On the distribution of the second elementary symmetric function of the roots of a matrix

by

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1. Summary and Introduction: Distribution problems in multivariate analysis are often related to the joint distribution of the characteristic roots of a matrix derived from sample observations taken from multivariate normal populations. This joint distribution (under certain null hypotheses) of s non-null characteristic roots given by Fisher [4], Girshick [5], Hsu [6], and Roy [15] can be expressed in the form

$$(1.1)\qquad f(\theta_1,\ldots,\theta_s) = C(s,m,n)\prod_{i=1}^{s}\theta_i^m(1-\theta_i)^n\prod_{i>j}(\theta_i-\theta_j)$$

$$0<\theta_1\le\ldots\le\theta_s<1$$

where

$$(1.2)\qquad C(s,m,n) = \Pi^{s/2}\prod_{i=1}^{s}\Gamma\left(\frac{2m+2n+s+i+2}{2}\right) \Big/ \left[\Gamma\left(\frac{2m+i+1}{2}\right)\Gamma\left(\frac{2n+i+1}{2}\right)\Gamma(i/2)\right]$$

and m and n are defined differently for various situations described in [9], [11]. Nanda [7] has shown that if $\xi_i = n\theta_i (i=1,\ldots,s)$, then the limiting

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distribution of ξ_i's as n tends to infinity is given by

$$(1.3) \qquad f_1(\xi_1,\xi_2,\ldots,\xi_s) = K(s,m) \prod_{i=1}^{s} \xi_i^m e^{-\xi_i} \prod_{i>j} (\xi_i-\xi_j)$$

$$0 < \xi_1 \le \ldots \le \xi_s < \infty$$

where

$$(1.4) \qquad K(s,m) = \Pi^{s/2} \Big/ \left[\prod_{i=1}^{s} \Gamma\left(\frac{2m+i+1}{2}\right) \Gamma(i/2) \right] .$$

The distribution (1.3) can also be arrived at as that of $\xi_i = \frac{1}{2}\gamma_i$ $(i=1,2,\ldots,s)$ where γ_i's are the roots of the equation $|S - \gamma\, \Sigma| = o$ where S is the variance-covariance matrix computed from a sample taken from an s-variate normal population with dispersion matrix Σ. In this paper, the first four moments of $W_2^{(s)}$, the second elementary symmetric function (esf) in s ξ's, have been obtained and approximations to its distribution suggested. In addition, the variances of the third and fourth esf's are also obtained.

2. Values of some determinants.

The joint distribution (1.3) can be thrown into a determinantal form of the Vandermonde type and integrated over the range $R, 0 < \xi_1 \le \ldots \le \xi_s < \infty$, giving

$$(2.1) \qquad \int_R f_1(\xi_1,\xi_2,\ldots,\xi_s) \prod_{i=1}^{s} d\xi_i = K(s,m) \begin{vmatrix} \int_0^\infty \xi_s^{m+s-1} e^{-\xi_s} d\xi_s \ldots \int_0^\infty \xi_s^m e^{-\xi_s} d\xi_s \\ \\ \int_0^{\xi_2} \xi_1^{m+s-1} e^{-\xi_1} d\xi_1 \ldots \int_0^{\xi_2} \xi_1^m e^{-\xi_1} d\xi_1 \end{vmatrix} .$$

Now denote by $W(s-1,s-2,\ldots,1,0)$ the determinant on the right side of (2.1). It has been shown by the present authors in an earlier paper [13] that the moments of $W_i^{(s)}(i=1,\ldots s)$, the ith esf in the s ξ's, can be obtained as linear compounds of determinants of the type $W(q_s,\ldots,q_1)$, $q_i \geq 0$. Further the authors have evaluated in that paper [13], (See also [12]) the values of each determinant involved in the first four moments of $W_2^{(s)}$. However, the evaluation of the determinants was done in successive stages using a reduction formula [10] which reduced the original determinant into two parts, the first part consisting of a linear compound of lower order determinants and the second a determinant of the same order with q_s changed to q_s-1. The second part vanishes if $q_s = q_{s-1}+1$, but otherwise successive reductions should be carried on the second part as for the original determinant. In the previous paper the values of the determinants were presented giving the results for each stage separately but these are now consolidated for determinants with $q_s > q_{s-1}+1$ and presented below.

(2.2) $K(s,m)W(s+2,s,s-3,\ldots,1,0)$

$$= \binom{s}{2} \frac{M(s,s+1)}{2^6 \cdot 3!} [8s(s+1)(s+2)m^3+12s(s+2)(s^2+3s+6)m^2$$

$$+2(s+2)(3s^4+15s^3+47s^2+59s+24)m$$

$$+(s^6+9s^5+43s^4+123s^3+196s^2+204s+144)] ,$$

(2.3) $K(s,m)W(s+2,s+1,s,s-4,\ldots,1,0)$

$$= \binom{s+2}{5} \frac{M(s-1,\ldots,s+3)}{2^{11} \cdot 18} [16(s-1)s^2(s+1)m^4$$

$$+16s^2(s-1)(s^2+2s+10)m^3$$

$$+4s(s-1)(6s^4+18s^3+125s^2+77s+432)m^2$$

$$+2(s-1)(4s^6+17s^5+107s^4+353s^3+169s^2+1278s+2520)m$$

$$+s(s+1)\{s(s-1)(s^4+4s^3+41s^2+38s+432)+1440\}\Big] ,$$

(2.4) $K(s,m)W(s+2,s+1,s,s-3,s-5,\ldots 1,0)$

$$= \binom{s+2}{6} \frac{M(s-2,\ldots,s+3)}{2^{11} \cdot 12} \Big[16(s-1)s^2(s+1)m^4$$

$$+8(s-1)s^2(2s^2+4s+23)m^3$$

$$+4s(s-1)(6s^4+18s^3+143s^2+89s+624)m^2$$

$$+2(s-1)(4s^6+17s^5+125s^4+373s^3+643s^2+838s+5880)m$$

$$+s(s+1)\{s(s-1)(s^4+4s^3+47s^2+44s+624)+2880\}\Big] ,$$

(2.5) $K(s,m)W(s+2,s,s-2,s-4,\ldots 1,0) = \Big[\binom{s}{3} \frac{(s+2)M(s,s+1)}{2 \cdot 5!}\Big] \Big[16s(s+1)m^4 +$

$$+8s(4s^2+9s+20)m^3+4(6s^4+21s^3+65s^2+50s+45)m^2$$

$$+2(4s^5+19s^4+70s^3+95s^2+70s+90)m$$

$$+(s^6+6s^5+25s^4+45s^3+19s^2+39s-135)\Big] ,$$

(2.6) $K(s,m)W(s+2,s-1,s-2,s-3,s-5,\ldots 1,0)$

$$= \Big[\binom{s}{4} \frac{(s+2)M(s-2,s-1)}{2^5 \cdot 3!}\Big] \Big[4(s+1)m^2+2(2s^2+7s+17)m +$$

$$+(s^3+6s^2+23s+42)\Big] ,$$

(2.7) $K(s,m)W(s+3,s+1,s-2,s-4,\ldots 1,0)$

$$= 5\binom{s+2}{5}\frac{M(s-1,\ldots,s+3)}{2^8 \cdot 4!}\Big[8s(s+1)(s+3)m^3 +$$

$$+4s(3s^3+19s^2+69s+117)m^2$$

$$+2(3s^5+26s^4+143s^3+450s^2+786s+576)m$$

$$+(s^6+11s^5+77s^4+341s^3+946s^2+1824s+2880)\Big],$$

(2.8) $K(s,m)W(s+3,s,s-1,s-4,\ldots,1,0)$

$$= \binom{s}{3}\frac{M(s-1,\ldots,s+1)}{2^8 \cdot 5!}\Big[32(s-1)s(s+1)(s+2)(s+3)m^5$$

$$+16s(5s^5+35s^4+105s^3+145s^2-50s-240)m^4$$

$$+8(s-1)(10s^6+100s^5+505s^4+1550s^3+2785s^2+2670s+1080)m^3$$

$$+4(10s^8+110s^7+635s^6+2265s^5+4765s^4+5125s^3+1550s^2-4020s-6120)m^2$$

$$+2(5s^9+65s^8+450s^7+2000s^6+5679s^5+10115s^4+11770s^3+7860s^2$$

$$+1656s+1440)m$$

$$+(s^{10}+15s^9+120s^8+630s^7+2193s^6+5115s^5+8630s^4+10800s^3$$

$$+12096s^2+25200s+30240)\Big],$$

(2.9) $K(s,m)W(s+3,s,s-2,s-3,s-5,\ldots 1,0)$

$$=3\binom{s}{4}\frac{M(s-2,\ldots,s+1)}{2^{11} \cdot 7}\Big[16s(s+1)(s+2)(s+3)m^4$$

$$+8s(4s^4+34s^3+132s^2+274s+228)m^3$$

$$+4(6s^6+66s^5+365s^4+1220s^3+2357s^2+2226s+672)m^2$$

$$+2(4s^7+54s^6+378s^5+1692s^4+4862s^3+8670s^2+9428s+5152)m$$

$$+s^8+16s^7+134s^6+740s^5+2757s^4+6924s^3+12212s^2+15040s+9408\Big] \quad ,$$

(2.10) $K(s,m)W(s+2,s,s-1,s-3,s-5,\ldots,1,0)$

$$= \binom{s}{4}\frac{M(s-2,\ldots,s+1)}{2^8 \, . \, 4!}\Big[16(s-1)s(s+1)(s+2)m^4$$

$$+8s(4s^4+14s^3+32s^2+10s-60)m^3$$

$$+4\,(6s^6+30s^5+113s^4+160s^3+97s^2-22s-384)m^2$$

$$+2s(4s^6+26s^5+126s^4+288s^3+458s^2+598s+228)m$$

$$+(s+2)(s^7+6s^6+34s^5+72s^4+157s^3+306s^2+1728)\Big] \; ,$$

(2.11) $K(s,m)W(s+2,s,s-2,s-3,s-4,s-6,\ldots,1,0)$

$$= \binom{s}{5}\frac{M(s-3,\ldots,s+1)}{2^6 \, . \, 7.3}\Big[8s(s+1)(s+2)m^3$$

$$+4s(3s^3+15s^2+45s+54)m^2$$

$$+2(3s^5+21s^4+95s^3+225s^2+295s+210)m$$

$$+s^6+9s^5+52s^4+177s^3+376s^2+663s+630\Big] \; ,$$

(2.12) $K(s,m)W(s+3,s-1,s-2,s-3,s-4,s-6,\ldots 1,0)$

$$= 5\binom{s}{5}\frac{M(s-3,\ldots,s+1)}{2^{12} \, . \, 3}\Big[8(s+1)(s+2)(s+3)m^3$$

$$+4(3s^4+27s^3+111s^2+237s+198)m^2$$

$$+2(3s^5+36s^4+215s^3+780s^2+1570s+1260)m$$

$$+(s^6+15s^5+115s^4+573s^3+1804s^2+3108s+2160)\Big] \quad ,$$

(2.13) $$K(s,m)W(s+2,s-1,s-2,s-3,s-4,s-5,s-7,\ldots 1,0)$$

$$= 3.\binom{s}{6}\frac{M(s-4,\ldots,s+1)}{2^{11}}\Big[4(s+1)(s+2)m^2$$

$$+2(2s^3+11s^2+35s+42)m$$

$$+s^4+8s^3+39s^2+108s+108\Big]$$

and

(2.14) $$K(s,m)W(s+1,s,s-1,s-2,s-5,\ldots,1,0)$$

$$= \Big[\binom{s}{4}M(s-2,\ldots,s+1)/2^8.5!\Big]\Big[16(s+1)s(s-1)(s-2)m^4 +$$

$$+16s(s-1)(s-2)(2s^2+3s+11)m^3$$

$$+4(s-1)(s-2)(6s^4+12s^3+65s^2-s+240)m^2$$

$$+2(s-2)(4s^6+6s^5+54s^4-68s^3+462s^2-698s+1680)m$$

$$+s^8+14s^6-60s^5+269s^4-900s^3\_2596s^2-4800s+5760\Big] \quad ,$$

3. Moments of the second, third and fourth esf's. Using the values of the determinants evaluated in the precious paper [13] and in the preceding section, the raw moments of the second esf, $W_2^{(s)}$, are obtained as follows:

(3.1) $$\mu_1'\{W_2^{(s)}\} = \binom{s}{2}M(s,s+1)\ /\ 2^2 \quad ,$$

(3.2) $$\mu_2'\left\{W_2^{(s)}\right\} = \left[\binom{s}{2}M(s,s+1)/2^5\right]\left[s(s-1)(2m)^2+(2s^3-s^2+7s-8)(2m)\right.$$
$$\left.+s^4+7s^2-8s+12\right],$$

(3.3) $$\mu_3'\left\{W_2^{(s)}\right\} = \left[\binom{s}{2}M(s,s+1)/2^8\right]\left[s^2(s-1)^2(2m)^4+2s(s-1)^2(2s^2+s+12)(2m)^3\right.$$
$$+(s-1)(6s^5+67s^3-49s^2+172s-160)(2m)^2$$
$$+2(2s^7-s^6+33s^5-47s^4+185s^3-314s^2+462s-368)(2m)$$
$$\left.+s^8+22s^6-24s^5+173s^4-296s^3+764s^2-832s+672\right],$$

and

(3.4) $$\mu_4'\left\{W_2^{(s)}\right\} = \left[\binom{s}{2}M(s,s+1)/2^{11}\right]\left[s^3(s-1)^3(2m)^6\right.$$
$$+s^2(s-1)^2(6s^3-3s^2+45s-48)(2m)$$
$$+s(s-1)^2(15s^5+228s^3-147s^2+808s-832)(2m)^4$$
$$+(s-1)(20s^8-10s^7+452s^6-569s^5+3386s^4-5679s^3+10080s^2$$
$$-13440s+5376)(2m)^3$$
$$+(15s^{10}-15s^9+453s^8-840s^7+5295s^6-12105s^5+32517s^4-62960s^3$$
$$+92536s^2-104048s+50304)(2m)^2$$
$$+(6s^{11}-3s^{10}+225s^9-330s^8+3340s^7-7155s^6+27645s^5-58688s^4$$
$$+128240s^3-198448s^2+213456s-132480)(2m)$$
$$+s^{12}+45s^{10}-48s^9+811s^8-1568s^7+8415s^6-18416s^5+54520s^4$$
$$\left.-99776s^3+164304s^2-161280s+93312\right].$$

Using the raw moments given above, the first four central moments of $W_2^{(s)}$ are obtained in the following simple forms:

$$(3.5) \qquad \mu_2\left\{W_2^{(s)}\right\} = \left[\binom{s}{2} M(s,s+1)/2^3\right]\left[4(s-1)m+2s^2-2s+3\right] ,$$

$$(3.6) \qquad \mu_3\left\{W_2^{(s)}\right\} = \left[\binom{s}{2} M(s,s+1)/2^3\right]\left[5(s-1)^2(2m)^2 +(10s^3-20s^2+30s-23)(2m) +5s^4-10s^3+25s^2-26s+21\right] ,$$

and

$$(3.7) \qquad \mu_4\left\{W_2^{(s)}\right\} = \left[3\binom{s}{2} M(s,s+1)/2^7\right]\left[4s(s-1)^3(2m)^4 +4(s-1)^2(4s^3-3s^2+30s-28)(2m)^3 +(24s^6-60s^5+408s^4-1056s^3+1833s^2-2173s+1048)(2m)^2 +(16s^7-36s^6+384s^5-1036s^4+2634s^3-4209s^2+4503s-2760)(2m) +4s^8-8s^3+124s^2-340s^5+1145s^4-2148s^3+3479s^2-3360s+1944\right] .$$

Further, the results of the previous paper [13] and the preceding section can also be used to obtain the first two raw moments (and hence the central moments) of $W_3^{(s)}$ and $W_4^{(s)}$, the third and fourth esf's respectively in the s, ξ's. It may be observed in general that

$$(3.8) \qquad \mu_1'\left\{W_i^{(s)}\right\} = K(s,m)W(s,s-1,s-2,\ldots,s-i+1,s-i-1,\ldots,1,0) , \qquad i = 1,\ldots,s$$

and the value of the right side of (3.8) is given in (4.3) of [13]. Now using the methods in section 3 of [13] we get

$$(3.9)\qquad \mu_2'\left\{W_3^{(s)}\right\} = K(s,m)[W(s+1,s,s-1,s-4,\ldots,1,0)+W(s+1,s,s-2,s-3,s-5,\ldots,1,0)$$
$$+W(s+1,s-1,s-2,s-3,s-4,s-6,\ldots,1,0)$$
$$+W(s,s-1,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)]$$

and

$$(3.10)\qquad \mu_2'\left\{W_4^{(s)}\right\} = K(s,m)[W(s+1,s,s-1,s-2,s-5,\ldots,1,0)$$
$$+W(s+1,s,s-1,s-3,s-4,s-6,\ldots,1,0)$$
$$+W(s+1,s,s-2,s-3,s-4,s-5,s-7,\ldots,1,0)$$
$$+W(s+1,s-1,s-2,s-3,s-4,s-5,s-6,s-8,\ldots,1,0)$$
$$+W(s,s-1,s-2,s-3,s-4,s-5,s-6,s-7,s-9,\ldots,1,0)]\ .$$

It may be pointed out that the values of the determinants on the right side of (3.9) and (3.10) are available in the preceding section and using these values and (3.8), the variances of $W_3^{(s)}$ and $W_4^{(s)}$ were obtained and are given below.

$$(3.11)\qquad \mu_2\left\{W_3^{(s)}\right\} = [3\binom{s}{3}M(s-1,s,s+1)/2^6]\,[(s-1)(s-2)(2m)^2$$
$$+(s-2)(2s^2-3s+7)2m+s^4-4s^3+11s^2-20s+20]\ ,$$

and

$$(3.12)\qquad \mu_2'\left\{W_3^{(s)}\right\} = \left[\binom{s}{3} M(s-1,s,s+1)/2^6 \;.\; 3!\right]\Big[s(s-1)(s-2)(2m)^3$$
$$+3(s-1)(s-2)(s^2+6)(2m)^2$$
$$+(s-2)(3s^4-3s^3+35s^2-53s+126)(2m)$$
$$+s^6-3s^5+19s^4-69s^3+196s^2-360s+360\Big] ,$$

$$(3.13)\qquad \mu_2'\left\{W_4^{(s)}\right\} = \left[\binom{s}{4} M(s-2,\ldots s+1)/2^{11} \;.3\right]\Big[s(s-1)(s-2)(s-3)(2m)^4$$
$$+2(s-1)(2s^4-11s^3+33s^2-86s+96)(2m)^3$$
$$+(6s^6-42s^5+197s^4-768s^3+2089s^2-3786s+3168)(2m)^2$$
$$+2(2s^7-15s^6+87s^5-422s^4+1569s^3-4327s^2+7906s-7536)(2m)$$
$$+s^8-8s^7+54s^6-308s^5+1413s^4-5060s^3+13508s^2-24000s+21888\Big] ,$$

and

$$(3.14)\qquad \mu_2\left\{W_4^{(s)}\right\} = \left[\binom{s}{4} M(s-2,\ldots,s+1)/4!2^4\right]\Big[2(s-1)(s-2)(s-3)(2m)^3$$
$$+3(s-2)(s-3)(2s^2-4s+11)(2m)^2$$
$$+(s-3)(6s^4-30s^3+106s^2-225s+314)2m$$
$$+2s^6-18s^5+89s^4-318s^3+845s^2-1500s+1368\Big] .$$

4. Approximations to the distribution of $W_2^{(s)}$. Using the results on moments of $W_2^{(s)}$ given in (3.1), (3.5), (3.6) and (3.7) the following approximation to the distribution of $W_2^{(s)}$ is suggested:

$$(4.1) \qquad f(W_2^{(s)}) = \frac{\alpha^{\nu}}{2\Gamma(\nu)} e^{-\alpha(W_2^{(s)})^{\frac{1}{2}}} (W_2^{(s)})^{\frac{1}{2}\nu-1} \qquad 0 < W_2^{(s)} < \infty \quad ,$$

where

$$(4.2) \qquad \nu = s(2m+s+1)/2$$

and

$$(4.3) \qquad \alpha^2 = 2[s(2m+s+1)+2] \;/\; (s-1)(2m+s) \quad .$$

It may be pointed out that the first moment is the same for the exact and approximate distributions. For further comparison, numerical values of the first four moments from the exact and approximate distributions and the ratios of the respective approximate and exact moments and the moment quotients are presented in Tables 1 to 2 for values of $s = 3, 4, 5, 7$ and 10 and selected values of m. The tables show that the ratio of the respective approximate to the exact moments tend to unity as m increases or s increases or both. On the basis of these ratios the approximate distribution might be recommended for $m = 5$ and above when $s = 3$, $m = 3$ and above for $s = 4$, $m = 2$ and above for $s = 5$ and $m = 0$ and above for $s = 7$ and all values of m and all values of s beyond 7. The values of the approximate and exact standard deviations, β_1's and β_2's practically agree in the first two places at the smallest values of m recommended for each value of s and this in turn almost guarantees sufficient accuracy for upper or lower percentage points from the approximate distribution. It may further be observed that an interesting feature of the distribution of $W_2^{(s)}$ is that it is asymptotically normal for large values of m or s.

Table 1

Ratios of moments (central) of $W_2^{(s)}$ from the exact and approximate distributions for $s = 3$ and different values of m

Moments	$m = 2$ Exact	Approximate	Ratio (A/E)	$m = 5$ Exact	Approximate	Ratio (A/E)
μ_1'	$.42000000\text{X}10^{2}$	$.42000000\text{X}10^{2}$	1.0000	$.13650000\text{X}10^{3}$	$.13650000\text{X}10^{3}$	1.0000
μ_2	$.65100000\text{X}10^{3}$	$.61061538\text{X}10^{3}$	.9380	$.37537500\text{X}10^{4}$	$.36296590\text{X}10^{4}$	.9669
μ_3	$.26271000\text{X}10^{5}$	$.22869372\text{X}10^{5}$	.8705	$.26433224\text{X}10^{6}$	$.24565092\text{X}10^{6}$	.9293
μ_4	$.31091445\text{X}10^{7}$	$.25952045\text{X}10^{7}$	.8347	$.74113715\text{X}10^{8}$	$.67887768\text{X}10^{8}$	.9160
$\sqrt{\mu_2}$	$.25514701\text{X}10^{2}$	$.24710632\text{X}10^{2}$	.9685	$.61267854\text{X}10^{2}$	$.60246652\text{X}10^{2}$	.9833
β_1	.25015561X10	.22972342X10	.9183	.13210043X10	.12619417X10	.9553
β_2	.73363312X10	.69604304X10	.9488	.52597837X10	.51529966X10	.9797

Moments	$m = 10$ Exact	Approximate	Ratio (A/E)	$m = 20$ Exact	Approximate	Ratio (A/E)
μ_1'	$.41400000\text{X}10^{3}$	$.414000000\text{X}10^{3}$	1.0000	$.14190000\text{X}10^{4}$	$.14190000\text{X}10^{4}$	1.0000
μ_2	$.19665000\text{X}10^{5}$	$.19301351\text{X}10^{5}$	.9815	$.12416250\text{X}10^{6}$	$.12294470\text{X}10^{6}$	.9902
μ_3	$.23687010\text{X}10^{7}$	$.22738869\text{X}10^{7}$	.9600	$.27374639\text{X}10^{8}$	$.26789059\text{X}10^{8}$	.9786
μ_4	$.16445728\text{X}10^{10}$	$.15718388\text{X}10^{10}$	.9558	$.56447347\text{X}10^{11}$	$.55202852\text{X}10^{11}$	.9780
$\sqrt{\mu_2}$	$.14023194\text{X}10^{3}$	$.13892930\text{X}10^{3}$	.9907	$.35236699\text{X}10^{3}$	$.35063471\text{X}10^{3}$	.9951
β_1	.73779996	.71907568	.9746	.39149430	.38617617	.9864
β_2	.42527045X10	.42192239X10	.9921	.36615303X10	.36520935X10	.9974

Moments	$m = 40$ Exact	Approximate	Ratio (A/E)	$m = 100$ Exact	Approximate	Ratio (A/E)
μ_1'	$.52290000\text{X}10^{4}$	$.52290000\text{X}10^{4}$	1.0000	$.31059000\text{X}10^{5}$	$.31059000\text{X}10^{5}$	1.0000
μ_2	$.87585749\text{X}10^{6}$	$.87143137\text{X}10^{6}$	.9949	$.12656542\text{X}10^{8}$	$.12630491\text{X}10^{8}$	.9979
μ_3	$.36828631\text{X}10^{9}$	$.36421020\text{X}10^{9}$	.9889	$.12915931\text{X}10^{11}$	$.12857552\text{X}10^{11}$	.9955
μ_4	$.25623342\text{X}10^{13}$	$.25345972\text{X}10^{13}$	.9892	$.50273540\text{X}10^{15}$	$.50060149\text{X}10^{15}$	.9958
$\sqrt{\mu_2}$	$.93587258\text{X}10^{3}$	$.93350488\text{X}10^{3}$	.9975	$.35576034\text{X}10^{4}$	$.35539402\text{X}10^{4}$	.9990
β_1	.20186954	.20044934	.9930	$.82282265\text{X}10^{-1}$	$.82045702\text{X}10^{-1}$	.9971
β_2	.33401722X10	.33376636X10	.9992	.31384072X10	.31379906X10	.9999

Table 2

Ratios of moments (central) of $W_2^{(s)}$ from the exact and approximate distributions for s = 4, 5, 7 and 10 and different values of m.

Moments	s = 4 m = 0			s = 4 m = 3		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio(A/E)
μ_1'	.30000000X10^{2}	.30000000X10^{2}	1.0000	.16500000X10^{3}	.16500000X10^{3}	1.0000
μ_2	.40500000X10^{3}	.37636364X10^{3}	.9293	.51975000X10^{4}	.50576086X10^{4}	.9731
μ_3	.14354999X10^{5}	.12228099X10^{5}	.8518	.41901749X10^{6}	.39426890X10^{6}	.9409
μ_4	.13799024X10^{7}	.11130166X10^{7}	.8066	.13880369X10^{9}	.12914122X10^{9}	.9304
$\sqrt{\mu_2}$	.20124611X10^{2}	.19400093X10^{2}	.9640	.72093689X10^{2}	.71116866X10^{2}	.9865
β_1	.31019966X10	.28047550X10	.9042	.12504917X10	.12015708X10	.9609
β_2	.84127571X10	.78575356X10	.9340	.51382120X10	.50486403X10	.9826

Moments	s = 4 m = 20			s = 4 m = 100		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	.29700000X10^{4}	.29700000X10^{4}	1.0000	.62730000X10^{5}	.62730000X10^{5}	1.0000
μ_2	.39649499X10^{6}	.39419406X10^{6}	.9942	.38484855X10^{8}	.38437464X10^{8}	.9988
μ_3	.13311094X10^{9}	.13137293X10^{9}	.9869	.59102856X10^{11}	.58938151X10^{11}	.9972
μ_4	.54702853X10^{12}	.53999866X10^{12}	.9871	.45958868X10^{16}	.45842690X10^{16}	.9975
$\sqrt{\mu_2}$	.62967848X10^{3}	.62784875X10^{3}	.9971	.62036163X10^{4}	.61997955X10^{4}	.9994
β_1	.28425910	.28176142	.9912	.61284040X10^{-1}	.61168642X10^{-1}	.9981
β_2	.34796418X10	.34751417X10	.9987	.31030560X10	.31028490X10	.9999

Moments	s = 5 m = 0			s = 5 m = 2		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	.75000000X10^{2}	.75000000X10^{2}	1.0000	.22500000X10^{3}	.22500000X10^{3}	1.0000
μ_2	.16125000X10^{4}	.15468749X10^{4}	.9593	.84375000X10^{4}	.82557692X10^{4}	.9785
μ_3	.89662500X10^{5}	.81738281X10^{5}	.9116	.80763750X10^{6}	.76890088X10^{6}	.9520
μ_4	.16395075X10^{8}	.14589294X10^{8}	.8899	.34549841X10^{9}	.32636309X10^{9}	.9446
$\sqrt{\mu_2}$	.40155946X10^{2}	.39330331X10^{2}	.9794	.91855864X10^{2}	.90861263X10^{2}	.9892
β_1	.19174432X10	.18050338X10	.9414	.10859043X10	.10506742X10	.9676
β_2	.63054191X10	.60971074X10	.9670	.48530915X10	.47883505X10	.9867

Table 2 (Cont'd.)

Moments	s = 5 m = 20			s = 7 m = 0		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	$.51750000X10^4$	$.51750000X10^4$	1.0000	$.29400000X10^3$	$.29400000X10^3$	1.0000
μ_2	$.93926249X10^6$	$.93551508X10^6$	.9960	$.12789000X10^5$	$.12560896X10^5$	.9822
μ_3	$.42815620X10^9$	$.42425269X10^9$	.9909	$.14169329X10^7$	$.13600565X10^7$	.9599
μ_4	$.29756611X10^{13}$	$.29497474X10^{13}$	.9913	$.75814050X10^9$	$.72364864X10^9$	.9545
$\sqrt{\mu_2}$	$.96915556X10^3$	$.96722028X10^3$	.9980	$.11308846X10^3$	$.11207540X10^3$	.9910
β_1	.22122975	.21983500	.9937	.95981795	.93336613	.9724
β_2	.33729468X10	.33704139X10	.9992	.46352859X10	.45865535X10	.9895

Moments	s = 7 m = 5			s = 7 m = 20		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	$.16065000X10^4$	$.16065000X10^4$	1.0000	$.11844000X10^5$	$.11844000X10^5$	1.0000
μ_2	$.16627274X10^6$	$.16514318X10^6$	.9932	$.33577739X10^7$	$.33498896X10^7$	.9977
μ_3	$.43382729X10^8$	$.42705234X10^8$	.9844	$.23872706X10^{10}$	$.23742615X10^{10}$	.9946
μ_4	$.10207592X10^{12}$	$.10046898X10^{12}$	.9843	$.36682109X10^{14}$	$.36498082X10^{14}$	.9950
$\sqrt{\mu_2}$	$.40776555X10^3$	$.40637812X10^3$	.9966	$.18324229X10^4$	$.18302704X10^4$	.9988
β_1	.40942136	.40493031	.9890	.15053897	.14995660	.9961
β_2	.36921655X10	.36839245X10	.9978	.32535042X10	.32524380X10	.9997

Moments	s = 10 m = 0			s = 10 m = 500		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	$.12375000X10^4$	$.12375000X10^4$	1.0000	$.11487487X10^8$	$.11487487X10^8$	1.0000
μ_2	$.11323125X10^6$	$.11236942X10^6$	.9924	$.10443849X10^{12}$	$.10443158X10^{12}$	.9999
μ_3	$.26148994X10^8$	$.25690653X10^8$	.9825	$.23740000X10^{16}$	$.23736303X10^{16}$	.9998
μ_4	$.48682272X10^{11}$	$.47808616X10^{11}$	.9821	$.32812861X10^{23}$	$.32808512X10^{23}$	.9999
$\sqrt{\mu_2}$	$.33649851X10^3$	$.33521548X10^3$	.9962	$.32316944X10^6$	$.32315876X10^6$	1.0000
β_1	.47098932	.46516379	.9876	$.49474294X10^{-2}$	$.49468701X10^{-2}$	.9999
β_2	.37969793X10	.37862553X10	.9972	.30083123X10	.30083112X10	1.0000

An alternate approximation (which is exact for $s = 2$) is obtained by replacing the value of ν in (4.2) by $s(2m+s)/2$ and α^2 in (4.3) by $2[s(2m+s)+2]/(s-1)(2m+s+1)$. But this second approximation is not as good as the one suggested in (4.1) even for $s = 3$.

7. Some remarks. It may be pointed that $2 \sum_{i=1}^{s} \xi_i$ is distributed [8] as a chi-square with $s(2m+s+1)$ degrees of freedom and hence the distribution problem in this case is very simple. The results of this paper show that we can also have a simple approximation to the distribution of the second esf in the s ξ's. While the former chi-square distribution can be interpreted as the limiting distribution of Pillai's $V^{(s)}$ criterion [8,11, 14], the same is also true in the present case that the distribution of $W_2^{(s)}$ can also be considered as the limiting distribution of the second esf in the s θ's following the joint density (1.1). It might also be pointed out that the distribution problem studied in this paper has great use since it has been shown that several tests based on the esf's of the characteristic roots have been observed to have monotonicity of power [1], [2], [3] .

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