An Alternative Proof of a Theorem of Takacs on the GI|M|1 Queue*

by

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Marcel F. Neuts

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Abstract

An analytic proof is given of the fact that the stationary distribution for the imbedded Markov chain in a GI|M|1 queue is geometric. A generating function for the stationary transition probabilities is obtained as the unique solution to an integro-differential equation, which may be solved by reduction to a Wiener-Hopf equation.

We assume that customers arrive at a counter at the instants $\tau_0, \tau_1, \ldots$ and that the interarrival times $\tau_{n+1}-\tau_n$, $(n=0,1,\ldots)$ are independent, identically distributed, positive random variables with distribution function $F(x)$. The service times are independent, identically distributed, negative exponential random variables with parameter μ.

This note supplies an analytic proof to the following theorem, due to Takács [2,3] :

Theorem:

The queue lengths $\xi_n, n = 0,1,\ldots$ immediately before arrivals form an irreducible, aperiodic Markov chain. This Markov chain is positive recurrent if and only if $\mu\alpha > 1$. The stationary probability distribution $\{\pi_k , k \geq 0\}$ is geometric with parameter δ, where δ is the root of the equation:

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(1) $$z = \varphi\left[\mu(1-z)\right],$$

with smallest absolute value. $\varphi(\cdot)$ is the Laplace - Stieltjes transform of $F(\cdot)$.

Remark

This theorem was proved in [2] by Takács, using the backward equations for the Markov chain, together with a renewal argument. In [3] a combinatorial proof, together with a renewal argument was given, also by Takács.

We use the forward equations and obtain a recursive system of integro-differential equations for the generating functions of the n-step transition probabilities. The same method may be applied to the study of an M|G|1 queue in series with a queue with negative exponential service times.

Proof:

The queue-lengths $\{\xi_n, n \geq o\}$ clearly form an irreducible, aperiodic Markov chain with transition probability matrix P, given by:

(2) $$\left\| \begin{matrix} 1-p_o & p_o & 0 & 0 & 0 & --- \\ 1-p_o-p_1 & p_1 & p_o & 0 & 0 & --- \\ 1-p_o-p_1-p_2 & p_e & p_1 & p_o & 0 & --- \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix} \right\|$$

in which $p_j = \int_o^\infty e^{-\mu y} \frac{(\mu y)^j}{j!} \, dF(y)$, We denote the n-step transition probabilities by $P_y^{(n)}$. They satisfy the following recurrence relations:

$$(3) \qquad P_{io}^{(n+1)} = \sum_{\nu=o}^{\infty} P_{i\nu}^{(n)} \left[1- \sum_{\rho=o}^{\nu} p_\rho\right], \quad n \geq o$$

and

$$(4) \qquad P_{ij}^{(n+1)} = \sum_{\nu=j-1}^{\infty} P_{i\nu}^{(n)} p_{\nu-j+1}, \quad \text{for} \quad j > o, \; n \geq o .$$

We introduce the generating functions:

$$(5) \qquad U^{(n)}(z) = \sum_{j=o}^{\infty} P_{ij}^{(n)} \frac{z^j}{j!}, \qquad |z| < \infty$$

and obtain the system of integro-differential equations:

$$(6) \qquad \frac{d}{dz} U^{(n+1)}(z) = \int_o^{\infty} e^{-\mu y} U^{(n)}(z+\mu y)\, dF(y),$$

The initial conditions are obtained from equation (3), via the following manipulations:

$$(7) \qquad U^{(n+1)}(o) \doteq P_{io}^{(n+1)} = 1 - \sum_{\nu=o}^{\infty} P_{i\nu}^{(n)} \sum_{\rho=o}^{\nu} p_\rho =$$

$$1 - \sum_{\rho=o}^{\infty} \sum_{\nu=\rho}^{\infty} P_{i\nu}^{(n)} p_{\nu-\rho} = 1 - \sum_{\rho=o}^{\infty} \int_o^{\infty} e^{-\mu y} D_{\mu y}^{\rho} U^{(n)}(\mu y) \, dF(y)$$

$$= 1 - \int_o^{\infty} e^{-\mu y} (1-D_{\mu y})^{-1} U^{(n)}(\mu y) \, dF(y) =$$

$$= \int_0^\infty dF(y) \int_0^{\mu y} e^{-\zeta} U^{(n)}(\zeta)\, d\zeta ,$$

It is known from the theory of Markov chains that $\lim_{n \to \infty} U^{(n)}(z) = U(z)$ exists. We have:

$$(8) \qquad U(z) = \sum_{\nu=0}^{\infty} \frac{\pi_\nu}{\nu!} z^\nu , \quad |z| < \infty$$

The function $U(z)$ must satisfy the equations:

$$(9) \qquad \frac{d}{dz} U(z) = \int_0^\infty e^{-\mu y}\, U(z+\mu y)\, dF(y) , \quad \text{for all } |z| < \infty$$

and

$$(10) \qquad U(o) = \int_0^\infty dF(y) \int_0^{\mu y} e^{-\zeta} U(\zeta)\, d\zeta$$

Equation (9) may be simplified through the following substitutions:

(11) a. $\mu y = u$

b. $U(z) = e^z R(z)$

c. $F(\frac{u}{\mu}) - I(u) = -K(u)$

$I(u)$ is a distribution with a unit-jump at zero.

We obtain:

$$(12) \qquad R(z) = \int_0^\infty R(z+u)\, dK(u) , \quad \text{for all } |z| < \infty$$

Lemma 1

If (12) holds for all $x > o$, for some entire function $R(x)$, then it holds for all $|z| < \infty$.

Proof:

This may be verified by series expansion, using the fact that the distribution $K(\cdot)$ has moments of all order.

Partial integration in the right-hand side of (12) yields:

$$(13)\qquad R'(x) = \int_0^\infty R'(x+u)\, K(u)\, du, \qquad x > 0 ,$$

Setting $x + u = y$ and $K(-u) = H(u)$, we obtain:

$$(14)\qquad R'(x) = \int_0^\infty R'(y)\, H(x-y)\, dy , \qquad x > 0$$

which is a Wiener-Hopf equation for $R'(x)$.

It follows from the standard Wiener - Hopf theory [1,4] that equation (14) has a unique entire solution, which may be seen to be either an exponential function e^{-ax} for some $0 < a < 1$ or a constant.

Knowing this, we examine under which conditions there is an exponential solution

$$R'(x) = e^{-ax}\, C ,$$

to equation (14). Substituting in (12) we obtain that a **must** be a solution to the equation:

$$(15)\qquad 1 - a = \varphi(a\mu)$$

or if we set $1 - a = \delta$, then

(16) $$\delta = \varphi \left[\mu (1-\delta) \right]$$

which is Takács' equation. The usual argument yields that (16) has a solution in $(o,1)$ if and only if $\alpha \mu > 1$.

So we obtain:

(17) $$U(x) = C e^{\delta x}$$

The initial condition does not determine the constant C, since we have already included in the calculation of the initial condition that the $\{ \pi_\nu \}$ must sum to one.

This same condition applied to (17) yields $C = 1 - \delta$, which concludes the proof of the theorem.

Acknowledgement

The author is deeply indebted to Professor Harry Pollard for his many suggestions on the proof given above.

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AN ALTERNATIVE PROOF OF A THEOREM OF TAKÁCS ON THE *GI/M/1* QUEUE*

Marcel F. Neuts

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An analytic proof is given of the fact that the stationary distribution for the imbedded Markov chain in a $GI/M/1$ queue is geometric. A generating function for the stationary transition probabilities is obtained as the unique solution to an integro-differential equation, which may be solved by reduction to a Wiener-Hopf equation.

WE ASSUME that customers arrive at a counter at the instants $\tau_0, \tau_1, \cdots$ and that the interarrival times $\tau_{n+1}-\tau_n$, $(n=0, 1, \cdots)$ are independent, identically distributed, positive random variables with distribution function $F(x)$. The service times are independent, identically distributed, negative exponential random variables with parameter μ.

This note supplies an analytic proof to the following theorem, due to TAKÁCS:[2,3]

THEOREM. *The queue lengths* ξ_n, $n=0, 1, \cdots$ *immediately before arrivals form an irreducible, aperiodic Markov chain. This Markov chain is positive recurrent if and only if* $\mu\alpha>1$. *The stationary probability distribution* $\{\pi_k, k\geqq 0\}$ *is geometric with parameter* δ, *where* δ *is the root of the equation*:

$$z=\varphi[\mu(1-z)], \tag{1}$$

with smallest absolute value. $\varphi(\cdot)$ *is the Laplace-Stieltjes transform of* $F(\cdot)$

REMARK. This theorem was proved in reference 2 by Takács, using the backward equations for the Markov chain, together with a renewal argument. In reference 3 a combinatorial proof, together with a renewal argument was given, also by Takács.

We use the forward equations and obtain a recursive system of integro-differential equations for the generating functions of the n-step transition probabilities. The same method may be applied to the study of an $M/G/1$ queue in series with a queue with negative exponential service times.

Proof. The queue-lengths $\{\xi_n, n \geqq 0\}$ clearly form an irreducible, aperiodic Markov chain with transition probability matrix P, given by:

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$$\left\|\begin{array}{cccccc} 1-p_0 & p_0 & 0 & 0 & 0 & \cdots \\ 1-p_0-p_1 & p_1 & p_0 & 0 & 0 & \cdots \\ 1-p_0-p_1-p_2 & p_l & p_1 & p_0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \end{array}\right\|, \tag{2}$$

in which $p_j=\int_0^\infty e^{-\mu y}\frac{(\mu y)^j}{j!}\,dF(y)$.

We denote the n-step transition probabilities by $P_{ij}^{(n)}$. They satisfy the following recurrence relations:

$$P_{i0}^{(n+1)}=\sum_{\nu=0}^{\nu=\infty} P_{i\nu}^{(n)}[1-\sum_{\rho=0}^{\rho=\nu} p_\rho], \qquad (n\geqq 0) \tag{3}$$

and
$$P_{ij}^{(n+1)}=\sum_{\nu=j-1}^{\nu=\infty} P_{i\nu}^{(n)}\, p_{\nu-j+1}, \quad \text{for} \quad j>0,\ n\geqq 0. \tag{4}$$

We introduce the generating functions:

$$U^{(n)}(z)=\sum_{j=0}^{j=\infty} P_{ij}^{(n)} z^j/j!, \qquad (|z|<\infty), \tag{5}$$

and obtain the system of integro-differential equations:

$$\frac{d}{dz}U^{(n+1)}(z)=\int_0^\infty \bar{e}^{\mu y}\, U^{(n)}(z+\mu y)\, dF(y). \tag{6}$$

The initial conditions are obtained from equation (3), via the following manipulations:

$$\begin{aligned} U^{(n+1)}(0)=P_{i0}^{(n+1)}&=1-\sum_{\nu=0}^{\infty} P_{i\nu}^{(n)}\sum_{\rho=0}^{\nu} p_\rho \\ &=1-\sum_{\rho=0}^{\infty}\sum_{\nu=\rho}^{\infty} P_{i\nu}^{(n)}\, p_{\nu-\rho}=1-\sum_{\rho=0}^{\infty}\int_0^\infty \bar{e}^{\mu y}\, D_{\mu y}^{\rho}\, U^{(n)}(\mu y)\, dF(y) \\ &=1-\int_0^\infty \bar{e}^{\mu y}\,(1-D_{\mu y})^{-1}U^{(n)}(\mu y)\, dF(y) \\ &=\int_0^\infty dF(y)\int_0^{\mu y}\bar{e}^{\zeta}\, U^{(n)}(\zeta)\, d\zeta. \end{aligned} \tag{7}$$

It is known from the theory of Markov chains that $\lim_{n\to\infty} U^{(n)}(z)=U(z)$ exists. We have:

$$U(z)=\sum_{\nu=0}^{\nu=\infty}(\pi_\nu/\nu!)z^\nu. \qquad (|z|<\infty) \tag{8}$$

The function $U(z)$ must satisfy the equations:

$$\frac{d}{dz}U(z)=\int_0^\infty \bar{e}^{\mu y}\, U(z+\mu y)\, dF(y), \quad \text{for all} \quad |z|<\infty, \tag{9}$$

and
$$U(0)=\int_0^\infty dF(y)\int_0^{\mu y}\bar{e}^{\zeta}\, U(\zeta)\, d\zeta. \tag{10}$$

Equation (9) may be simplified through the following substitutions:

$$\begin{aligned} &(a) && \mu y = u, \\ &(b) && U(z) = e^{z} R(z), \qquad (11) \\ &(c) && F(u/\mu) - I(u) = -K(u). \end{aligned}$$

$I(u)$ is a distribution with a unit-jump at zero. We obtain:

$$R(z) = \int_0^\infty R(z+u)\, dK(u), \quad \text{for all} \quad |z| < \infty. \qquad (12)$$

LEMMA 1. *If (12) holds for all $x>0$, for some entire function $R(x)$, then it holds for all $|z|<\infty$.*

Proof. This may be verified by series expansion, using the fact that the distribution $K(\cdot)$ has moments of all order.

Partial integration in the right-hand side of (12) yields:

$$R'(x) = \int_0^\infty R'(x+u)\, K(u)\, du. \qquad (x>0) \quad (13)$$

Setting $x+u=y$ and $K(-u)=H(u)$, we obtain:

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It follows from the standard Wiener-Hopf theory[1,4] that equation (14) has a unique entire solution, which may be seen to be either an exponential function $\bar{e}^{ax}$ for some $0<a<1$ or a constant.

Knowing this, we examine under which conditions there is an exponential solution

$$R'(x) = e^{-ax} C,$$

to equation (14). Substituting in (12) we obtain that a must be a solution to the equation:

$$1 - a = \varphi(a\mu), \qquad (15)$$

or if we set $1-a=\delta$, then

$$\delta = \varphi[\mu(1-\delta)], \qquad (16)$$

which is Takács' equation. The usual argument yields that (16) has a solution in (0, 1) if and only if $\alpha\mu > 1$. So we obtain:

$$U(x) = Ce^{\delta x}. \qquad (17)$$

The initial condition does not determine the constant C, since we have already included in the calculation of the initial condition that the $\{\pi_\nu\}$ must sum to one.

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