nu+4)(\nu-1)]^2}{2^4 f_2(f_2+2)(f_2+4)(f_2-1)} \; \frac{1}{4!7} \; [3b_1^2b_2-4b_2^2]$$

$$A_{43} = \frac{[\nu(\nu+2)(\nu-1)(\nu+1)]^2}{2^4 f_2(f_2+2)(f_2-1)(f_2+1)} \; \frac{2}{3.5} \; b_2^2 ,$$

$$A_{31} = \frac{[\nu(\nu+2)(\nu+4)]^2}{2^3 f_2(f_2+2)(f_2+4)} \; \frac{1}{5!4} \; [5b_1^3-12b_1b_2] ,$$

$$A_{32} = \frac{[\nu(\nu+2)(\nu-1)]^2}{2^3 f_2(f_2+2)(f_2-1)} \; \frac{1}{5} \; b_1b_2 ,$$

$$A_{21} = \frac{[\nu(\nu+2)]^2}{2^2 f_2(f_2+2)\ \ 4!2}\ [3b_1^2-4b_2]$$

$$A_{22} = \frac{[\nu(\nu-1)]^2}{2^2 f_2(f_2-1)}\ \frac{4b_2}{3!}\ , \quad b_1 = \rho_1^2+\rho_2^2, \quad \text{and} \quad b_2 = \rho_1^2\rho_2^2\ .$$

In obtaining the cdf of r_2^2 in (3.1), zonal polynomials of degrees 1 to 6 were included. The expression for the cdf of r_2^2 in (3.1) has been used to compute the power of test (iii) for various pairs of values of (ρ_1^2, ρ_2^2) and the results are presented in Table 1.

4. Non-central cdf of r_3^2 under a single non-zero ρ. When there is only a single non zero population root, ρ^2, the joint density function (2.3) can be expressed in the following simple form:

$$(4.1) \quad (1-\rho^2)^{\frac{1}{2}\nu} C(p,m,n) \sum_{i=0}^{\infty} \frac{[\nu(\nu+2)\ldots(\nu+2(i-1))]^2}{2^i f_2(f_2+2)\ldots(f_2+2(i-1))}\ \frac{\rho^{2i}}{p(p+2)\ldots(p+2(i-1))}\ \frac{Z_i(R^2)}{i!}$$

where $Z_i(R^2)$ are ith degree zonal polynomials which are given a different normalizing constant than that of $C_i(R^2)$, [5]. For $p = 3$, using methods as above, the cdf of the largest root is obtained from (4.1) and given below.

$$(4.2) \quad \Pr\{r_3^2 \leq x\} = K_1\Big\{ 2I(x;m,n)\Big[(B_6^{(3)}-C_6^{(3)}-E_6^{(3)}-\frac{225A_6}{m+n+6})\ I(x;2m+9,2n+1)$$
$$+(B_5^{(3)}-C_5^{(3)}-E_5^{(3)})\ I(x;2m+8,2n+1)$$
$$+(B_4^{(3)}-C_4^{(3)}-E_4^{(3)})\ I(x;2m+7,2n+1)$$
$$+(B_3^{(3)}-C_3^{(3)})\ I(x;2m+6,2n+1)$$
$$+(B_2^{(3)}-C_2^{(3)})I(x;2m+5,2n+1)+B_1^{(3)}I(x;2m+4,2n+1)+B_0^{(3)}I(x;2m+3,2n+1)\Big]$$

$$-2I(x;m+1,n)\Big[(B_6^{(3)}-D_6^{(3)}-F_6^{(3)})I(x;2m+8,2n+1)+(B_5^{(3)}-D_5^{(3)}-F_5^{(3)})I(x;2m+7,2n+1)$$

$$+(B_4^{(3)}-D_4^{(3)})I(x;2m+6,2n+1)+(B_3^{(3)}-D_3^{(3)})I(x;2m+5,2n+1)$$

$$+B_2^{(3)}I(x;2m+4,2n+1)+B_1^{(3)}I(x;2m+3,2n+1)+B_0^{(3)}I(x;2m+2,2n+1)\Big]$$

$$+2I(x;m+2,n)\Big[(C_6^{(3)}-D_6^{(3)}-\frac{45A_6}{m+n+5})I(x;2m+7,2n+1)+(C_5^{(3)}-D_5^{(3)})I(x;2m+6,2n+1)$$

$$+(C_4^{(3)}-D_4^{(3)})I(x;2m+5,2n+1)+(C_3^{(3)}-D_3^{(3)})I(x;2m+4,2n+1)+C_2^{(3)}I(x;2m+3,2n+1$$

$$+2I(x;m+3,n)\Big[(E_6^{(3)}-F_6^{(3)}+\frac{45A_6}{m+n+5})I(x;2m+6,2n+1)+(E_5^{(3)}-F_5^{(3)})I(x;2m+5,2n+1)$$

$$+E_4^{(3)}\ I(x;2m+4,2n+1)\Big]$$

$$+\frac{450A_6}{m+n+6}\ I(x;2m+5,2n+1)\ I(x;m+4,n)$$

$$-V(x;1,0;n)\ I_0(x;m+2,n+1)\Big[B_6^{(3)}x^6+B_5^{(3)}x^5+\ldots+B_0^{(3)}\Big]$$

$$+V(x;2,0;n)\ I_0(x;m+3,n+1)\Big[C_6^{(3)}x^4+C_5^{(3)}x^3+C_4^{(3)}x^2+C_3^{(3)}x+C_2^{(3)}\Big]$$

$$-V(x;2,1;n)\ I_0(x;m+3,n+1)\Big[D_6^{(3)}x^3+D_5^{(3)}x^2+D_4^{(3)}x+D_3^{(3)}\Big]$$

$$+V(x;3,0;n)\ I_0(x;m+4,n+1)\Big[E_6^{(3)}x^2+E_5^{(3)}x+E_4^{(3)}\Big]$$

$$-V(x;3,1;n)\ I_0(x;m+4,n+1)\Big[F_6^{(3)}x+F_5^{(3)}\Big]$$

$$+\frac{225A_6}{m+n+6}\ I_0(x;m+5,n+1)+\frac{45A_6}{m+n+5}\ I_0(x;m+4,n+1)\Big\}$$

where

$$B_6^{(3)}=\frac{10395A_6}{m+n+9},\qquad B_5^{(3)}=\frac{(m+8)B_6^{(3)}+945A_5}{m+n+8},$$

$$B_4^{(3)}=\frac{(m+7)B_5^{(3)}+105A_4}{m+n+7},\qquad B_3^{(3)}=\frac{(m+6)B_4^{(3)}+15A_3}{m+n+6},$$

$$B_2^{(3)} = \frac{(m+5)B_3^{(3)}+3A_2}{m+n+5}, \quad B_1^{(3)} = \frac{(m+4)B_2^{(3)}+A_1}{m+n+4}, \quad B_0^{(3)} = \frac{(m+3)B_1^{(3)}+A_0}{m+n+3},$$

$$C_6^{(3)} = \frac{4725A_6}{m+n+8}, \quad C_5^{(3)} = \frac{(m+7)C_6^{(3)}+420A_5}{m+n+7}, \quad C_4^{(3)} = \frac{(m+6)C_5^{(3)}+45A_4}{(m+n+6)},$$

$$C_3^{(3)} = \frac{(m+5)C_4^{(3)}+6A_3}{m+n+5}, \quad C_2^{(3)} = \frac{(m+4)C_3^{(3)}+A_2}{m+n+4},$$

$$D_6^{(3)} = \frac{3150A_6}{m+n+6}, \quad D_5^{(3)} = \frac{(m+6)D_6^{(3)}+270A_5}{m+n+5},$$

$$D_4^{(3)} = \frac{(m+5)D_5^{(3)}+27A_4}{m+n+4}, \quad D_3^{(3)} = \frac{(m+4)D_4^{(3)}+3A_3}{m+n+3},$$

$$E_6^{(3)} = \frac{945A_6}{m+n+7}, \quad E_5^{(3)} = \frac{(m+6)E_6^{(3)}+75A_5}{m+n+6}, \quad E_4^{(3)} = \frac{(m+5)E_5^{(3)}+6A_4}{m+n+5},$$

$$F_6^{(3)} = \frac{720A_6}{m+n+6}, \quad F_5^{(3)} = \frac{(m+5)F_6^{(3)}+45A_5}{m+n+5},$$

$$K_1 = C(3,m,n)(1-\rho^2)^{\nu/2}, \quad A_0 = 1$$

$$A_i = \frac{[\nu\ldots(\nu+2(i-1))]^2}{2^i f_2\ldots(f_2+2(i-1))} \; \frac{\rho^{2i}}{3.5\ldots(3+2(i-1))i!}$$

$$i = 1,2,\ldots$$

The determinants of the $V(x;q_2,q_1;n)$ type occuring in (4.2) can further be reduced in terms of simple incomplete beta functions using (2.6). The cdf (4.2) also has been obtained as before using the zonal ploynomials of degrees one to six. Power function tabulations from (4.2) are presented in Table 2.

5. <u>Non-central cdf of the largest root for test (ii)</u>. Let X be a $p \times f_2$ matrix variate $(p \leq f_2)$ and Y a $p \times f_1$ matrix variate $(p \leq f_1)$

and the columns be all independently normally distributed with covariance Σ, $E(X) = M$ and $E(Y) = 0$. Let $\ell_1,\ldots,\ell_p$ be the characteristic roots of

$$(5.1) \qquad |XX' - \ell(YY'+XX')| = 0 \;,$$

and $\omega_1,\ldots,\omega_p$ those of

$$(5.2) \qquad |MM' - \omega\Sigma| = 0 \;,$$

then the joint density function of $\ell_1,\ldots\ell_p$ is given by Constantine [4], [5] in the form

$$(5.3) \qquad e^{-\frac{1}{2}\mathrm{tr}\Omega}\,{}_1F_1(\tfrac{1}{2}\nu;\tfrac{1}{2}f_2;\,\tfrac{1}{2}\Omega,L)\;C(p,m,n)|L|^m|I-L|^n\prod_{i>j}(\ell_i-\ell_j)$$

where $L = X'(YY'+XX')^{-1}X$, $\Omega = M'\Sigma^{-1}M$,

$m = \frac{1}{2}(f_2-p-1)$, $n = \frac{1}{2}(f_1-p-1)$ and $\nu = f_1+f_2$.

In the context of test (ii) $f_2 = \ell-1$ and $f_1 = N-\ell$, N being the pooled sample size of the samples from the ℓ populations. It should be pointed out that the same symbols m and n are used in connection with both tests because of the reason that these definitions of m and n will leave the joint distribution of $\ell_1,\ldots\ell_p$ the same as that of $r_1^2,\ldots,r_p^2$ under the null hypotheses (ii) and (iii) respectively as may be seen from (5.3) and (2.3). Now the cdf of the largest root, ℓ_p, can be derived from (5.3) by the method outlined in Section 2.

6. Non-central cdf of ℓ_2. Now putting $p = 2$ in (5.3) and following the method described in Section 2, the cdf of the largest of two roots can be obtained in the same form as (3.1) with the following changes:

$$r_2^2 \longrightarrow \ell_2^2 ,$$

$$K \longrightarrow K' = e^{-\frac{1}{2}(\omega_1+\omega_2)} C(2,m,n) ,$$

$A_{ij} \longrightarrow A'_{ij}$ where A'_{ij} is obtained from A_{ij} by multiplying A_{ij} by 2^i and each linear factor involving ν in the numerator of A_{ij} is raised only to a single power instead of power two in A_{ij}, and defining b_1 as $\frac{1}{2}(\omega_1+\omega_2)$ and b_2 as $\omega_1\omega_2/4$. For example,

$$A'_{21} = \frac{2^2 A_{21} [3(\frac{1}{2}(\omega_1+\omega_2))^2 - 4(\omega_1\omega_2/4)]}{\nu(\nu+2)(3b_1^2 - 4b_2)} .$$

Further, $a_{ij} \longrightarrow a'_{ij}$ where a'_{ij} is the same function of A'_{ij}s as a_{ij} is of A'_{ij}s. Similarly

$$(B_i, C_i, D_i, E_i) \longrightarrow (B'_i, C'_i, D'_i, E'_i) ,$$

where the meaning of (') is obvious. In other words, the basic changes are in the A_{ij} coefficients, K coefficient and the different definitions of the b's. Now, tabulation of the power of test (ii) based on ℓ_2 is presented in Table 3.

7. Non-central cdf of ℓ_3 under a single non-zero ω. When there is only one non-zero population root, ω, the joint density function (5.3) takes the simple form

$$(7.1) \qquad e^{-\frac{1}{2}\omega} C(p,m,n) \sum_{i=0}^{\infty} \frac{[\nu(\nu+2)\dots(\nu+2(i-1))]}{f_2(f_2+2)\dots(f_2+2(i-1))} \frac{(\frac{1}{2}\omega)^i Z_i(L)}{p(p+2)\dots(p+2(i-1))\, i!} .$$

As before, the cdf of the largest root ℓ_3 can be obtained in the same form as (4.2) with the following changes:

$$r_3^2 \longrightarrow \ell_3 ,$$

$$K_1 \longrightarrow K_1' = e^{-\frac{1}{2}\omega} C(3,m,n) ,$$

$$A_i \longrightarrow A_i' \quad \text{where} \quad A_i' = 2^i(\omega/2\rho^2)^i A_i/[\nu(\nu+2)\ldots(\nu+2(i-1))] ,$$

$$i = 1, 2, \ldots$$

$$(B_i^{(3)}, C_i^{(3)}, D_i^{(3)}, E_i^{(3)}, F_i^{(3)}) \longrightarrow (B_i^{(3)'}, C_i^{(3)'}, D_i^{(3)'}, E_i^{(3)'}, F_i^{(3)'})$$

where (') means that $A_i's$ are replaced by $A_i's$ in the corresponding original coefficients to get the primed ones. Now for the cdf of ℓ_3 as well as ℓ_2, the degrees of the zonal polynomials used were as before. Numerical values of the power function from the cdf of ℓ_3 are presented in Table 4.

8. The cdf of the largest root under the three null hypotheses. It has been shown [6], [8], that when each of the three null hypotheses is true, the cdf of the largest of s non-null characteristic roots of a matrix is given by

$$(8.1) \qquad C(s,m,n)\ V(x;m+s-1,\ m+s-2,\ldots,m;n) .$$

A general expression approximating (8.1) for computing the upper percentage points has been obtained by Pillai [15] which is given below:

For odd values of s, (8.1) is approximated by

$$(8.2) \qquad \frac{I(x;m,n)}{\beta(m+1,n+1)} + \sum_{i=1}^{s-1} (-1)^i k_{m+s-i} I_0(x;m+s-i,n+1)$$

where

$$(8.3) \qquad k_{m+s-i} = (m+n+s-i+1)^{-1}[C(s,m,n)V(1;m+s-1,\ldots m+s-i+1,m+s-i-1,\ldots,m;n) - (m+s-i+1)k_{m+s-i+1}]$$

where $k_{m+s} = 0$.

For even values of s, (8.1) is approximated by

$$(8.4) \qquad 1 + \sum_{i=1}^{s-1} (-1)^i k_{m+s-i} I_0(x;m+s-i,n+1) \quad .$$

Further it has been shown that [17]

$$(8.5) \qquad V(1;m+s-1,m+s-2,\ldots,m+s-i+1,m+s-i-1,\ldots,m;n)$$

$$= \left[\binom{s-1}{i-1} \prod_{j=1}^{i-1} \frac{(2m+s-j+1)}{(2m+2n+2s-j+1)} \right] / C(s-1,m,n) \quad .$$

Using (8.2), upper 5 and 1 per cent points of the largest root for $s = 11$ have been computed and presented in Tables 5 and 6. Similarly (8.4) has been used to compute similar percentage points for $s = 12$ which are given in Tables 7 and 8. The computations were carried out on IBM 7094 but a trial value was first extrapolated from Pillai's tables [11], [12], [15] for each computed value in order to be fed into the machine. The error of approximation practically remains the same as discussed previously [16], [12], [15].

9. Some remarks. The tabulation of power functions presented in Tables 1 to 4 reveal, in addition to monotonicity with respect to individual population roots, the following facts: a) the power is not monotonic with respect to either the sum of the population roots or their product b) the power seems to be a minimum when the two population roots are equal and increases as they move apart.

Further, the tabulations of the power functions made in this paper are being used to compare the powers of the largest root with those of Hotelling's T_0^2, and Pillai's $V^{(s)}$ criterion and Wilks' Λ criterion and a report on this study will be soon forthcoming.

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Table 1

Powers of the r_2^2 test for testing: $\rho_1 = 0$, $\rho_2 = 0$ against different simple alternative hypotheses

$m = 0$, $\alpha = .05$

ρ_1^2	ρ_2^2	n = 5	10	15	20	25	30	40	60
0	$.0^51$	.0500004	.0500007	.0500011	.0500015	.0500018	.0500022	.050003	.050004
0	.0001	.0500394	.0500756	.0501120	.0501484	.0501848	.0502213	.050294	.050441
$\rho_1^2+\rho_2^2=.0025$									
0	.0025	.0509921	.0519112	.0528426	.0537831	.054732	.0556888	.057626	.061593
.0001	.0024	.0509916	.0519097	.0528395	.0537779	.0547240	.0556777	.057607	.061552
.0002	.0023	.0509912	.0519083	.0528366	.0537731	.0547168	.0556676	.057590	.061514
.0003	.0022	.0509907	.0519070	.0528340	.0537687	.0547102	.0556584	.057574	.061480
.0005	.0020	.0509900	.0519048	.0528296	.0537613	.0546991	.0556428	.057547	.061422
.001	.0015	.0509890	.0519016	.0528231	.0537504	.0546827	.0556198	.057508	.061337
.00125	.00125	.0509888	.0519012	.0528223	.0537490	.0546807	.0556170	.057503	.061326
$\rho_1^2+\rho_2^2=.01$									
0	.01	.0540580	.0579188	.0619218	.0660524	.0703052	.0746772	.083768	.103263
.0001	.0099	.0540558	.0579123	.0619088	.0660306	.0702725	.0746315	.083690	.103097
.0002	.0098	.0540537	.0579059	.0618960	.0660092	.0702405	.0745867	.083613	.102935
.0003	.0097	.0540517	.0578997	.0618834	.0659883	.0702091	.0745428	.083539	.102776
.0005	.0095	.0540477	.0578876	.0618592	.0659477	.0701483	.0744579	.083394	.102468
.001	.009	.0540385	.0578598	.0618031	.0658541	.0700079	.0742616	.083059	.101757
.002	.008	.0540234	.0578139	.0617108	.0656999	.0697765	.0739382	.082508	.100581
.003	.007	.0540126	.0577811	.0616448	.0655897	.0696112	.0737071	.082114	.099742
.005	.005	.0540040	.0577549	.0615920	.0655015	.0694790	.0735221	.081798	.099069
ρ_1^2=constant (.0015)									
.0015	.0015	.0511876	.0522844	.0533928	.0545090	.0556321	.0567618	.059040	.063670
.0015	.0025	.0515865	.0530555	.0545433	.0560447	.0575585	.0590843	.062170	.068478
.0015	.0035	.0519879	.0538339	.0557083	.0576047	.0595215	.0614580	.065388	.073471
.0015	.0045	.0523916	.0546195	.0568880	.0591892	.0615211	.0638827	.068693	.078648
.0015	.0055	.0527977	.0554125	.0580824	.0607982	.0635572	.0663583	.072083	.084005
.0015	.0065	.0532062	.0562127	.0592914	.0624315	.0656298	.0688846	.075559	.089539
.0015	.0075	.0536171	.0570203	.0605152	.0640893	.0677388	.0714614	.079119	.095248
.0015	.0085	.0540304	.0578352	.0617536	.0657715	.0698840	.0740884	.082764	.101127
$\rho_1^2\rho_2^2=.00005$									
.005	.01	.0560686	.0618237	.0677713	.0738876	.0801632	.0865914	.099884	.128065
.002	.025	.0604076	.0727491	.0849060	.097800	.111378	.12559	.15574	.2214
.001	.05	.0732128	.09792	.12552	.15553	.1876	.2211	.2915	.433

Table 1 (Cont'd.)

$m = 1 \quad \alpha = .05$

ρ_1^2	ρ_2^2	n = 5	10	15	20	25	30	40	60
0	$.0^51$	.0500003	.0500005	.0500008	.0500011	.0500013	.0500016	.050002	.050003
0	.0001	.0500289	.0500549	.0500810	.0501073	.0501335	.0501598	.050212	.050318
$\rho_1^2+\rho_2^2=.0025$									
0	.0025	.0507271	.0513874	.0520570	.0527333	.0534157	.0541039	.055497	.058350
.0001	.0024	.0507268	.0513863	.0520548	.0527297	.0534103	.0540963	.055484	.058322
.0002	.0023	.0507264	.0513854	.0520529	.0527264	.0534053	.0540894	.055472	.058296
.0003	.0022	.0507261	.0513844	.0520511	.0527234	.0534008	.0540831	.055461	.058273
.0005	.0020	.0507256	.0513829	.0520480	.0527183	.0533932	.0540723	.055443	.058233
.001	.0015	.0507249	.0513807	.0520435	.0527108	.0533819	.0540565	.055416	.058174
.00125	.00125	.0507248	.0513804	.0520430	.0527099	.0533805	.0540545	.055412	.058166
$\rho_1^2+\rho_2^2=.01$									
0	.01	.0529726	.0557452	.0586222	.0615927	.0646534	.0678026	.074362	.088499
.0001	.0099	.0529711	.0557407	.0586131	.0615775	.0646305	.0677706	.074307	.088380
.0002	.0098	.0529696	.0557362	.0586041	.0615626	.0646081	.0677392	.074253	.088264
.0003	.0097	.0529681	.0557319	.0585954	.0615480	.0645862	.0677084	.074200	.088150
.0005	.0095	.0529653	.0557234	.0585785	.0615197	.0645437	.0676488	.074098	.087929
.001	.009	.0529588	.0557040	.0585394	.0614543	.0644455	.0675112	.073862	.087418
.002	.008	.0529481	.0556719	.0584750	.0613467	.0642837	.0672844	.073473	.086577
.003	.007	.0529405	.0556490	.0584290	.0612698	.0641681	.0671224	.073195	.085975
.005	.005	.0529344	.0556307	.0583922	.0612083	.0640756	.0669928	.072973	.086493
ρ_1^2=Constant(.0015)									
.0015	.0015	.0508704	.0516587	.0524560	.0532593	.0540677	.0548810	.056522	.059857
.0015	.0025	.0511628	.0522185	.0532886	.0543691	.0554589	.0565575	.058780	.063326
.0015	.0035	.0514569	.052783	.0541317	.0554964	.0568762	.0582706	.061102	.066930
.0015	.0045	.0517527	.0533537	.0549852	.0566412	.0583199	.0600205	.063486	.070669
.0015	.0055	.0520502	.0539291	.0558493	.0578036	.0597898	.0618072	.065933	.074543
.0015	.0065	.0523494	.0545097	.0567239	.0589836	.0612863	.0636310	.0684438	.078551
.0015	.0075	.0526504	.0550956	.0576091	.0601812	.0628092	.0654918	.071017	.082693
.0015	.0085	.0529531	.0556868	.0585049	.0613967	.0643588	.0673897	.073654	.086967
$\rho_1^2\rho_2^2=.00005$									
.005	.01	.0544469	.0585846	.0628672	.0672776	.0718096	.0764603	.086105	.10670
.002	.025	.0583535	.0665110	.075286	.084633	.094531	.10496	.1273	.1770
.001	.05	.0669962	.084870	.10507	.1274	.1515	.1774	.2333	.355

Table 1 (Cont'd.)

m = 2 α = .05

ρ_1^2	ρ_2^2	n = 5	10	15	20	25	30	40	60
0	$.0^5 1$	.0500003	.0500005	.0500007	.0500009	.0500011	.0500013	.050002	.050003
0	.0001	.0500240	.0500451	.0500662	.0500874	.0501087	.0501299	.050172	.050258
$\rho_1^2 + \rho_2^2$=.0025									
0	.0025	.0506041	.0511385	.0516797	.0522259	.0527768	.0533322	.054456	.056755
.0001	.0024	.0506038	.0511377	.0516780	.0522231	.0527727	.0533264	.054446	.056733
.0002	.0023	.0506036	.0511369	.0516765	.0522206	.0527689	.0533211	.054437	.056713
.0003	.0022	.0506033	.0511362	.0516751	.0522183	.0527654	.0533163	.054429	.056696
.0005	.0020	.0506029	.0511350	.0516727	.0522144	.0527595	.0533080	.0544145	.056665
.001	.0015	.0506023	.0511333	.0516692	.0522086	.0527509	.0532959	.054394	.056620
.00125	.00125	.0506022	.0511330	.0516688	.0522078	.0527498	.0532944	.054391	.056614
$\rho_1^2+\rho_2^2$=.01									
0	.01	.0524680	.0547084	.0570277	.0594194	.0618813	.0644124	.069681	.081030
.0001	.0099	.0524668	.0547048	.0570206	.0594076	.0618636	.0643876	.069638	.080938
.0002	.0098	.0524656	.0547014	.0570137	.0593960	.0618462	.0643633	.069596	.080847
.0003	.0097	.0524645	.0546979	.0570069	.0593847	.0618292	.0643395	.069556	.080759
.0005	.0005	.0524622	.0546913	.0569937	.0593628	.0617963	.0642934	.069476	.080587
.001	.009	.0524570	.0546761	.0569633	.0593121	.0617203	.0641869	.069293	.080189
.002	.008	.0524485	.0546509	.0569132	.0592286	.0615951	.0640116	.068992	.079534
.003	.007	.0524424	.0546330	.0568774	.0591690	.0615057	.0638862	.068777	.079066
.005	.005	.0524375	.0546186	.05684877	.0591213	.0614341	.0637859	.068605	.078691
ρ_1^2=Constant(.0015)									
.0015	.0015	.0507233	.0513614	.0520060	.0526552	.0533084	.0539654	.055290	.057982
.0015	.0025	.0519662	.0518206	.0526756	.0535586	.0544388	.0553258	.057120	.060786
.0015	.0035	.0512104	.0522840	.0533734	.0544756	.0555895	.0567147	.058998	.063695
.0015	.0045	.0514561	.0527514	.0540695	.0554063	.0567608	.0581323	.060926	.066710
.0015	.0055	.0517031	.0532230	.0547737	.0563507	.0579526	.0595788	.062902	.069831
.0015	.0065	.0519515	.0536987	.0554863	.0573090	.0591653	.0610542	.064929	.073060
.0015	.0075	.0522013	.0541785	.0562072	.0582812	.0603988	.0625589	.067006	.076396
.0015	.0085	.0524525	.0546625	.0569365	.0592674	.0616532	.0640930	.069132	.079839
$\rho_1^2\rho_2^2$=.00005									
.005	.01	.0536927	.0570378	.0604935	.0640489	.0677007	.0714462	.079215	.09582
.002	.025	.0569281	.0635085	.070570	.078090	.086055	.09446	.11254	.1532
.001	.05	.0640729	.07848	.09477	.1128	.1325	.1538	.200	.305

Table 1 (Cont'd.)

m = 5 α = .05

ρ_1^2	ρ_2^2	n = 5	10	15	20	25	30	40	60
0	$.0^5 1$	.0500002	.0500003	.0500005	.0500006	.0500008	.0500009	.050001	.050002
0	.0001	.0500180	.0500326	.0500471	.0500617	.0500762	.0500907	.050120	.050178
$\rho_1^2+\rho_2^2=.0025$									
0	.0025	.0504521	.0509221	.0511932	.0515664	.0519420	.0523199	.053083	.054638
.0001	.0024	.0504519	.0508215	.0511922	.0515647	.0519394	.0523164	.053077	.054626
.0002	.0023	.0504517	.0508210	.0511912	.0515631	.0519371	.0523132	.053072	.054614
.0003	.0022	.0504515	.0508206	.0511903	.0515617	.0519350	.0523102	.053067	.054603
.0005	.0020	.0504513	.0508198	.0511888	.0515593	.0519314	.0523053	.053058	.054585
.001	.0015	.0504509	.0508187	.0511866	.0515557	.0519262	.0522980	.053046	.054559
.00125	.00125	.0504508	.0508185	.0511864	.0515553	.0519255	.0522970	.053044	.054556
$\rho_1^2+\rho_2^2=.01$									
0	.01	.0518441	.0533889	.0549688	.0565876	.0582464	.0599453	.063465	.070999
.0001	.0099	.0518433	.0533866	.0549643	.0565803	.0582355	.0599302	.063440	.070948
.0002	.0098	.0518425	.0533843	.0549600	.0565731	.0582248	.0599154	.063414	.070888
.0003	.0097	.0518417	.0533821	.0549557	.0565661	.0582144	.0599008	.063390	.070835
.0005	.0095	.0518401	.0533778	.0549473	.0565525	.0581942	.0598727	.063342	.070730
.001	.009	.0518365	.0533679	.0549281	.0565210	.0581475	.0598078	.063231	.070491
.002	.008	.0518306	.0533515	.0548965	.0564692	.0580706	.0597008	.063049	.070096
.003	.007	.0518263	.0533398	.0548739	.0564322	.0580157	.0596244	.062918	.069813
.005	.005	.0518229	.0533305	.0548558	.0564026	.0579717	.0595632	.062814	.069587
ρ_1^2=Constant(.0015)									
.0015	.0015	.0505414	.0509834	.0514258	.0518699	.0523160	.0527639	.053666	.055494
.0015	.0025	.0507231	.0513147	.0519081	.0525047	.0531049	.0537088	.054928	.057411
.0015	.0035	.0509057	.0516487	.0523955	.0531479	.0539064	.0546711	.056219	.059392
.0015	.0045	.0510893	.0519854	.0528881	.0537996	.0547206	.0556511	.0575410	.061437
.0015	.0055	.0512739	.0523247	.0533859	.0544599	.0555476	.0566489	.058893	.063548
.0015	.0065	.0514594	.0526668	.0538890	.0551289	.0563874	.0576646	.060276	.065724
.0015	.0075	.0516459	.0530116	.0543974	.0558067	.0572403	.0586983	.061689	.067968
.0015	.0085	.0518333	.0533591	.0549112	.0564933	.0581063	.0597505	.063133	.070278
$\rho_1^2\rho_2^2=.00005$									
.005	.01	.0527597	.0550674	.0574242	.0598350	.0623011	.0648223	.070034	.081121
.002	.025	.0551626	.0596736	.0644430	.069482	.074801	.08038	.09237	.1196
.001	.05	.06044	.07025	.08118	.0933	.1064	.1209	.152	.225

Table 1 (Cont'd.)

$m = 0 \quad \alpha = .01$

ρ_1^2	ρ_2^2	n = 5	10	15	20	25	30	40	60
0	$.0^5 1$	.0100001	.0100002	.0100003	.0100004	.0100005	.0100006	.0100008	.010001
0	.0001	.0100100	.0100201	.0100305	.0100409	.0100514	.0100618	.0100828	.010125
$\rho_1^2 + \rho_2^2 = .0025$									
0	.0025	.0102519	.0105123	.0107815	.0110566	.0113367	.0116214	.0122044	.013423
.0001	.0024	.0102518	.0105117	.0107803	.0110545	.0113335	.0116168	.0121961	.013405
.0002	.0023	.0102516	.0105112	.0107792	.0110526	.0113305	.0116125	.0121886	.013388
.0003	.0022	.0102515	.0105107	.0107782	.0110508	.0113278	.0116086	.0121817	.013373
.0005	.0020	.0102512	.0105099	.0107765	.0110478	.0113231	.0116020	.0121701	.013346
.001	.0015	.0102509	.0105087	.0107740	.0110434	.0113163	.0115923	.0121530	.013308
.00125	.00125	.0102509	.0105086	.0107736	.0110429	.0113155	.0115911	.0121508	.013303
$\rho_1^2 + \rho_2^2 = .01$									
0	.01	.0110401	.0121657	.0133803	.0146726	.0160400	.0174820	.020591	.027723
.0001	.0099	.0110394	.0121632	.0133750	.0146634	.0160257	.0174614	.020554	.027638
.0002	.0098	.0110387	.0121609	.0133699	.0146543	.0160116	.0174412	.020518	.027556
.0003	.0097	.0110381	.0121585	.0133698	.0146455	.0159979	.0174214	.020583	.027474
.0005	.0095	.0110370	.0121540	.0133551	.0146284	.0159712	.0173830	.020414	.027317
.001	.009	.0110338	.0121436	.0133325	.0145888	.0159097	.0172944	.020256	.026953
.002	.008	.0110289	.0121265	.0132953	.0145236	.0158083	.0171484	.019994	.026354
.003	.007	.0110255	.0121142	.0132688	.0144770	.0157359	.0170441	.019808	.025925
.005	.005	.0110227	.0121044	.0132475	.0144398	.0156779	.0169606	.019659	.025582
ρ_1^2=Constant(.0015)									
.0015	.0015	.0103014	.0106117	.0109314	.0112568	.0115868	.0119211	.012602	.015010
.0015	.0025	.0104031	.0108109	.0112512	.0116921	.0121411	.0125978	.013533	.015491
.0015	.0035	.0105056	.0110312	.0115777	.0121391	.0127135	.0133005	.014511	.017076
.0015	.0045	.0106090	.0112455	.0119110	.0125978	.0133043	.0140295	.015535	.018767
.0015	.0055	.0107132	.0114630	.0122510	.0130686	.0139136	.0147852	.016607	.020566
.0015	.0065	.0108183	.0116836	.0125979	.0135514	.0145417	.0155679	.017727	.022476
.0015	.0075	.0109243	.0119074	.0129517	.0140465	.0151889	.0163782	.018897	.024496
.0015	.0085	.0110312	.0121344	.0133126	.0145538	.0158553	.0172162	.020116	.026632
$\rho_1^2 \rho_2^2 = .00005$									
.005	.01	.0115579	.0132435	.0150614	.0169938	.0190363	.0211877	.025816	.036374
.002	.025	.0129902	.0164772	.0205164	.0250761	.030154	.035755	.04853	.08025
.001	.05	.0162674	.024602	.035134	.04784	.06268	.0796	.119	.218

Table 1 (Cont'd.)

m = 1 α = .01

ρ_1^2	ρ_2^2	n = 5	10	15	20	25	30	40	60
0	$.0^5 1$	.0100001	.0100002	.0100002	.0100003	.0100004	.0100005	.010001	.010001
0	.0001	.0100071	.0100142	.0100215	.0100288	.0100361	.0100435	.010058	.010088
$\rho_1^2+\rho_2^2$=.0025									
0	.0025	.0101798	.0103613	.0105492	.0107412	.0109364	.0111348	.011540	.012385
.0001	.0024	.0101797	.0103610	.0105485	.0107399	.0109344	.0111319	.011535	.012373
.0002	.0023	.0101796	.0103606	.0105478	.0107386	.0109325	.0111292	.011530	.012363
.0003	.0022	.0101795	.0103603	.0105471	.0107375	.0109308	.0111267	.011526	.012353
.0005	.0020	.0101794	.0103598	.0105460	.0107356	.0109279	.0111226	.011519	.012336
.001	.0015	.0101791	.0103591	.0105444	.0107328	.0109236	.0111164	.011508	.012312
.00125	.00125	.0101791	.0103590	.0105442	.0107325	.0109231	.0111157	.011506	.012309
$\rho_1^2+\rho_2^2$=.01									
0	.01	.0107403	.0115198	.0123587	.0132485	.0141872	.0151744	.017296	.022141
.0001	.0099	.0107399	.0115183	.0123554	.0132426	.0141780	.0151612	.017272	.022087
.0002	.0098	.0107394	.0115168	.0123521	.0132369	.0141691	.0151483	.017249	.022033
.0003	.0097	.0107390	.0115153	.0123489	.0132313	.0141603	.0151357	.017226	.021981
.0005	.0095	.0107382	.0115124	.0123427	.0132203	.0141434	.0151112	.017182	.021879
.001	.009	.0107362	.0115057	.0123283	.0131952	.0141042	.0150548	.017081	.021644
.002	.008	.0107330	.0114948	.0123046	.0131537	.0140396	.0149618	.016914	.021257
.003	.007	.0107307	.0114869	.0122876	.0131241	.0139936	.0148953	.016795	.020980
.005	.005	.0107289	.0114806	.0122742	.0131004	.0139567	.0148421	.016699	.020859
ρ_1^2=Constant(.0015)									
.0015	.0015	.0102152	.0104317	.0106550	.0108824	.0111129	.0113462	.011821	.012800
.0015	.0025	.0102877	.0105783	.010879	.0111869	.0114999	.0118180	.012469	.013826
.0015	.0035	.0103607	.0107269	.0111078	.0114988	.0118984	.0123063	.013146	.014917
.0015	.0045	.0104343	.0108774	.0113406	.0118182	.0123086	.0128113	.013852	.016077
.0015	.0055	.0105085	.0110300	.0115777	.0121452	.0127306	.0133334	.014589	.017307
.0015	.0065	.0105833	.0111846	.0118192	.0124800	.0131647	.0138728	.015358	.018609
.0015	.0075	.0106586	.0113412	.0120652	.0128225	.0136109	.0144299	.016159	.019985
.0015	.0085	.0107345	.0114998	.0123156	.0131730	.0140396	.0150049	.016991	.021437
$\rho_1^2\ \rho_2^2$=.00005									
.005	.01	.0111090	.0122766	.0135327	.0148644	.0162680	.0177431	.020908	.028108
.002	.025	.0121112	.0145091	.0172645	.0203647	.023813	.027618	.03632	.0582
.001	.05	.0144067	.020063	.02720	.03584	.04603	.0578	.0860	.159

Table 1 (Cont'd.)

		m = 2	α = .01						
					n				
ρ_1^2	ρ_2^2	5	10	15	20	25	30	40	60
0	$.0^{5}1$	.0100001	.0100001	.0100002	.0100002	.0100003	.0100004	.010001	.010001
0	.0001	.0100058	.0100115	.0100173	.0100231	.0100289	.0100348	.010047	.010070
$\rho_1^2+\rho_2^2$=.0025									
0	.0025	.0101471	.0102916	.0104409	.0105932	.0107480	.0109051	.011226	.011892
.0001	.0024	.0101471	.0102914	.0104403	.0105922	.0107465	.0109030	.011222	.011883
.002	.0023	.0101471	.0102911	.0104398	.0105913	.0107451	.0109010	.011218	.011875
.003	.0022	.0101470	.0102909	.0104393	.0105905	.0107439	.0108999	.011215	.011868
.005	.0020	.0101469	.0102905	.0104385	.0105891	.0107417	.0108962	.011210	.011856
.001	.0015	.0101467	.0102899	.0104373	.0105871	.0107386	.0108917	.011202	.011839
.00125	.00125	.0101467	.0102899	.0104372	.0105868	.0107382	.0108912	.011201	.011837
$\rho_1^2+\rho_2^2$=.01									
0	.01	.0106050	.0112219	.0118826	.0125807	.0133147	.0140842	.015730	.019467
.0001	.0099	.0106047	.0112207	.0118801	.0125765	.0133080	.0140746	.015714	.019427
.0002	.0098	.0106044	.0112196	.0118777	.0125722	.0133015	.0140652	.015697	.019388
.0003	.0097	.0106040	.0112185	.0118753	.0125681	.0132951	.0140559	.015680	.019350
.0005	.0095	.0106034	.0112163	.0118709	.0125601	.0132826	.0140380	.015648	.019275
.001	.009	.0106019	.0112113	.0118601	.0125416	.0132539	.0139967	.015574	.019103
.002	.008	.0105994	.0112032	.0118427	.0125112	.0132067	.0139287	.015452	.018820
.003	.007	.0105977	.0111973	.0118302	.0124894	.0131730	.0138801	.015365	.018618
.005	.005	.0105963	.0111926	.0118202	.0124720	.0131460	.0138413	.015296	.018456
ρ_1^2=Constant(.0015)									
.0015	.0015	.0101762	.0103485	.0105260	.0108067	.0108897	.0110748	.011451	.012225
.0015	.0025	.0102356	.0104667	.0107057	.0109498	.0111979	.0114499	.011964	.013034
.0015	.0035	.0102953	.0105864	.0108886	.0111984	.0115146	.0118369	.012499	.0138905
.0015	.0045	.0103555	.0108075	.0110745	.0114524	.0118598	.0122362	.013055	.014797
.0015	.0055	.0104161	.0108311	.0112637	.0117121	.0121737	.0126480	.013634	.015755
.0015	.0065	.0104772	.0109542	.0114561	.0119774	.0125163	.0130724	.014235	.016766
.0015	.0075	.0105387	.0110798	.0116518	.0122484	.0128679	.0135098	.014860	.017832
.0015	.0085	.0106006	.0112070	.0118508	.0125253	.0132287	.0139603	.015509	.018951
$\rho_1^2\ \rho_2^2$=.00005									
.005	.01	.0109063	.0118303	.0128197	.0138648	.0149629	.0161134	.018573	.024133
.002	.025	.0117193	.0136010	.0157450	.0181433	.020773	.023721	.03039	.0472
.001	.05	.0135679	.017958	.02340	.03007	.0379	.0475	.0688	.127

Table 1 (Cont'd.)

		m = 5	α = .01						
		n							
ρ_1^2	ρ_2^2	5	10	15	20	25	30	40	60
0	$.0^51$	.0100001	.0100001	.0100001	.0100002	.0100002	.0100003	.0100003	.0100005
0	.0001	.0100043	.0100081	.0100120	.0100159	.0100198	.0100237	.010032	.010047
$\rho_1^2+\rho_2^2=.0025$									
0	.0025	.0101079	.0102053	.0103048	.0104061	.0105086	.0106124	.010823	.011259
.0001	.0024	.0101078	.0102051	.0103045	.0104055	.0105078	.0106112	.010821	.011254
.0002	.0023	.0101078	.0102050	.0103042	.0104050	.0105070	.0106101	.010819	.011250
.0003	.0022	.0101077	.0102048	.0103039	.0104045	.0105063	.0106091	.010818	.011246
.0005	.0020	.0101077	.0102046	.0103035	.0104037	.0105051	.0106074	.010815	.011240
.001	.0015	.0101075	.0102043	.0103028	.0104026	.0105033	.0106049	.010810	.011230
.00125	.00125	.0101075	.0102042	.0103027	.0104024	.0105031	.0106047	.010810	.011229
$\rho_1^2+\rho_2^2=.01$									
0	.01	.0104419	.0108540	.0112880	.0117418	.0122147	.0127067	.013748	.016070
.0001	.0099	.0104416	.0108533	.0112865	.0117393	.0122109	.0127013	.013739	.016048
.0002	.0098	.0104415	.0108526	.0112851	.0117369	.0122073	.0126961	.013729	.016027
.0003	.0097	.0104413	.0108519	.0112838	.0117346	.0122037	.0126909	.013720	.016006
.0005	.0095	.0104408	.0108507	.0112811	.0117300	.0121967	.0126810	.013702	.015965
.001	.009	.0104399	.0108477	.0112749	.0117195	.0121806	.0126580	.013662	.015871
.002	.008	.0104383	.0108427	.0112648	.0117022	.0121540	.0126201	.013595	.015717
.003	.007	.0104371	.0108392	.0112575	.0116898	.0121351	.0125931	.013547	.015607
.005	.005	.0104362	.0108364	.0112517	.0116800	.0121199	.0125715	.013508	.015518
ρ_1^2=Constant(.0015)									
.0015	.0015	.0101292	.0102454	.0103640	.0104843	.0106058	.0107285	.010977	.011486
.0015	.0025	.0101726	.0103285	.0104879	.0105500	.0108143	.0109807	.011319	.012018
.0015	.0035	.0102163	.0104123	.0106135	.0108188	.0110275	.0112394	.011673	.012576
.0015	.0045	.0102603	.0104971	.0107410	.0109907	.0112454	.0115050	.012038	.013160
.0015	.0055	.0103045	.0105827	.0108703	.0111657	.0114682	.0117775	.012416	.013773
.0015	.0065	.0103491	.0106692	.0110015	.0113439	.0116959	.0120570	.012807	.014415
.0015	.0075	.0103939	.0107567	.0111345	.0115254	.0119286	.0123437	.013210	.015086
.0015	.0085	.0104390	.0108450	.0112695	.0117102	.0121664	.0126378	.013626	.015788
$\rho_1^2\ \rho_2^2=.00005$									
.005	.01	.0106619	.0112791	.0119290	.0126084	.0133162	.0140521	.015609	.019070
.002	.025	.0112481	.0124853	.0138589	.0153690	.017018	.01790	.02286	.0329
.001	.05	.012564	.015386	.018808	.02283	.02803	.0330	.046	.082

Table 2

Powers of the r_3^2 test for testing $\rho = 0$ against different simple alternative hypotheses

ρ	n = 5	10	15	20	25	30	40	60
				$m = 0 \quad \alpha = .05$				
.001	.0500003	.0500005	.0500007	.0500009	.0500011	.0500014	.0500019	.0500028
.005	.0500062	.0500118	.0500175	.0500232	.0500289	.0500346	.0500461	.0500690
.01	.0500246	.0500472	.0500700	.0500928	.0501157	.0501386	.0501845	.0502763
.02	.0500986	.0501891	.0502805	.0503722	.0504642	.0505565	.0507413	.0511127
.03	.0502223	.0504265	.0506332	.0508410	.0510597	.0512593	.0516805	.0525310
.05	.0506203	.0511937	.0517769	.0523667	.0529622	.0535630	.0547804	.0572756
				$m = 1 \quad \alpha = .05$				
.001	.0500002	.0500004	.0500006	.0500007	.0500009	.0500011	.0500014	.0500022
.005	.0500048	.0500091	.0500135	.0500178	.0500222	.0500266	.0500354	.0500530
.01	.0500192	.0500365	.0500539	.0500714	.0500889	.0501065	.0501416	.0502120
.02	.0500769	.0501460	.0502158	.0502858	.0503560	.0504262	.0505664	.0508459
.03	.0501729	.0503283	.0504849	.0506415	.0507977	.0509531	.0512604	.0518547
.05	.0504756	.0508970	.0513109	.0517102	.0520898	.0524448	.0530640	.0538247
				$m = 2 \quad \alpha = .05$				
.001	.0500002	.0500003	.0500004	.0500006	.0500007	.0500009	.0500012	.0500018
.005	.0500041	.0500077	.0500113	.0500149	.0500186	.0500222	.0500295	.0500441
.01	.0500163	.0500307	.0500452	.0500597	.0500743	.0500889	.0501181	.0501766
.02	.0500654	.0501130	.0501811	.0502394	.0502980	.0503566	.0504743	.0507105
.03	.0501474	.0502773	.0504086	.0505407	.0506734	.0508065	.0510742	.0516141
.05	.0504112	.0507754	.0511454	.0515195	.0518969	.0522775	.0530476	.0546224
				$m = 5 \quad \alpha = .05$				
.001	.0500001	.0500002	.0500003	.0500004	.0500005	.0500006	.0500008	.0500012
.005	.0500031	.0500057	.0500082	.0500108	.0500134	.0500159	.0500211	.0500313
.01	.0500125	.0500227	.0500330	.0500432	.0500535	.0500638	.0500843	.0501255
.02	.0500499	.0500909	.0501320	.0501731	.0502144	.0502557	.0503384	.0505045
.03	.0501125	.0502050	.0502978	.0503909	.0504842	.0505779	.0507658	.0511445
.05	.0503136	.0505729	.0508339	.0510968	.0513616	.0516281	.0521663	.0532631

Table 2(Cont'd.)

ρ	n = 5	10	15	20	25	30	40	60
				m = 0 α = .01				
.001	.01000006	.01000012	.01000018	.01000025	.01000031	.01000037	.01000050	.01000075
.005	.0100015	.0100030	.0100046	.0100062	.0100077	.0100093	.0100125	.0100189
.01	.0100060	.0100121	.0100184	.0100247	.0100310	.0100374	.0100502	.0.00758
.02	.0100242	.0100486	.0100738	.0100992	.0101247	.0101504	.0102021	.0103062
.03	.0100545	.0101098	.0101667	.0102245	.0102827	.0103414	.0104599	.0107008
.05	.0101522	.0103082	.0104701	.0106358	.0108044	.0109757	.0113259	.0120561
				m = 1 α = .01				
.001	.01000004	.01000010	.01000014	.01000020	.01000023	.01000028	.01000038	.01000057
.005	.0100012	.0100023	.0100035	.0100047	.0100058	.0100071	.0100094	.0100143
.01	.0100046	.0100092	.0100140	.0100187	.0100234	.0100282	.0100378	.0100572
.02	.0100186	.0100369	.0100558	.0100749	.0100941	.0101135	.0101523	.0102308
.03	.0100418	.0100833	.0101261	.0101695	.0102132	.0102574	.0103464	.0105273
.05	.0101167	.0102335	.0103550	.0104790	.0106052	.0107334	.0109949	.0115382
				m = 2 α = .01				
.001	.01000003	.01000008	.01000011	.01000016	.01000020	.01000023	.01000030	.01000047
.005	.0100010	.0100019	.0100029	.0100038	.0100048	.0100058	.0100078	.0100117
.01	.0100039	.0100077	.0100115	.0100154	.0100193	.0100233	.0100312	.0100471
.02	.0100156	.0100307	.0100462	.0100619	.0100777	.0100936	.0101255	.0101899
.03	.0100352	.0100693	.0101044	.0101400	.0101759	.0102121	.0102852	.0104334
.05	.0100983	.0101942	.0102936	.0103952	.0104985	.0106032	.0108167	.0112589
				m = 5 α = .01				
.001	.01000002	.01000005	.01000008	.01000011	.01000014	.01000016	.01000021	.01000032
.005	.0100007	.0100014	.0100021	.0100027	.0100034	.0100041	.0100055	.0100082
.01	.0100029	.0100056	.0100083	.0100110	.0100137	.0100164	.0100219	.0100328
.02	.0100118	.0100223	.0100331	.0100439	.0100548	.0100658	.0100879	.0101323
.03	.0100265	.0100503	.0100746	.0190992	.0101240	.0101489	.0101992	.0103010
.05	.0100739	.0101408	.0102094	.0102792	.0103501	.0104217	.0105673	.0108671

Table 3

Powers of ℓ_2 test for testing: $\omega_1 = 0$, $\omega_2 = 0$ against

different simple alternative hypotheses

$m = 0 \quad \alpha = .05$

n

ω_1	ω_2	5	10	15	20	25	30	40	60
0	$.0^41$	.0500003	.0500003	.0500003	.0500003	.0500003	.0500003	.0500003	.0500003
0	.0001	.0500025	.0500029	.0500031	.0500032	.0500033	.0500034	.0500034	.0500035
.0005	.0005	.0500246	.0500291	.0500311	.0500322	.0500330	.0500335	.0500342	.0500349
0	.01	.0502465	.0502909	.0503112	.0503228	.0503302	.0503355	.0503423	.0503495
ω_1=Constant(.005)									
.005	.015	.0504931	.0505822	.0506228	.0506459	.0506609	.0506713	.0506850	.0506994
.005	.025	.0507404	.0508743	.0509354	.0509702	.0509927	.0510085	.0510290	.0510507
.005	.045	.0512367	.0514610	.0515633	.0510217	.0516595	.0516859	.0517204	.0517567
$\omega_1+\omega_2=.1$									
0	.1	.0524889	.0529447	.0531527	.0532716	.0533485	.0534023	.0534726	.0535466
.01	.09	.0524846	.0529384	.0531455	.0532639	.0533404	.0533939	.0534639	.0535375
.02	.08	.0524813	.0529336	.0531400	.0532578	.0533341	.0533874	.0534571	.0535304
.03	.07	.0524789	.0529302	.0531360	.0532536	.0533296	.0533828	.0534523	.0535254
.05	.05	.0524770	.0529274	.0531328	.0532501	.0533260	.0533790	.0534484	.0535213
$\omega_1+\omega_2=.5$									
0	.5	.0629750	.0654892	.0666565	.0673103	.0677405	.0680420	.0684364	.0688519
.01	.49	.0629519	.0654559	.0666080	.0672687	.0676969	.0679970	.0683896	.0688031
.02	.48	.0629297	.0654238	.0665710	.0672289	.0676552	.0679539	.0683446	.0687562
.03	.47	.0629085	.0653932	.0665356	.0671907	.0676152	.0679126	.0683016	.0687113
.05	.45	.0628689	.0653359	.0664696	.0671195	.0675405	.0678354	.0682212	.0686275
.10	.40	.0627864	.0652166	.0663320	.0669710	.0673849	.0676747	.0680538	.0684530
.15	.35	.0627274	.0651314	.0662337	.0668649	.0672737	.0675599	.0679342	.0683282
.20	.30	.0626920	.0650803	.0661727	.0668013	.0672069	.0674910	.0678054	.0682534
.25	.25	.0626802	.0650632	.0661550	.0667801	.0671847	.0674681	.0678624	.0682284
0	1	.077227	.082817	.085408	.086899	.087866	.088545	.089434	.0903708
0	2	.1092	.1224	.1286	.1321	.1344	.1361	.1381	.1404
1	1	.1048	.1160	.1212	.1242	.1262	.1275	.1293	.1312
0	3	.145	.168	.178	.184	.188	.191	.194	.198
$\omega_1\omega_2=.01$									
.01	1	.077478	.083115	.085727	.087229	.088205	.088889	.089785	.090729
.02	.5	.0634740	.0660798	.0672788	.0679665	.0684122	.0687245	.0691330	.0695633
.03	.3333	.0592508	.0609990	.0618008	.0622600	.0625573	.0627656	.0630379	.0633246
.04	.25	.0573177	.0586838	.0593092	.0596670	.0598987	.0600609	.0602729	.0604960
.05	.02	.0562752	.0574380	.0579698	.0582740	.0584708	.0586086	.0587887	.0589782
.08	.125	.0551124	.0560512	.0564799	.0567296	.0568834	.0569944	.0571394	.0572919
.1	.1	.0549839	.0558980	.0563155	.0565540	.0567083	.0568163	.0569574	.0571059

Table 3 (Cont'd.)

$m = 1 \quad \alpha = .05$

ω_1	ω_2	n = 5	10	15	20	25	30	40	60
0	$.0^41$	.0500002	.0500002	.0500002	.0500002	.0500002	.0500002	.0500002	.0500002
0	.0001	.0500016	.0500019	.0500021	.0500022	.0500023	.0500023	.0500024	.0500025
.0005	.0005	.0500161	.0500196	.0500213	.0500223	.0500230	.0500235	.0500241	.0500248
0	.01	.0501606	.0501961	.0502134	.0502236	.0502303	.0502350	.0502413	.0502480
ω_1=Constant(.005)									
.005	.015	.0503213	.0503924	.0504271	.0504474	.0504608	.0504703	.0504829	.0504963
.005	.025	.0504823	.0505893	.0506413	.0506720	.0506921	.0507064	.0507254	.0507456
.005	.045	.0508053	.0509846	.0510717	.0511230	.0511568	.0511808	.0512125	.0512464
$\omega_1+\omega_2=.1$									
0	.1	.0516194	.0519830	.0521598	.0522641	.0523329	.0523817	.0524463	.0525152
.01	.09	.0516170	.0519793	.0521554	.0522593	.0523278	.0523763	.0524406	.0525092
.02	.08	.0516152	.0519765	.0521520	.0522556	.0523238	.0523722	.0524362	.0525046
.03	.07	.0516139	.0519744	.0521150	.0522528	.0523209	.0523691	.0524331	.0525012
.05	.05	.0516128	.0519728	.0521476	.0522507	.0523186	.0523668	.0524305	.0524986
$\omega_1\omega_2=.5$									
0	.5	.0584008	.0603901	.0613676	.0619473	.0623309	.0626034	.0629647	.0633512
.01	.49	.0583880	.0603702	.0613436	.0619209	.0623028	.0625741	.0629338	.0633185
.02	.48	.0583758	.0603520	.0613216	.0618956	.0622759	.0625459	.0629041	.0632871
.03	.47	.0583640	.0603326	.0612986	.0618713	.0622500	.0625190	.0628756	.0632571
.05	.45	.0583421	.0602983	.0612576	.0618260	.0622018	.0624687	.0628226	.0632009
.10	.40	.0582965	.0602983	.0611719	.0617315	.0621014	.0623640	.0627120	.0630840
.15	.35	.0582639	.0601758	.0611107	.0616641	.0620297	.0622892	.0626330	.0630005
.20	.30	.0582444	.0601452	.0610740	.0616236	.0619866	.0622443	.0625856	.0629503
.25	.25	.0582378	.0601349	.0610618	.0616101	.0619723	.0622293	.0625698	.0629336
0	1	.067551	.071952	.074137	.075439	.076303	.076918	.077734	.078609
0	2	.0880	.0984	.1037	.1068	.1089	.1104	.1124	.1145
1	1	.0854	.0944	.0988	.1015	.1033	.1045	.1062	.1079
0	3	.111	.129	.138	.144	.147	.150	.153	.157
$\omega_1\omega_2=.01$									
.01	1	.067715	.072154	.074357	.075670	.076541	.077161	.077985	.078867
.02	.5	.0587265	.0607895	.0618027	.0624037	.0628012	.0630836	.0634580	.0638585
.03	.3333	.0560032	.0573918	.0580708	.058473	.0587383	.0588927	.0591765	.0594434
.04	.25	.0547540	.0558411	.0563716	.0566852	.0568922	.0570391	.0572337	.0574416
.05	.2	.0540795	.0550061	.0554575	.0557243	.0559004	.0560252	.0561906	.0563672
.08	.125	.0533264	.0540757	.0544401	.0546553	.0547972	.0548978	.0550311	.0551733
.1	.1	.0532431	.0539729	.0543278	.0545373	.0546755	.0547734	.0549031	.0550416

Table 3 (Cont'd.)

ω_1	ω_2	5	10	15	20	25	30	40	60
				$m = 2 \quad \alpha = .05$					
0	$.0^41$	.0500001	.0500002	.0500002	.0500002	.0500002	.0500002	.0500002	.0500002
0	.0001	.0500012	.0500015	.0500017	.0500017	.0500018	.0500019	.0500019	.0500020
.0005	.0005	.0500120	.0500150	.0500166	.0500175	.0500181	.0500185	.0500191	.0500198
0	.01	.0501201	.0501503	.0501656	.0501749	.0501811	.0501856	.0501916	.0501980
ω_1=Constant(.005)									
.005	.015	.0502402	.0503008	.0503314	.0503500	.0503625	.053714	.0503833	.0503963
.005	.025	.0503606	.0504515	.0504977	.0505256	.0505444	.0505578	.0505758	.0505952
.005	.045	.0506020	.0507540	.0508314	.0508782	.0509097	.0509325	.0509622	.0509948
$\omega_1+\omega_2=.1$									
0	.1	.0512096	.0515174	.0516744	.0517695	.0518332	.0518789	.0519401	.0520063
.01	.09	.0512081	.0515150	.0516714	.0517661	.0518296	.0518751	.0519360	.0520019
.02	.08	.0512070	.0515131	.0516690	.0517634	.0518267	.0518721	.0519328	.0519985
.03	.07	.0512061	.0515117	.0516673	.0517615	.0518247	.0518700	.0519305	.0519960
.05	.05	.0512055	.0515106	.0516660	.0517600	.0518231	.0518682	.0519287	.0519941
$\omega_1+\omega_2=.5$									
0	.5	.0562452	.0579126	.0587728	.0592968	.0596495	.0599029	.0602428	.0606115
.01	.49	.0562370	.0578991	.0587561	.0592781	.0596293	.0598817	.0602202	.0605873
.02	.48	.0562292	.0578862	.0587401	.0592601	.0596100	.0598614	.0601985	.0605641
.03	.47	.0562217	.0578738	.0587248	.0592430	.0595915	.0598420	.0601778	.0605418
.05	.45	.0562077	.0578506	.0586962	.0592109	.0595770	.0598057	.0601390	.0605004
.10	.40	.0561786	.0578023	.0586366	.0591441	.0594851	.0597300	.0600583	.0604140
.15	.35	.0561578	.0577678	.0585941	.0590963	.0594337	.0596760	.0600006	.0603522
.20	.30	.0561453	.0577471	.0585686	.0590677	.0594029	.0596436	.0599660	.0603152
.25	.25	.0561411	.0577402	.0585601	.0590581	.0593926	.0596328	.0599545	.0603028
0	1	.062981	.066637	.068546	.069716	.070506	.071075	.071840	.072672
0	2	.0779	.0865	.0911	.0939	.0958	.0972	.0990	.1011
1	1	.0762	.0837	.0876	.0900	.0916	.0928	.0943	.0960
0	3	.095	.109	.117	.122	.125	.128	.131	.134
$\omega_1\omega_2=.01$									
.01	1	.0631040	.0667925	.0687178	.0698978	.0706945	.0712684	.0720397	.0728782
.02	.5	.0564886	.0582185	.0591104	.0596385	.0600195	.0602822	.0606346	.0610167
.03	.3333	.0544726	.0556415	.0562414	.0566060	.0568509	.0570268	.0572624	.0575177
.04	.25	.0535456	.0544628	.0549322	.0552172	.0554086	.0555459	.0557297	.0559288
.05	.2	.0530444	.0538272	.0542273	.0544699	.0546328	.0547496	.0549060	.0550753
.08	.125	.0524843	.0531183	.0534418	.0536378	.0537693	.0538636	.0539897	.0541262
.1	.1	.0524223	.0530400	.0533550	.0535459	.0536740	.0537658	.0538886	.0540215

Table 3 (Cont'd.)

m = 5 α = .05

ω_1	ω_2	n = 5	10	15	20	25	30	40	60
0	$.0^41$	.0500001	.0500001	.0500001	.0500001	.0500001	.0500001	.0500001	.0500001
0	.0001	.0500007	.0500009	.0500010	.0500011	.0500012	.0500012	.0500012	.0500013
.0005	.0005	.0500069	.0500091	.0500102	.0500110	.0500116	.0500119	.0500125	.0500131
0	.01	.0500691	.0500905	.0501024	.0501101	.0501155	.0501194	.0501248	.0501308
ω_1 = Constant (.005)									
.005	.015	.0501383	.0501811	.0502050	.0502203	.0502310	.0502389	.0502497	.0502618
.005	.025	.0502076	.0502718	.0503078	.0503308	.0503469	.0503587	.0503749	.0503931
.005	.045	.0503464	.0504537	.0505138	.0505524	.0505793	.0505991	.0506263	.0506567
$\omega_1 + \omega_2$=.1									
0	.1	.0506950	.0509115	.0510331	.0511111	.0511655	.0512056	.0512607	.0513223
.01	.09	.0506944	.0509104	.0510317	.0511095	.0511623	.0512036	.0512586	.0513200
.02	.08	.0506939	.0509096	.0510306	.0511082	.0511613	.0512021	.0512569	.0513182
.03	.07	.0506936	.0509090	.0510298	.0511073	.0511605	.0512011	.0512557	.0513169
.05	.05	.0506933	.0509085	.0510292	.0511066	.0523552	.0512002	.0512548	.0513158
$\omega_1 + \omega_2$=.5									
0	.5	.0535538	.0547018	.0553540	.0557753	.0560702	.0562880	.0565885	.0569256
.01	.49	.0535506	.0546958	.0553462	.0557663	.0560602	.0562773	.0565768	.0569127
.02	.48	.0535475	.0546902	.0553388	.0557576	.0560506	.0562671	.0565655	.0569003
.03	.47	.0535446	.0546847	.0553317	.0557494	.0560415	.0562573	.0565547	.0568884
.05	.45	.0535391	.0546746	.0553184	.0557339	.0560244	.0562389	.0565346	.0568663
.10	.40	.0535276	.0546535	.0552908	.0557017	.0559888	.0562007	.0564927	.0568201
.15	.35	.0535194	.0546384	.0552711	.0556786	.0559633	.0561734	.0564628	.0567871
.20	.30	.0535145	.0546294	.0552592	.0556648	.0559480	.0561570	.0564449	.0567673
.25	.25	.0535128	.0546264	.0552553	.0556602	.0559430	.0561516	.0564389	.0567606
0	1	.057306	.059768	.061185	.062107	.062756	.063236	.063901	.064649
0	2	.0654	.0710	.0743	.0765	.0780	.0792	.0807	.0825
1	1	.0647	.0698	.0727	.0746	.0759	.0769	.0782	.0798
0	3	.074	.084	.089	.093	.096	.098	.101	.104
$\omega_1 \omega_2$=.01									
.01	1	.0573769	.0598613	.0612912	.0622215	.0628753	.0633599	.0640303	.0647852
.02	.5	.0536935	.0548853	.0555622	.0559994	.0563053	.0565313	.0568430	.0571927
.03	.3333	.0525563	.0533686	.0538278	.0541236	.0543302	.0544827	.0546927	.0549280
.04	.25	.0520306	.0526710	.0530320	.0532643	.0534264	.0535459	.0537105	.0538947
.05	.2	.0517457	.0522937	.0526022	.0528006	.0529389	.0530409	.0531812	.0538383
.08	.125	.0514266	.0518720	.0521224	.0522831	.0523952	.0524777	.0525913	.0527183
.1	.1	.0513913	.0518254	.0520693	.0522259	.0523350	.0524155	.0525261	.0526498

Table 3 (Cont'd.)

m = 0 α = .01

ω_1	ω_2	n = 5	10	15	20	25	30	40	60
0	$.0^41$	.0100001	.0100001	.0100001	.0100001	.0100001	.0100001	.0100001	.0100001
0	.0001	.0100006	.0100008	.0100008	.0100009	.0100009	.0100009	.0100009	.0100010
.0005	.0005	.0100062	.0100077	.0100085	.0100089	.0100092	.0100094	.0100096	.0100099
0	.01	.0100624	.0100775	.0100848	.0100890	.0100918	.0100937	.0100963	.0100991
ω_1= Constant (.005)									
.005	.015	.0101250	.0101552	.0101697	.0101782	.0101838	.0101877	.0101929	.0101984
.005	.025	.0101878	.0102334	.0102552	.0102680	.0102764	.0102823	.0102901	.0102985
.005	.045	.0103140	.0103907	.0104275	.0104490	.0104632	.0104731	.0104863	.0105003
$\omega_1 + \omega_2$ = .1									
0	.1	.0106344	.0107915	.0108672	.0109116	.0109408	.0109614	.0109885	.0110174
.01	.09	.0106329	.0107891	.0108643	.0109084	.0109373	.0109578	.0109847	.0110134
.02	.08	.0106317	.0107872	.0108620	.0109059	.0109347	.0109550	.0109817	.0110102
.03	.07	.0106308	.0107858	.0108604	.0109041	.0109327	.0109530	.0109796	.0110080
.05	.05	.0106302	.0107848	.0108591	.0109026	.0109312	.0109514	.0109779	.0110062
$\omega_1 + \omega_2$ = .5									
0	.5	.0133981	.0143221	.0147770	.0150464	.0152244	.0153505	.0155175	.0156960
.01	.49	.0133897	.0143085	.0147605	.0150281	.0152049	.0153301	.0154960	.0156731
.02	.48	.0133816	.0142955	.0147447	.0150106	.0151861	.0153106	.0154753	.0156511
.03	.47	.0133739	.0142830	.0147296	.0149938	.0151682	.0152919	.0154555	.0156301
.05	.45	.0133596	.0142597	.0147013	.0149625	.0151348	.0152569	.0154185	.0155909
.10	.40	.0133297	.0142111	.0146424	.0148972	.0150651	.0151841	.0153414	.0155092
.15	.35	.0133083	.0141764	.0146003	.0148505	.0150153	.0151320	.0152863	.0154509
.20	.30	.0132955	.0141555	.0145752	.0148225	.0149854	.0151008	.0152533	.0154159
.25	.25	.0132912	.0141586	.0145667	.0148131	.0149755	.0150904	.0152423	.0154042
0	1	.017372	.019582	.020692	.021356	.021796	.022110	.022526	.022973
0	2	.0271	.0331	.0361	.0380	.0392	.0401	.0413	.0425
1	1	.0253	.0300	.0324	.0339	.0348	.0355	.0364	.0373
0	3	.039	.051	.056	.060	.062	.064	.066	.069
$\omega_1\omega_2$ = .01									
.01	1	.017440	.019668	.020787	.021457	.021900	.022217	.022636	.023086
.02	.5	.0135287	.0144864	.0149577	.0152367	.0154210	.0155517	.0157246	.0159093
.03	.3333	.0123958	.0130219	.0133274	.0135076	.0136263	.0137104	.0138215	.0139399
.04	.25	.0118847	.0123677	.0126024	.0127404	.0128313	.0128955	.0129804	.0130708
.05	.2	.0116112	.0120193	.0122170	.0123332	.0124096	.0124636	.0125348	.0126108
.08	.125	.0113078	.0116342	.0117919	.0118844	.0119451	.0119881	.0120447	.0121050
.1	.1	.0112743	.0115919	.0117452	.0118351	.0118942	.0119359	.0119909	.0120496

Table 3 (Cont'd.)

$m = 1 \quad \alpha = .01$

ω_1	ω_2	n = 5	10	15	20	25	30	40	60
0	$.0^41$	.01000003	.01000005	.01000005	.01000006	.01000006	.01000006	.01000006	.01000006
0	.0001	.0100004	.0100005	.0100006	.0100006	.0100006	.0100006	.0100007	.0100007
.0005	.0005	.0100040	.0100051	.0100056	.0100060	.0100062	.0100064	.0100066	.0100068
0	.01	.0100396	.0100509	.0100564	.0100600	.0100623	.0100640	.0100662	.0100685
ω_1 = Constant (.005)									
.005	.015	.0100793	.0101018	.0101132	.0101202	.0101248	.0101281	.0101325	.0101372
.005	.025	.0101191	.0101530	.0101702	.0101806	.0101876	.0101926	.0101992	.0102064
.005	.045	.0101991	.0102559	.0102849	.0103024	.0103141	.0103225	.0101033	.0103457
$\omega_1 + \omega_2 = .1$									
0	.1	.0104014	.0105173	.0105767	.0106126	.0106366	.0106539	.0106769	.0107017
.01	.09	.0104006	.0105160	.0105750	.0106107	.0106346	.0106518	.0106746	.0106993
.02	.08	.0104000	.0105150	.0105737	.0106093	.0106331	.0106501	.0106729	.0106974
.03	.07	.0103996	.0105143	.0105728	.0106083	.0106320	.0106489	.0106716	.0106960
.05	.05	.0103993	.0105136	.0105772	.0106074	.0106311	.0106480	.0106706	.0106949
$\omega_1 + \omega_2 = .5$									
0	.5	.0121216	.0127856	.0131332	.0133460	.0134895	.0135928	.0137313	.0138816
.01	.49	.0121173	.0127782	.0131239	.0133354	.0134780	.0135807	.0137184	.0138677
.02	.48	.0121133	.0127711	.0131149	.0133253	.0134671	.0135691	.0137059	.0138542
.03	.47	.0121094	.0127643	.0131064	.0133156	.0134566	.0135580	.0136940	.0138415
.05	.45	.0121022	.0127517	.0130904	.0132975	.0134370	.0135373	.0136718	.013[illegible]75
.10	.40	.0120871	.0127253	.0130572	.0132598	.0133961	.0134941	.0136254	.0137676
.15	.35	.0120764	.0127064	.0130335	.0132329	.0133669	.0134633	.0135923	.0137320
.20	.30	.0120699	.0126951	.0130192	.0132167	.0133494	.0134448	.0135724	.0137106
.25	.25	.0120678	.0126914	.0130144	.0132113	.0133436	.0134386	.0135658	.0137035
0	1	.014537	.016088	.016919	.017433	.017783	.018035	.018375	.018747
0	2	.0203	.0244	.0266	.0280	.0290	.0297	.0307	.0317
1	1	.0194	.0227	.0245	.0256	.0263	.0269	.0276	.0284
0	3	.027	.035	.039	.042	.044	.045	.047	.049
$\omega_1\omega_2 = .01$									
.01	1	.014579	.016144	.016981	.017500	.017853	.018107	.018450	.018824
.02	.5	.0122038	.0128924	.013253	.0134733	.0136220	.0137290	.0138725	.0140282
.03	.3333	.0115044	.0119593	.012195	.0123389	.0124355	.0125049	.0125979	.0126985
.04	.25	.0111869	.0115397	.011722	.0118325	.0119068	.0119601	.0120315	.0121086
.05	.2	.0110163	.0113154	.011469	.0115628	.0116255	.0116705	.0117306	.0117955
.08	.125	.0108266	.0110669	.0111902	.0112649	.0113149	.0113508	.0113987	.0114504
.1	.1	.0108056	.0110395	.0111595	.0112321	.0112808	.0113157	.0113623	.0114126

Table 3 (Cont'd.)

m = 2 α = .01

ω_1	ω_2	n = 5	10	15	20	25	30	40	60
0	$.0^{4}1$	.01000002	.01000003	.01000004	.01000005	.01000005	.01000005	.01000005	.01000005
0	.0001	.0100003	.0100003	.0100004	.0100005	.0100005	.0100005	.0100005	.0100005
.0005	.0005	.0100029	.0100038	.0100043	.0100046	.0100048	.0100050	.0100051	.0100054
0	.01	.0100292	.0100383	.0100432	.0100462	.0100483	.0100498	.0100517	.0100539
ω_1=Constant (.005)									
.005	.015	.0100585	.0100767	.0100865	.0100925	.0100966	.0100996	.0101036	.0101080
.005	.025	.0100878	.0101153	.0101300	.0101391	.0101452	.0101497	.0101557	.0101623
.005	.045	.0101467	.0101928	.0102174	.0102327	.0102430	.0102506	.0102607	.0102718
$\omega_1+\omega_2$=.1									
0	.1	.0102953	.0103891	.0104393	.01047055	.0104918	.0105072	.0105279	.0105507
.01	.09	.0102948	.0103883	.0104383	.0104693	.0104904	.0105057	.0105264	.0105490
.02	.08	.0102944	.0103876	.0104374	.0104683	.0104894	.0105046	.0105252	.0105477
.03	.07	.0102942	.0103871	.0104368	.0104677	.0104886	.0105038	.0105243	.0105467
.05	.05	.0102939	.0103868	.0104363	.0104671	.0104880	.0105032	.0105236	.0105460
$\omega_1+\omega_2$=.5									
0	.5	.0115461	.0120729	.0123612	.0125424	.0126666	.0127571	.0128799	.0130150
.01	.49	.0115436	.0120681	.0123551	.0125353	.0126589	.0127488	.0128709	.0130052
.02	.48	.0115411	.0120636	.0123492	.0125286	.0126514	.0127409	.0128623	.0129959
.03	.47	.0115388	.0120593	.0123436	.0125221	.0126443	.0127333	.0128541	.0129869
.05	.45	.0115344	.0120512	.0123333	.0125100	.0126310	.0127192	.0128387	.0129701
.10	.40	.0115252	.0120344	.0123113	.0124847	.0126034	.0126897	.0128066	.0129352
.15	.35	.0115187	.0120223	.0122957	.0124667	.0125836	.0126686	.0127838	.0129102
.20	.30	.0115148	.0120151	.0122863	.0124559	.0125718	.0126560	.0127700	.0128953
.25	.25	.0115135	.0120127	.0122832	.0124523	.0125678	.0126518	.0127654	.0128903
0	1	.013272	.014477	.015153	.015583	.015879	.016098	.016394	.016723
0	2	.0173	.0204	.0221	.0233	.0241	.0247	.0255	.0264
1	1	.0167	.0193	.0207	.0217	.0223	.0228	.0234	.0241
0	3	.022	.028	.031	.033	.035	.036	.038	.040
$\omega_1\omega_2$ = .01									
.01	1	.013302	.014519	.015200	.015634	.015934	.016153	.016452	.016783
.02	.5	.0116064	.0121529	.0124519	.0126397	.0127685	.0128623	.0129895	.0131296
.03	.3333	.0111008	.0114648	.0116621	.0117855	.0118698	.0119311	.0120141	.0121052
.04	.25	.0108702	.0111537	.0113067	.0114021	.0114672	.0115145	.0115785	.0116486
.05	.2	.0107460	.0109869	.0111166	.0111974	.0112524	.0112924	.0113464	.0114056
.08	.125	.0106076	.0108018	.0109060	.0109707	.0110148	.0110468	.0110900	.0111373
.1	.1	.0105923	.0107814	.0108828	.0109458	.0109887	.0110198	.0110618	.0111078

Table 3 (Cont'd.)

$m = 5 \quad \alpha = .01$

ω_1	ω_2	n = 5	10	15	20	25	30	40	60
0	$.0^{4}1$	.01000001	.01000002	.01000002	.01000003	.01000003	.01000003	.01000003	.01000003
0	.0001	.0100002	.0100003	.0100003	.0100003	.0100003	.0100003	.0100003	.0100004
.0005	.0005	.0100016	.0100023	.0100026	.0100028	.0100030	.0100031	.0100033	.0100035
0	.01	.0100165	.0100226	.0100261	.0100284	.0100300	.0100312	.0100329	.0100348
ω_1=Constant (.005)									
.005	.015	.0100330	.0100451	.0100522	.0100568	.0100600	.0100625	.0100659	.0100697
.005	.025	.0100495	.0100677	.0100783	.0100853	.0100902	.0100939	.0100989	.0101047
.005	.045	.0100826	.0101131	.0101309	.0101426	.0101508	.0101569	.0101654	.0101751
$\omega_1+\omega_2$=.1									
0	.1	.0101659	.0102276	.0102637	.0102874	.0103042	.0103167	.0103340	.0103537
.01	.09	.0101658	.0102273	.0102632	.0102869	.0103036	.0103160	.0103333	.0103529
.02	.08	.0101656	.0102270	.0102629	.0102865	.0103031	.0103155	.0103327	.0103522
.03	.07	.0101656	.0102268	.0102626	.0102862	.0103028	.0103151	.0103323	.0103517
.05	.05	.0101655	.0102267	.0102624	.0102859	.0103025	.0103148	.0103320	.0103514
$\omega_1+\omega_2$=.5									
0	.5	.0108553	.0111892	.0113886	.0115210	.0116153	.0116859	.0117844	.0118967
.01	.49	.0108544	.0111874	.0113860	.0115179	.0116118	.0116821	.0117801	.0118919
.02	.48	.0108535	.0111855	.0113835	.0115149	.0116084	.0116784	.0117761	.0118873
.03	.47	.0108526	.0111838	.0113831	.0115120	.0116052	.0116749	.0117722	.0118829
.05	.45	.0108510	.0111806	.0113766	.0115067	.0115990	.0116684	.0117649	.0118747
.10	.40	.0108477	.0111738	.0113673	.0114956	.0115867	.0116547	.0117496	.0118576
.15	.35	.0108453	.0111690	.0113607	.0114876	.0115777	.0116450	.0117387	.0118453
.20	.30	.0108439	.0111661	.0113568	.0114828	.0115723	.0116391	.0117322	.0118380
.25	.25	.0108434	.0111651	.0113554	.0114813	.0115705	.0116372	.0117300	.0118355
0	1	.011777	.012511	.012960	.013262	.013479	.013642	.013871	.014134
0	2	.0138	.0156	.0167	.0175	.0180	.0185	.0191	.0198
1	1	.0136	.0152	.0161	.0167	.0172	.0176	.0180	.0186
0	3	.016	.019	.022	.023	.024	.025	.026	.027
$\omega_1\omega_2$ = .01									
.01	1	.011794	.0125359	.012988	.013292	.013511	.013676	.013907	.014172
.02	.5	.0108890	.0112357	.0114425	.0115799	.0116777	.0117510	.0118532	.0119697
.03	.3333	.0106132	.0108475	.0109862	.0110779	.0111430	.0111916	.0112593	.0113362
.04	.25	.0104864	.0106702	.0107786	.0108502	.0109008	.0109387	.0109913	.0110510
.05	.2	.0104177	.0105747	.0106671	.0107279	.0107710	.0108031	.0108477	.0108984
.08	.125	.0103410	.0104683	.0105429	.0105919	.0106266	.0106525	.0106884	.0107291
.1	.1	.0103325	.0104565	.0105292	.0105769	.0106107	.0106359	.0106709	.0107105

Table 4

Powers of the ℓ_3 test for testing: $\omega = 0$ against different simple alternative hypotheses

ω	n = 5	10	15	20	25	30	40	60
				$m = 0$ $\alpha = .05$				
$.0^41$	.0500001	.0500002	.0500002	.0500002	.0500002	.0500002	.0500002	.0500002
.0001	.0500014	.0500017	.0500019	.0500019	.0500020	.0500020	.0500021	.0500022
.001	.0500137	.0500169	.0500184	.0500193	.0500199	.0500204	.0500209	.0500216
.01	.0501370	.0501688	.0501843	.0501935	.0501996	.0502039	.0502096	.0502158
.02	.0502742	.0503379	.0503691	.0503876	.0503998	.0504085	.0504200	.0504323
.03	.0504117	.0505075	.0505544	.0505822	.0506006	.0506136	.0506309	.0506494
.05	.0506875	.0508479	.0509266	.0509732	.0510040	.0510259	.0510548	.0510859
.1	.0513816	.0517063	.0518657	.0519603	.0520228	.0520673	.0521262	.0521892
.2	.0527894	.0534543	.0537818	.0539763	.0541051	.0541967	.0543181	.0544481
.3	.0542234	.0552440	.0557483	.0560482	.0562470	.0563883	.0565759	.0567768
.5	.0571701	.0589486	.0598322	.0603593	.0607090	.0609580	.0612888	.0616434
1	.0649981	.0689415	.0709237	.0721124	.0729036	.0734680	.0742194	.0750266
2	.082720	.092108	.096905	.099800	.101736	.103118	.104964	.106949
3	.10345	.11965	.12798	.13303	.13641	.13882	.14203	.14551
				$m = 1$ $\alpha = .05$				
$.0^41$	.0500001	.0500001	.0500001	.0500002	.0500002	.0500002	.0500002	.0500002
.0001	.0500010	.0500012	.0500014	.0500014	.0500015	.0500015	.0500016	.0500016
.001	.0500096	.0500122	.0500135	.0500142	.0500148	.0500152	.0500157	.0500163
.01	.0500960	.0501217	.0501349	.0501429	.0501483	.0501522	.0501574	.0501631
.02	.0501923	.0502436	.0502700	.0502861	.0502970	.0503048	.0503152	.0503266
.03	.0502887	.0503658	.0504055	.0504297	.0504461	.0504578	.0504735	.0504906
.05	.0504819	.0506109	.0506775	.0507181	.0507454	.0507651	.0507915	.0508200
.1	.0509677	.0512284	.0513632	.0514455	.0515009	.0515407	.0515942	.0516522
.2	.0519551	.0524831	.0527593	.0529281	.0530420	.0531238	.0532338	.0533532
.3	.0529499	.0537643	.0541885	.0544481	.0546233	.0547495	.0549190	.0551031
.5	.0549946	.0564065	.0571463	.0576006	.0579077	.0581291	.0584269	.0587510
1	.0603880	.0634791	.0651252	.0661426	.0668332	.0673325	.0680057	.0687402
2	.072320	.079631	.083596	.086070	.087760	.088985	.090643	.092459
3	.08579	.09841	.10537	.10973	.11274	.11498	.11787	.12110

Table 4 (Cont'd.)

ω	n: 5	10	15	20	25	30	40	60
				m = 2 α = .05				
$.0^41$	.05000006	.05000008	.05000010	.05000011	.05000011	.05000013	.05000013	.05000013
.0001	.0500007	.0500010	.0500011	.0500012	.0500012	.0500012	.0500013	.0500013
.001	.0500074	.0500096	.0500108	.0500115	.0500120	.0500123	.0500128	.0500134
.01	.0500743	.0500960	.0501076	.0501149	.0501200	.0501235	.0501284	.0501337
.02	.0501487	.0501922	.0502155	.0502301	.0502400	.0502473	.0502471	.0502678
.03	.0502232	.0502885	.0506236	.0503455	.0503605	.0503714	.0503861	.0504023
.05	.0503726	.0504818	.0505406	.0505772	.0506023	.0506205	.0506452	.0506722
.1	.0507478	.0509682	.0510870	.0511611	.0512119	.0512487	.0512987	.0513537
.2	.0515058	.0519547	.0521974	.0523492	.0524532	.0525290	.0526313	.0527441
.3	.0522744	.0529597	.0533314	.0535644	.0537240	.0538403	.0539981	.0541717
.5	.0538427	.0550254	.0556708	.0560767	.0563554	.0565585	.0568347	.0571392
1	.0579464	.0605173	.0619402	.0628416	.0634634	.0639179	.0645378	.0652235
2	.068942	.072927	.076316	.078488	.079997	.081105	.082622	.084308
3	.07761	.08725	.09318	.09699	.09965	.10161	.10431	.10731
				m = 5 α = .05				
$.0^41$	.05000003	.05000006	.05000007	.05000007	.05000008	.05000008	.05000008	.05000008
.0001	.0500004	.0500006	.0500007	.0500007	.0500008	.0500008	.0500009	.0500009
.001	.0500044	.0500060	.0500069	.0500074	.0500079	.0500082	.0500086	.0500091
.01	.0500445	.0500598	.0500687	.0500745	.0500787	.0500817	.0500860	.0500909
.02	.0500891	.0501196	.0501375	.0501492	.0501575	.0501636	.0501722	.0501820
.03	.0501338	.0501796	.0502064	.0502239	.0502364	.0502457	.0502586	.0502733
.05	.0502232	.0502997	.0503445	.0503739	.0503948	.0504103	.0504320	.0504565
.1	.0504475	.0506015	.0506918	.0507512	.0507933	.0508247	.0508684	.0509180
.2	.0508994	.0512114	.0513948	.0515157	.0516016	.0516656	.0517548	.0518562
.3	.0513555	.0518297	.0521092	.0522938	.0524249	.0525229	.0526595	.0528148
.5	.0522809	.0530916	.0535722	.0538908	.0541175	.0542872	.0545241	.0547940
1	.0546711	.0563965	.0574334	.0581261	.0586216	.0589936	.0595151	.0601118
2	.059784	.063666	.066059	.067674	.068842	.069235	.070968	.072407
3	.06534	.07183	.07586	.07871	.08074	.08228	.08446	.08697

Table 4 (Cont'd.)

ω	n: 5	10	15	20	25	30	40	60
				m = 0 α = .01				
$.0^41$	.01000003	.01000004	.01000004	.01000005	.01000005	.01000005	.01000006	.01000006
.0001	.0100003	.0100004	.0100005	.0100005	.0100005	.0100005	.0100006	.0100006
.001	.0100033	.0100043	.0100048	.0100051	.0100053	.0100055	.0100057	.0100059
.01	.0100335	.0100434	.0100484	.0100515	.0100536	.0100550	.0100580	.0100591
.02	.0100672	.0100869	.0100971	.0101032	.0101074	.0101103	.0101143	.0101186
.03	.0101009	.0101306	.0101459	.0101552	.0101614	.0101659	.0101719	.0101783
.05	.0101687	.0102186	.0102443	.0102598	.0102703	.0102778	.0102879	.0102987
.1	.0103397	.0104412	.0104936	.0105255	.0105470	.0105623	.0105829	.0106052
.2	.0106887	.0108988	.0110080	.0110747	.0111195	.0111517	.0111949	.0112417
.3	.0110472	.0113731	.0115436	.0116479	.0117182	.0117687	.0118367	.0119102
.5	.0117931	.0123729	.0126796	.0128683	.0129960	.0130880	.0132118	.0133464
1	.0138375	.0151882	.0159204	.0163765	.0166872	.0169124	.0172167	.0175491
2	.018864	.022383	.024364	.025621	.026487	.027119	.027977	.028922
3	.02556	.03214	.03596	.03840	.04010	.04133	.04303	.04489
				m = 1 α = .01				
$.0^41$	.01000002	.01000003	.01000004	.01000004	.01000004	.01000004	.01000004	.01000004
.0001	.0100003	.0100003	.0100004	.0100004	.0100004	.0100004	.0100004	.0100004
.001	.0100024	.0100031	.0100035	.0100037	.0100039	.0100040	.0100041	.0100044
.01	.0100232	.0100307	.0100348	.0100374	.0100391	.0100404	.0100420	.0100439
.02	.0100464	.0100615	.0100698	.0100749	.0100783	.0100809	.0100843	.0100881
.03	.0100697	.0100924	.0101049	.0101125	.0101177	.0101216	.0101267	.0101324
.05	.0101164	.0101546	.0101753	.0101882	.0101970	.0102035	.0102120	.0102217
.1	.0102340	.0103115	.0103537	.0103801	.0103980	.0104111	.0104288	.0104483
.2	.0104732	.0106328	.0107200	.0107747	.0108121	.0108394	.0108764	.0109171
.3	.0107175	.0109638	.0110991	.0111842	.0112426	.0112852	.0113430	.0114066
.5	.0112222	.0116558	.0118967	.0120491	.0121540	.0122310	.0123350	.0124503
1	.0125790	.0135676	.0141304	.0144913	.0147419	.0149259	.0151779	.0154576
2	.015720	.018237	.019715	.020688	.021374	.021877	.022577	.023364
3	.01945	.02413	.02694	.02884	.03018	.03118	.03256	.03412

Table 4 (Cont'd)

ω	n = 5	10	15	20	25	30	40	60
				m = 2 α = .01				
$.0^4 1$	.01000002	.01000002	.01000002	.01000003	.01000003	.01000003	.01000003	.01000003
.0001	.0100002	.0100002	.0100002	.0100003	.0100003	.0100003	.0100003	.0100004
.001	.0100017	.0100024	.0100027	.0100030	.0100031	.0100032	.0100034	.0100035
.01	.0100177	.0100240	.0100275	.0100296	.0100312	.0100323	.0100339	.0100356
.02	.0100355	.0100480	.0100550	.0100595	.0100626	.0100639	.0100680	.0100714
.03	.0100533	.0100721	.0100826	.0100894	.0100940	.0100975	.0101021	.0101074
.05	.0100891	.0101205	.0101381	.0101495	.0101573	.0101631	.0101709	.0101796
.1	.0101790	.0102426	.0102785	.0103015	.0103174	.0103291	.0103451	.0103630
.2	.0103612	.0104917	.0105656	.0106131	.0106461	.0106704	.0107036	.0107407
.3	.0105470	.0107473	.0108615	.0109350	.0109862	.0110239	.0110757	.0111335
.5	.0109283	.0112783	.0114800	.0116105	.0117018	.0117692	.0118619	.0119657
1	.0119424	.0127267	.0131895	.0134932	.0137073	.0138664	.0140865	.0143344
2	.014242	.016178	.017370	.018169	.018741	.019170	.019768	.020449
3	.01693	.02045	.02269	.02424	.02533	.02618	.02731	.02867
				m = 5 α = .01				
$.0^4 1$	.01000001	.01000001	.01000001	.01000001	.01000002	.01000002	.01000002	.01000002
.0001	.0100001	.0100002	.0100002	.0100002	.0100002	.0100002	.0100002	.0100002
.001	.0100010	.0100015	.0100017	.0100019	.0100020	.0100021	.0100022	.0100024
.01	.0100105	.0100147	.0100172	.0100189	.0100201	.0100210	.0100223	.0100238
.02	.0100210	.0100293	.0100344	.0100378	.0100403	.0100421	.0100447	.0100476
.03	.0100315	.0100440	.0100517	.0100568	.0100605	.0100632	.0100671	.0100716
.05	.0100526	.0100736	.0100863	.0100949	.0101010	.0101057	.0101122	.0101197
.1	.0101055	.0101478	.0101736	.0101909	.0102034	.0102128	.0102260	.0102411
.2	.0121221	.0102983	.0103510	.0103865	.0104121	.0104314	.0104586	.0104899
.3	.0103203	.0104516	.0105323	.0105869	.0106263	.0106560	.0106979	.0107463
.5	.0105404	.0107665	.0109068	.0110021	.0110711	.0111234	.0111973	.0112827
1	.0111143	.0111605	.0119151	.0121289	.0122850	.0124040	.0125731	.0127700
2	.012367	.013512	.014265	.014797	.015191	.015495	.015930	.016442
3	.01376	.01575	.01712	.01810	.01883	.01939	.02021	.02117

Table 5. Upper 5% Points of the Largest Root for s = 11

n \ m	0	1	2	3	4	5	7	10	15
5	.91618	.92506	.93220	.93809	.94302	.94722	.95398	.96139	.96954
10	.80023	.81700	.83104	.84301	.85333	.86234	.87734	.89449	.91433
15	.69988	.72072	.73863	.75422	.76796	.78016	.80094	.82546	.85496
20	.61840	.64100	.66075	.67823	.69385	.70790	.73225	.76169	.79822
25	.55248	.57557	.59602	.61433	.63086	.64589	.67227	.70479	.74615
30	.49857	.52149	.54199	.56052	.57739	.59284	.62026	.65457	.69911
40	.41635	.43806	.45777	.47582	.49245	.50786	.53563	.57122	.61887
60	.31222	.33087	.34808	.36409	.37907	.39314	.41899	.45310	.50067
80	.24945	.26546	.28038	.29438	.30758	.32010	.34335	.37458	.41924
100	.20760	.22153	.23459	.24692	.25861	.26976	.29061	.31896	.36020
130	.16581	.17741	.18836	.19874	.20866	.21815	.23605	.26067	.29712
160	.13800	.14792	.15731	.16626	.17483	.18306	.19867	.22032	.25272
200	.11277	.12107	.12895	.13649	.14373	.15071	.16401	.18257	.21067
300	.077378	.083257	.088870	.094261	.099463	.10450	.11415	.12777	.14859*
500	.047533	.051239	.054792	.058217	.061535	.064759	.070972	.07983*	.09349*
1000	.024197	.026121	.027971	.029760	.031498	.033192	.03647*	.04118*	.04850*

* Value extrapolated

Table 6. Upper 1% Points of the Largest Root for s = 11

n \ m	0	1	2	3	4	5	7	10	15
5	.94070	.94704	.95213	.95632	.95982	.96280	.96760	.97284	.97860
10	.83835	.85213	.86365	.87344	.88187	.88922	.90143	.91535	.93140
15	.74297	.76119	.77680	.79036	.80228	.81285	.83081	.85193	.87725
20	.66254	.68299	.70083	.71657	.73061	.74322	.76501	.79127	.82370
25	.59590	.61729	.63618	.65307	.66828	.68208	.70625	.73593	.77351
30	.54051	.56209	.58135	.59873	.61450	.62893	.65447	.68631	.72746
40	.45466	.47557	.49449	.51179	.52770	.54242	.56887	.60266	.64769
60	.34384	.36223	.37916	.39488	.40956	.42333	.44858	.48178	.52790
80	.27602	.29200	.30687	.32080	.33392	.34633	.36934	.40017	.44407
100	.23042	.24443	.25755	.26991	.28163	.29277	.31359	.34182	.38272
130	.18457	.19634	.20741	.21792	.22792	.23749	.25551	.28023	.31669
160	.15391	.16402	.17357	.18266	.19136	.19970	.21550	.23734	.26994
200	.12599	.13447	.14253	.15022	.15760	.16471	.17822	.19705	.22546
300	.086653	.092700	.098468	.10400	.10934	.11450	.12437	.13827	.1594*
500	.053335	.057164	.060832	.064365	.067785	.071105	.077494	.08650*	.1004*
1000	.027190	.029186	.031103	.032956	.034754	.036505	.03990*	.04470*	.05216*

* Value extrapolated.

Table 7. Upper 5% Points of the Largest Root for s = 12

n \ m	0	1	2	3	4	5	7	10	15
5	.92561	.93309	.93918	.94424	.94852	.95218	.95812	.96469	.97199
10	.81794	.83260	.84499	.85560	.86481	.87289	.88640	.90194	.92008
15	.72182	.74049	.75664	.77077	.78326	.79439	.81343	.83602	.86335
20	.64214	.66273	.68082	.69690	.71130	.72430	.74689	.77432	.80851
25	.57671	.59800	.61696	.63398	.64940	.66344	.68816	.71872	.75775
30	.52260	.54394	.56311	.58049	.59635	.61091	.63678	.66927	.71157
40	.43909	.45959	.47826	.49540	.51123	.52592	.55243	.58648	.63218
60	.33172	.34962	.36618	.38162	.39608	.40969	.43472	.46779	.51399
80	.26618	.28168	.29616	.30978	.32265	.33485	.35755	.38808	.43178
100	.22215	.23572	.24847	.26053	.27199	.28291	.30339	.33124	.37180
130	.17792	.18929	.20004	.21026	.22002	.22938	.24705	.27138	.30741
160	.14835	.15811	.16736	.17620	.18467	.19282	.20828	.22974	.26189
200	.12143	.12962	.13741	.14488	.15206	.15900	.17221	.19067	.2187*
300	.083511	.089338	.094914	.10028	.10546	.11049	.12012	.13371	.1545*
500	.051398	.055086	.058629	.062051	.065370	.068599	.07481*	.08362*	.09726*
1000	.026203	.028123	.029974	.031767	.03350*	.03520*	.03846*	.04313*	.05041*

* Value extrapolated.

Table 8. Upper 1% Points of the Largest Root for s = 12

n \ m	0	1	2	3	4	5	7	10	15
5	.94743	.95276	.95709	.96069	.96372	.96632	.97053	.97517	.98033
10	.85289	.86491	.87504	.88371	.89121	.89779	.90876	.92136	.93602
15	.76213	.77839	.79242	.80468	.81549	.82512	.84154	.86096	.88438
20	.68401	.70257	.71886	.73330	.74621	.75785	.77802	.80244	.83276
25	.61832	.63798	.65543	.67109	.68523	.69809	.72068	.74853	.78393
30	.56311	.58314	.60110	.61734	.63213	.64569	.66974	.69983	.73885
40	.47652	.49619	.51407	.53045	.54555	.55955	.58475	.61701	.66013
60	.36304	.38062	.39687	.41199	.42613	.43942	.46381	.49594	.54065
80	.29270	.30813	.32252	.33603	.34878	.36086	.38328	.41335	.45624
100	.24503	.25865	.27142	.28349	.29494	.30584	.32624	.35392	.39408
130	.19684	.20833	.21918	.22949	.23932	.24874	.26649	.29086	.32685
160	.16444	.17435	.18375	.19270	.20128	.20952	.22514	.24676	.27905
200	.13483	.14319	.15113	.15874	.16604	.17308	.18649	.20518	.2334*
300	.092952	.098932	.10465	.11015	.11545	.12059	.13043	.14425	.1653*
500	.057323	.061125	.064776	.068299	.071712	.075030	.08141*	.09042*	.1043*
1000	.029268	.031255	.033169	.035022	.03682*	.03858*	.04197*	.04680*	.05427*

* Value extrapolated.

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