Further Results on the Non-Central Multivariate Beta Distribution and Moments of Traces of Two Matrices

by

C. G. Khatri and K.C.S. Pillai

Gujarat University and Purdue University

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series No. 38

March 1965

Further Results on the Non-Central Multivariate Beta Distribution and Moments of Traces of Two Matrices

by

C. G. Khatri and K.C.S. Pillai[1]

Gujarat University and Purdue University

1. Introduction and Summary. Let A_1 and A_2 be two symmetric matrices of order p, A_1, positive definite and having a Wishart distribution [2,17] with f_1 degrees of freedom, and A_2, at least positive semi-definite and having a (pseudo) non-central (linear) Wishart distribution [1,3,4,17,18] with f_2 degrees of freedom. Now let

$$A_2 = C\, Y\, Y'\, C'$$

where Y is $p \times f_2$ and C is a lower triangular matrix such that

$$A_1 + A_2 = C\, C' \quad .$$

Now Pillai's $V^{(s)}$ criterion [9,10,11,12] is the sum of the non-zero characteristic roots of the matrix $Y\, Y'$. Here s is minimum (f_2,p). Also we may note that $V^{(s)}$ = trace $Y\, Y'$ = trace $Y'Y$. It can be shown that the density function of the characteristic roots of $Y'Y$ for $f_2 \leq p$ can be obtained from that of the characteristic roots of $Y\, Y'$ for $f_2 \geq p$ if in the latter case the following changes are made: [5,17]

$$(f_1,\ f_2, p) \longrightarrow (f_1+f_2-p,\ p,\ f_2).$$

1 The work of this author was supported in part by the National Science Foundation, Grant No. GP-4600.

Hence we only consider $V^{(s)}$ when $s = p$, i.e. $V^{(p)}$, based on the density function [7] of $\underset{\sim}{L} = \underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \geq p$. The first four moments of $V^{(2)}$ in the linear case were obtained by Pillai [13] and the first two moments of $V^{(p)}$ by the authors in an earlier paper [6]. In the present paper the third and fourth moments of $V^{(p)}$ are derived using a method which differs in certain respects from the previous one.

Further, if $U^{(p)} = \mathrm{tr}(\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} - p$, [9, 10, 11, 12, 16] the first four moments of $U^{(2)}$ were derived by Pillai [13] and the first two moments of $U^{(p)}$ by the authors [6]. These results are extended in the present paper, obtaining the third and fourth moments of $U^{(p)}$ and further, some approximations to the distribution of $U^{(p)}$ are suggested in the linear case.

2. <u>Moments of $V^{(p)}$</u>. In the previous paper [6] the authors have shown that

(2.1)
$$V^{(p)} = \ell_{11} + (1-\ell_{11})\, \underset{\sim}{u}'(\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})\, \underset{\sim}{u} + \mathrm{tr}\, \underset{\sim}{L}_{22}$$

$$= -(1-\ell_{11})(1-\underset{\sim}{u}'\underset{\sim}{u} + \underset{\sim}{u}'\underset{\sim}{L}_{22}\underset{\sim}{u}) + \mathrm{tr}\, \underset{\sim}{L}_{22} + 1$$

where ℓ_{11}, $\underset{\sim}{u}$: $(p-1) \times 1$ and $\underset{\sim}{L}_{22}$: $(p-1) \times (p-1)$ are independently distributed and their respective distributions are given by

(2.2)
$$\exp(-\lambda^2) \sum_{j=0}^{\infty} \frac{(\lambda^2)^j}{j\,!} \frac{\ell_{11}^{\frac{1}{2}f_2+j-1}(1-\ell_{11})^{\frac{1}{2}f_1-1}}{\beta\,[\frac{1}{2}f_2+j,\ \frac{1}{2}f_1]}\ d\,\ell_{11} ,$$

$$(2.3) \qquad \Pi^{-\frac{1}{2}(p-1)} \{\Gamma(\frac{f_1-p+1}{2})\}^{-1}\Gamma(\tfrac{1}{2}f_1)\ (1-\underset{\sim}{u}'\underset{\sim}{u})^{\frac{1}{2}(f_1-p+1)-1}\ d\,\underset{\sim}{u}$$

and

$$(2.4) \qquad \prod_{i=1}^{p-1} [\Gamma(\frac{f_1+f_2-i}{2}) / \{\Gamma(\frac{f_1-i+1}{2})\ \Gamma(\frac{f_2-i}{2})\}]\ \Pi^{-\frac{1}{4}(p-1)(p-2)}$$

$$\left|\underset{\sim}{L}_{22}\right|^{\frac{1}{2}\{f_2-1-(p-1)-1\}} \left|\underset{\sim}{I}_{p-1}-\underset{\sim}{L}_{22}\right|^{\frac{1}{2}\{f_1-(p-1)-1\}} d\,\underset{\sim}{L}_{22} \ .$$

Let λ_i's $(i=1,2,\ldots, p-1)$ be the characteristic roots of $\underset{\sim}{L}_{22}$. Then, noting that (2.3) is invariant under an orthogonal transformation of $\underset{\sim}{u}$ we can write without loss of generality the statistic $V^{(p)}$ of (2.1) to be equivalent to

$$(2.5) \qquad V^{(p)} - 1 = \sum_{i=1}^{p-1} \lambda_i - (1-\ell_{11})[(1-\underset{\sim}{u}'\underset{\sim}{u}) + \sum_{i=1}^{p-1} \lambda_i u_i^2] \ .$$

Let $\ell_{11,0}$ be a variate whose distribution is the same as that of ℓ_{11} when $\lambda = 0$ and independently distributed of $\underset{\sim}{u}$ and $\underset{\sim}{L}_{22}$. Let $V_0^{(p)}$ be the $V^{(p)}$ statistic when $\lambda = 0$. We may note that

$$(2.6) \qquad x_1 = E(1-\ell_{11,0}) - E(1-\ell_{11}) = f_1\delta(\nu) \ ,$$

$$(2.7) \qquad x_2 = E(1-\ell_{11,0})^2 - E(1-\ell_{11})^2 = \frac{f_1(f_1+2)}{2}\ \Delta_1 \ ,$$

$$(2.8) \qquad x_3 = E(1-\ell_{11,0})^3 - E(1-\ell_{11})^3 = \frac{1}{8}\ f_1(f_1+2)(f_1+4)\ \Delta_2$$

and

$$(2.9) \qquad x_4 = E(1-\ell_{11,0})^4 - E(1-\ell_{11})^4 = \frac{1}{48}\ f_1(f_1+2)(f_1+4)(f_1+6)\ \Delta_3$$

where $\nu = f_1+f_2$,

(2.10) $$\delta(\nu) = \frac{\lambda^2}{\nu} \exp(-\lambda^2) \sum_{i=0}^{\infty} \frac{(\lambda^2)^i}{i!} \frac{1}{\frac{1}{2}\nu+i+1} = \frac{\lambda^2}{\nu}\int_0^1 (1-y)^{\frac{1}{2}\nu} \exp(-\lambda^2 y)dy$$

$$= \frac{2\lambda^2}{\nu} \sum_{i=0}^{\infty} \frac{(2\lambda^2)^i \, (-1)^i}{(\nu+2)(\nu+4)\ldots(\nu+2i+2)} \quad \text{if} \quad \lambda^2 < \tfrac{1}{2}\nu+1 \;.$$

$$= \frac{1}{\nu} \Big[\sum_{i=0}^{\frac{1}{2}\nu} \binom{\frac{1}{2}\nu}{i} \frac{(-1)^i (i!)}{(\lambda^2)^i} - \frac{(-1)^{\frac{1}{2}\nu}(\frac{1}{2}\nu!)\exp(-\lambda^2)}{(\lambda^2)^{\frac{1}{2}\nu}} \Big] \quad \text{if} \quad \tfrac{1}{2}\nu \text{ is an}$$

integer,

(2.11) $\Delta_1 = \delta(\nu)-\delta(\nu+2), \Delta_2=\delta(\nu)-2\delta(\nu+2)+\delta(\nu+4)$ and

$\Delta_3=\delta(\nu)-3\delta(\nu+2)+3\delta(\nu+4)-\delta(\nu+6)$.

The results (2.6) ----(2.9) are obtained by using the partial fractions for $\frac{1}{\nu(\nu+2)(\nu+4)\ldots}$ ..... .

Let E_1 stand for the expectation over u_i's. Let $\beta_1 = \sum_{i=1}^{p-1} \lambda_i u_i^2$. It is easy to see from (2.3) that for $i \neq j \neq k \neq \ell$,

$$
(2.12)\quad
\begin{cases}
E(u_i^2) = \dfrac{1}{f_1}\ ,\ E(u_i^4) = \dfrac{3}{f_1(f_1+2)}\ ,\ E(u_i^6) = \dfrac{15}{f_1(f_1+2)(f_1+4)}\ , \\[2ex]
\qquad\qquad E(u_i^3) = \dfrac{105}{f_1(f_1+2)(f_1+4)(f_1+6)}\ , \\[2ex]
E(u_i^2\ u_j^2) = \dfrac{1}{f_1(f_1+2)}\ ,\ E(u_i^4\ u_j^2) = \dfrac{3}{f_1(f_1+2)(f_1+4)}\ , \\[2ex]
\qquad\qquad E(u_i^6\ u_j^2) = \dfrac{15}{f_1(f_1+2)(f_1+4)(f_1+6)}\ , \\[2ex]
E(u_i^4\ u_j^4) = \dfrac{9}{f_1(f_1+2)(f_1+4)(f_1+6)}\ ,\ E(u_i^2\ u_j^2\ u_k^2) = \dfrac{1}{f_1(f_1+2)(f_1+4)}\ , \\[2ex]
\qquad\qquad E(u_i^4\ u_j^2\ u_k^2) = \dfrac{3}{f_1(f_1+2)(f_1+4)(f_1+6)} \\[2ex]
\text{and}\quad E(u_i^2\ u_j^2\ u_k^2\ u_\ell^2) = \dfrac{1}{f_1(f_1+2)(f_1+4)(f_1+6)}\ .
\end{cases}
$$

Then putting $\sum_{i=1}^{p-1}\lambda_i = \text{tr}\ \underset{\sim}{L}_{22}$, $\sum_{i>j}\lambda_i\lambda_j = \text{tr}_2\underset{\sim}{L}_{22}$, $\sum_{i>j>k}\lambda_i\lambda_j\lambda_k = \text{tr}_3\ \underset{\sim}{L}_{22}$

and $\sum_{i>j>k>\ell}\lambda_i\lambda_j\lambda_k\lambda_\ell = \text{tr}_4\underset{\sim}{L}_{22}$ **and using (2.12), we get**

$$(2.13)\qquad E_1(\beta_1) = (\text{tr}\ \underset{\sim}{L}_{22})\ /\ f_1\ \ ,$$

$$(2.14)\qquad E_1(\beta_1^2) = \frac{1}{f_1(f_1+2)}\left[3\sum\lambda_i^2 + 2\sum_{i>j}\lambda_i\lambda_j\right]$$

$$= \frac{1}{f_1(f_1+2)}\left[3(\text{tr}\underset{\sim}{L}_{22})^2 - 4(\text{tr}_2\underset{\sim}{L}_{22})\right]\ \ ,$$

(2.15) $$E_1(\beta_1^3) = \frac{1}{f_1(f_1+2)(f_1+4)}\left[15\sum\lambda_i^3+9\sum_{i\neq j}\lambda_i^2\lambda_j+6\sum_{i>j>k}\lambda_i\lambda_j\lambda_k\right]$$

$$= \frac{1}{f_1(f_1+2)(f_1+4)}\left[15(\mathrm{tr}L_{22})^3-36(\mathrm{tr}L_{22})(\mathrm{tr}_2L_{22})+24(\mathrm{tr}_3L_{22})\right] ,$$

and

(2.16) $$E_1(\beta_1^4) = \frac{1}{f_1(f_1+2)(f_1+4)(f_1+6)}\left[105\sum\lambda_i^4+60\sum_{i\neq j}\lambda_i^3\lambda_j\right.$$

$$\left.+27\sum_{i\neq j}\lambda_i^2\lambda_j^2+18\sum_{i\neq j\neq k}\lambda_i^2\lambda_j\lambda_k+\sum_{i\neq j\neq k\neq \ell}\lambda_i\lambda_j\lambda_k\lambda_\ell\right] ,$$

$$= \frac{1}{f_1(f_1+2)(f_1+4)(f_1+6)}\left[105(\mathrm{tr}L_{22})^4-360(\mathrm{tr}L_{22})^2(\mathrm{tr}_2L_{22})\right.$$

$$\left.+288(\mathrm{tr}L_{22})(\mathrm{tr}_3L_{22})-192(\mathrm{tr}_4L_{22})+144(\mathrm{tr}_2L_{22})^2\right] ,$$

because

$$\sum\lambda_i^3 = (\mathrm{tr}L_{22})^3-3\mathrm{tr}L_{22}\mathrm{tr}_2L_{22}+3\mathrm{tr}_3L_{22}$$

$$\sum_{i\neq j}\lambda_i^2\lambda_j = (\mathrm{tr}L_{22})(\mathrm{tr}_2L_{22})-3(\mathrm{tr}_3L_{22}) ,$$

$$\sum_{i\neq j}\lambda_i^3\lambda_j = (\sum\lambda_i^3)(\sum\lambda_j) - \sum\lambda_i^4$$

$$= (\mathrm{tr}L_{22})^4-3(\mathrm{tr}L_{22})^2(\mathrm{tr}_2L_{22})+3(\mathrm{tr}L_{22})(\mathrm{tr}_3L_{22})-\sum\lambda_i^4 ,$$

$$\sum_{i\neq j} \lambda_i^2\lambda_j^2 = [(\mathrm{tr}\,\underset{\sim}{L}_{22}) - 2(\mathrm{tr}_2\underset{\sim}{L}_{22})]^2 - \sum \lambda_i^4 \quad ,$$

$$\sum_{i\neq j\neq k} \lambda_i^2\lambda_j\lambda_k = 2(\mathrm{tr}\,\underset{\sim}{L}_{22})^2(\mathrm{tr}_2\underset{\sim}{L}_{22}) - 4(\mathrm{tr}_2\underset{\sim}{L}_{22})^2 - 2\sum_{i\neq j}\lambda_i^3\lambda_j ,$$

and therefore

$$\sum\lambda_i^4 = (\mathrm{tr}\,\underset{\sim}{L}_{22})^4 - 4(\mathrm{tr}\,\underset{\sim}{L}_{22})^2(\mathrm{tr}_2\underset{\sim}{L}_{22}) + 4(\mathrm{tr}\,\underset{\sim}{L}_{22})(\mathrm{tr}_3\underset{\sim}{L}_{22}) - 4(\mathrm{tr}_4\underset{\sim}{L}_{22}) + 2(\mathrm{tr}_2\underset{\sim}{L}_{22})^2 \quad ,$$

$$\sum_{i\neq j}\lambda_i^3\lambda_j = (\mathrm{tr}\,\underset{\sim}{L}_{22})^2(\mathrm{tr}_2\underset{\sim}{L}_{22}) - (\mathrm{tr}\,\underset{\sim}{L}_{22})(\mathrm{tr}_3\underset{\sim}{L}_{22}) + 4(\mathrm{tr}_4\underset{\sim}{L}_{22}) - 2(\mathrm{tr}_2\underset{\sim}{L}_{22})^2 \quad ,$$

$$\sum_{i\neq j}\lambda_i^2\lambda_j^2 = 2(\mathrm{tr}_2\underset{\sim}{L}_{22})^2 - 4(\mathrm{tr}\,\underset{\sim}{L}_{22})(\mathrm{tr}_3\underset{\sim}{L}_{22}) + 4\mathrm{tr}_4\underset{\sim}{L}_{22}, \text{ and}$$

$$\sum_{i\neq j\neq k}\lambda_i^2\lambda_j\lambda_k = 2(\mathrm{tr}\,\underset{\sim}{L}_{22})(\mathrm{tr}_3\underset{\sim}{L}_{22}) - 8(\mathrm{tr}_4\underset{\sim}{L}_{22}) \quad .$$

With the help of above results, we can write

$$E[V^{(p)}-1] = E(V_0^{(p)}-1) + x_1E(\beta) \tag{2.17}$$

$$E[V^{(p)}-1]^2 = E(V_0^{(p)}-1)^2 - x_2E(\beta^2) + 2x_1\,E(\alpha\beta) \tag{2.18}$$

$$E(V^{(p)}-1)^3 = E(V_0^{(p)}-1)^3 + x_3E(\beta^3) - 3x_2E(\alpha\beta^2) + 3x_1E(\alpha^2\beta) \tag{2.19}$$

and

$$E(V^{(p)}-1)^4 = E(V_0^{(p)}-1)^4 - x_4 E(\beta^4) + 4x_3 E(\beta^3\alpha) - 6x_2 E(\beta^2\alpha^2) + 4x_1 E(\beta\alpha^3) \tag{2.20}$$

where $\alpha = (\mathrm{tr}\underset{\sim}{L}_{22})$ and $\beta = 1 - \underset{\sim}{u}'\underset{\sim}{u} + \beta_1$.

We note that

$$E(\alpha^i\beta) = \frac{1}{f_1} E[\mathrm{tr}\underset{\sim}{L}_{22})^i f^{(1)} + (\mathrm{tr}\underset{\sim}{L}_{22})^{i+1}] = \frac{1}{f_1} r_i^{(1)}; \quad \text{for } i=0,1,2,3 \tag{2.21}$$

$$E(\alpha^i\beta^2) = \frac{1}{f_1(f_1+2)} E(\mathrm{tr}\underset{\sim}{L}_{22})^i[f^{(2)} + 2f^{(1)}(\mathrm{tr}\underset{\sim}{L}_{22}) + 3(\mathrm{tr}\underset{\sim}{L}_{22})^2 - 4(\mathrm{tr}_2\underset{\sim}{L}_{22})] = \frac{1}{f_1(f_1+2)} r_i^{(2)} \text{ for } i=0,1,2 ; \tag{2.22}$$

$$E(\alpha^i\beta^3) = \frac{1}{f_1(f_1+2)(f_1+4)} E(\mathrm{tr}\underset{\sim}{L}_{22})^i[f^{(3)} + 3f^{(2)}(\mathrm{tr}\underset{\sim}{L}_{22}) + 3f^{(1)}\{3(\mathrm{tr}\underset{\sim}{L}_{22})^2 - 4(\mathrm{tr}_2\underset{\sim}{L}_{22})\} + \{15(\mathrm{tr}\underset{\sim}{L}_{22})^3 - 36(\mathrm{tr}\underset{\sim}{L}_{22})(\mathrm{tr}_2\underset{\sim}{L}_{22}) + 24\ \mathrm{tr}_3\underset{\sim}{L}_{22}] = \frac{1}{f_1(f_1+2)(f_1+4)} r_i^{(3)} \tag{2.23}$$

for $i = 0, 1,$

and

(2.24)
$$E(\beta^4) = \frac{1}{f_1(f_1+2)(f_1+4)(f_1+6)} \quad E[f^{(4)}+4f^{(3)}(\mathrm{tr}\underset{\sim}{L}_{22})$$

$$+6f^{(2)}\{3(\mathrm{tr}\underset{\sim}{L}_{22})^2-4(\mathrm{tr}_2\underset{\sim}{L}_{22})\}+4f^{(1)}\{15(\mathrm{tr}\underset{\sim}{L}_{22})^3$$

$$-36(\mathrm{tr}\underset{\sim}{L}_{22})(\mathrm{tr}_2\underset{\sim}{L}_{22})+24(\mathrm{tr}_3\underset{\sim}{L}_{22})\}+105(\mathrm{tr}\underset{\sim}{L}_{22})^4$$

$$-360(\mathrm{tr}\underset{\sim}{L}_{22})^2(\mathrm{tr}_2\underset{\sim}{L}_{22})+288(\mathrm{tr}\underset{\sim}{L}_{22})(\mathrm{tr}_3\underset{\sim}{L}_{22})-192(\mathrm{tr}_4\underset{\sim}{L}_{22})$$

$$+144(\mathrm{tr}_2\underset{\sim}{L}_{22})^2] = \frac{1}{f_1(f_1+2)(f_1+4)(f_1+6)} \quad r_0^{(4)} \quad ,$$

where

(2.25)
$$f^{(i)} = \prod_{j=1}^{i} (f_1-p-1+2j) \quad .$$

Hence, we have

(2.26)
$$E(V^{(p)}-1) = E(V_0^{(p)}-1) + \delta(\nu)\; r_0^{(1)} \quad ,$$

(2.27)
$$E(V^{(p)}-1)^2 = E(V_0^{(p)}-1)^2+\delta(\nu)[2r_1^{(1)}-\tfrac{1}{2}r_0^{(2)}]+(\tfrac{1}{2}r_0^{(2)})\delta(\nu+2) \quad ,$$

(2.28)
$$E(V^{(p)}-1)^3 = E(V_0^{(p)}-1)^3+\delta(\nu)[3r_2^{(1)}-\tfrac{3}{2}r_1^{(2)}+\tfrac{1}{8}r_0^{(3)}]$$

$$+\delta(\nu+2)[\tfrac{3}{2}r_1^{(2)}-\tfrac{1}{4}r_0^{(3)}]+\delta(\nu+4)[\tfrac{1}{8}r_0^{(3)}] \quad ,$$

and

(2.29) $$E(V^{(p)}-1)^4 = E(V_0^{(p)}-1)^4 + \delta(\nu)[4r_3^{(1)} - 3r_2^{(2)} + \tfrac{1}{2}r_1^{(3)} - \tfrac{1}{48}r_0^{(4)}]$$

$$+\delta(\nu+2)[9r_2^{(2)} - r_1^{(3)} + \tfrac{1}{16}r_0^{(4)}] + \delta(\nu+4)$$

$$[\tfrac{1}{2}r_1^{(3)} - \tfrac{1}{16}r_0^{(4)}] + \delta(\nu+6)[\tfrac{1}{48}r_0^{(4)}] \quad .$$

Expressions for the first two moments of $V^{(p)}$ are presented in [6]. For the third moment

(2.30) $$E(V^{(p)}-1)^3 = E(V_0^{(p)}-1)^3 + \Delta_2 f^{(3)}/8$$

$$+(3/8)[\{-3\delta(\nu)+2\delta(\nu+2)+\delta(\nu+4)\}f^{(2)}E(\mathrm{tr}\underset{\sim}{L}_{22})$$

$$+\{3\delta(\nu)+2\delta(\nu+2)+3\delta(\nu+4)\}f^{(1)}E(\mathrm{tr}\underset{\sim}{L}_{22})^2$$

$$+\{\delta(\nu)+2\delta(\nu+2)+5\delta(\nu+4)\}E(\mathrm{tr}\underset{\sim}{L}_{22})^3$$

$$+4\{\delta(\nu)+2\delta(\nu+2)-3\delta(\nu+4)\}E(\mathrm{tr}\underset{\sim}{L}_{22}\mathrm{tr}_2\underset{\sim}{L}_{22})$$

$$+4\Delta_2\{-f^{(1)}E(\mathrm{tr}_2\underset{\sim}{L}_{22})+2E(\mathrm{tr}_3\underset{\sim}{L}_{22})\}] \quad ,$$

where

(2.31) $$E(\mathrm{tr}_i\underset{\sim}{L}_{22}) = \binom{p-1}{i}\prod_{j=1}^{i}[(f_2-j)/(f_1+f_2-j)]$$

$$(i=1,2,\ldots,\ p-1;\ \mathrm{tr}_1\underset{\sim}{L}_{22} = \mathrm{tr}\underset{\sim}{L}_{22}) \quad ,$$

(2.32) $$E(trL_{22}tr_iL_{22}) = [E(tr_iL_{22})/\{(f_1+f_2+1)(f_1+f_2-i-1)\}] \cdot$$
$$[(f_1-p)\{(f_2-1)(p-1)+2i\}+(p-1)(f_2-p-1)(f_2+2p-i+1)$$
$$+2p^3-ip^2+(i-2)p+2i]$$

and

(2.33) $$E(trL_{22})^3 = E(trL_{22})\Big[\frac{2^3(f_1-f_2+1)f_1(f_1+f_2-2p+1)(f_1+f_2-p)}{(f_1+f_2-3)(f_1+f_2-2)(f_1+f_2-1)^2(f_1+f_2+1)(f_1+f_2+3)}$$
$$+3\,E(trL_{22})^2-2\{E(trL_{22})\}^2\Big] .$$

Now, for the fourth moment

(2.34) $$E(V^{(p)}-1)^4 = E(V_0^{(p)}-1)^4 -\Delta_3 f^{(4)}/48$$
$$+(1/12)\{5\delta(\nu)-9\delta(\nu+2)+3\delta(\nu+4)+\delta(\nu+6)\}f^{(3)}E(tr\underset{\sim}{L}_{22})$$
$$+(3/8)\{-5\delta(\nu)+19\delta(\nu+2)+\delta(\nu+4)+\delta(\nu+6)\}f^{(2)}E(tr\underset{\sim}{L}_{22})^2$$
$$+(\tfrac{1}{4})\{5\delta(\nu)+51\delta(\nu+2)+3\delta(\nu+4)+5\delta(\nu+6)\}f^{(1)}E(tr\underset{\sim}{L}_{22})^3$$
$$+(1/16)\{5\delta(\nu)+297\delta(\nu+2)+15\delta(\nu+4)+35\delta(\nu+6)\}E(tr\underset{\sim}{L}_{22})^4$$
$$+(3/2)\{\delta(\nu)-15\delta(\nu+2)+3\delta(\nu+4)-5\delta(\nu+6)\}E[(tr\underset{\sim}{L}_{22})^2tr_2\underset{\sim}{L}_{22}]$$
$$+3\Delta\{-f^{(1)}E(tr\underset{\sim}{L}_{22}tr_2\underset{\sim}{L}_{22})+2E(trL_{22}tr_3\underset{\sim}{L}_{22})\}$$
$$+\Delta_3\{(f^{(2)}/2)E(tr_2\underset{\sim}{L}_{22})-2f^{(1)}E(tr_3\underset{\sim}{L}_{22})+4E(tr_4\underset{\sim}{L}_{22})$$
$$-3E(tr_2\underset{\sim}{L}_{22})^2\} ,$$

where $\Delta = \delta(\nu)-\delta(\nu+2)-\delta(\nu+4)+\delta(\nu+6)$,

$$(2.35)\quad E[(\mathrm{tr}\underset{\sim}{L}_{22})^2\mathrm{tr}_2\underset{\sim}{L}_{22}] = \frac{(p-1)(p-2)(f_2-2)(f_2-1)}{12(f_1+f_2-3)(f_1+f_2-2)(f_1+f_2-1)(f_1+f_2)} \times$$

$$\Big[\frac{pf_2}{2(f_1+f_2+3)}\{(f_1-p)[(3p-5)(f_2-p-1)+3p^2-7p+14]$$

$$+(3p-5)(f_2-p-1)^2+(9p^2-17p+6)(f_2-p-1)$$

$$+2(3p^3-7p^2+6p+4)\}$$

$$+\frac{1}{(f_1+f_2-4)(f_1+f_2+1)}\{(f_1-p)^2[p(p-1)(f_2-p-1)^2+(p-1)(2p^2+p+6)(f_2-p-1)$$

$$+p^4+5p^2-6p+12]$$

$$+(f_1-p)[2p(p-1)(f_2-p-1)^3+(p-1)(8p^2-p+6)(f_2-p-1)^2$$

$$+(p-1)(10p^3+15p-6)(f_2-p-1)$$

$$+4p^5-3p^4+8p^3-15p^2+6p+12]$$

$$+p(p-1)(f_2-1)f_2(f_2+p-3)(f_2+p-2)\}\Big]$$

$$+\frac{(f_2+p+1)}{(f_1+f_2+3)}[E(\mathrm{tr}\underset{\sim}{L}_{22}\mathrm{tr}_2\underset{\sim}{L}_{22}) - E(\mathrm{tr}_3\underset{\sim}{L}_{22})]$$

$$+2E(\mathrm{tr}\underset{\sim}{L}_{22}\mathrm{tr}_3\underset{\sim}{L}_{22})-E(\mathrm{tr}_4\underset{\sim}{L}_{22}) \quad ,$$

$$(2.36)\quad E(\mathrm{tr}_2\underset{\sim}{L}_{22})^2=[(p-1)(p-2)(f_2-2)(f_2-1)/\{3!\prod_{j=1}^{6}(f_1+f_2-j+2)\}] \times$$

$$\Big[\{3(f_1-p)^2/2\}\{(p-1)(p-2)(f_2-p-1)^2+(p-2)(2p^2-3p+9)(f_2-p-1)$$

$$+p^4-4p^3+13p^2-26p+28\ \}$$

$$+\{3(f_1-p)/2\}\{2(p-1)(p-2)(f_2-p-1)^3+(p-2)(8p^2-11p+11)(f_2-p-1)^2$$

$$+(p-2)(10p^3-18p^2+33p-1)(f_2-p-1)$$

$$+4p^5-17p^4+40p^3-53p^2+10p+52\}$$

$$+\{p(p-1)(f_2-1)f_2(f_2+p-3)(f_2+p-2)/2\}$$

$$+(p-3)(f_2+p)(f_2-3)\{(p-1)(f_2-p-1)^2+(p-1)(3p-1)(f_2-p-1)$$

$$+2p^3-3p^2+p+6\Big] \quad ,$$

and

$$(2.37) \quad E(\mathrm{tr}\underset{\sim}{L}_{22})^4 = \frac{3.2^8(p-1)(f_2-1)f_1\{(\sum_{i=0}^{6} f_{i,p-1}d^{6-i})+\{(f_1-p)(f_2-p-1)/4\}(\sum_{i=0}^{5} g_{i,p-1}d^{5-i})\}}{(f_1+f_2-4)(f_1+f_2-3)(f_1+f_2-2)(f_1+f_2-1)^4(f_1+f_2)(f_1+f_2+1)(f_1+f_2+3)(f_1+f_2+5)}$$

$$+4E(\mathrm{tr}\underset{\sim}{L}_{22})^3E(\mathrm{tr}\underset{\sim}{L}_{22})-6E(\mathrm{tr}\underset{\sim}{L}_{22})^2[E(\mathrm{tr}\underset{\sim}{L}_{22})]^2+3[E(\mathrm{tr}\underset{\sim}{L}_{22})]^4 \quad ,$$

where $d = (f_1+f_2-2p-1)/2$,

$f_{i,p-1}(i=0,\ldots, 6)$ and $g_{i,p-1}(i=0,\ldots, 5)$ can be obtained from f_i's and g_i's (which are polynomials in $s = p$) presented in [12, pp.7-8] by substituting in the latter $p-1$ for s.

3. Moments of $U^{(p)}$. In the previous paper by the authors [6] it has been shown that

$$(3.1) \qquad 1+U^{(p)} = \{(1-\ell_{11})(1-\underset{\sim}{u}'\underset{\sim}{u})\}^{-1}+(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}(\underset{\sim}{u}'\underset{\sim}{M}\underset{\sim}{u})+\mathrm{tr}\underset{\sim}{M}$$

where $\underset{\sim}{M} = (\underset{\sim}{I}_{p-1}-\underset{\sim}{L}_{22})^{-1}-\underset{\sim}{I}_{p-1}$. The distribution of $\underset{\sim}{M}$ is given by

$$(3.2)\quad \prod_{i=1}^{p-1}[\Gamma(\frac{f_1+f_2-i}{2})/\{\Gamma(\frac{f_1-i+1}{2})\Gamma(\frac{f_2-i}{2})\}]\pi^{-\frac{1}{4}(p-1)(p-2)}\frac{|\underset{\sim}{M}|^{\frac{1}{2}\{f_2-1-(p-1)-1\}}}{|\underset{\sim}{I}_{p-1}+\underset{\sim}{M}|^{\frac{1}{2}(f_1+f_2-1)}}d\underset{\sim}{M} .$$

Now we have

$$(3.3)\quad E(1-\ell_{11})^{-1}-E(1-\ell_{11,0})^{-1}=2\lambda^2/(f_1-2), E(1-\ell_{11})^{-2}-E(1-\ell_{11,0})^{-2}$$

$$=1/[(f_1-2)(f_1-4)][(2\lambda^2)^2+2(\nu-2)(2\lambda^2)] ,$$

$$E(1-\ell_{11})^{-3}-E(1-\ell_{11,0})^{-3}=$$

$$=[\frac{(2\lambda^2)^3+3(\nu-2)(2\lambda^2)^2+3(\nu-2)(\nu-4)(2\lambda^2)}{(f_1-2)(f_1-4)(f_1-6)}]$$

and

$$E(1-\ell_{11})^{-4}-E(1-\ell_{11,0})^{-4}$$

$$=[\frac{(2\lambda^2)^4+4(\nu-2)(2\lambda^2)^3+6(\nu-2)(\nu-4)(2\lambda^2)^2+4(\nu-2)(\nu-4)(\nu-6)(2\lambda^2)}{(f_1-2)(f_1-4)(f_1-6)(f_1-8)}] .$$

Let $\alpha_1 = 1/(1-\underset{\sim}{u}'\underset{\sim}{u})$ and $\beta_2 = \mathrm{tr}\underset{\sim}{M}+(\underset{\sim}{u}'\underset{\sim}{M}\underset{\sim}{u})/(1-\underset{\sim}{u}'\underset{\sim}{u})$;

$$(3.4)\quad E(\alpha_1^i) = [(f_1-2)(f_1-4)\ \dots\ (f_1-2i)]/[(f_1-p-1)(f_1-p-3)\ \dots$$

$$(f_1-p-2i+1)] \quad \text{for } i=1,2,3,4 ;$$

$$= \eta_i^{(0)}(f_1-2)(f_1-4)\dots(f_1-2i) .$$

(3.5) $$E(\alpha_1^i\beta_2) = \frac{(f_1-2)(f_1-4)\ldots(f_1-2i)(f_1-p-2i)}{(f_1-p-1)(f_1-p-3)\ldots(f_1-p-2i-1)} E(tr\underset{\sim}{M})$$

$$=(f_1-2)(f_1-4)\ldots(f_1-2i)\ \eta_i^{(1)} \quad \text{for } i=1,2,3.$$

(3.6) $$E(\alpha_1^i\beta_2^2)$$

$$=\frac{(f_1-2)(f_1-4)\ldots(f_1-2i)}{(f_1-p-1)(f_1-p-3)\ldots(f_1-p-2i+1)}E[\frac{(f_1-p-2i)(f_1-p-2i-2)}{(f_1-p-2i-1)(f_1-p-2i-3)}(tr\underset{\sim}{M})^2$$

$$-\frac{4(tr_2\underset{\sim}{M})}{(f_1-p-2i-1)(f_1-p-2i-3)}]$$

$$= (f_1-2)(f_1-4)\ldots(f_1-2i)\ \eta_i^{(2)} \quad \text{for } i = 1,2 .$$

(3.7) $$E(\alpha_1\beta_2^3)$$

$$= \frac{(f_1-2)}{(f_1-p-1)(f_1-p-3)(f_1-p-5)(f_1-p-7)}E[(f_1-p-2)(f_1-p-4)(f_1-p-6)(tr\underset{\sim}{M})^3$$

$$-12(f_1-p-4)(tr\underset{\sim}{M})(tr_2\underset{\sim}{M})+24(tr_3\underset{\sim}{M})]$$

$$= (f_1-2)\ \eta_1^{(3)} .$$

Hence, we get

(3.8) $$E(1+U^{(p)}) = E(1+U_0^{(p)}) + (2\lambda^2)\ \eta_1^{(0)}$$

(3.9) $$E(1+U^{(p)})^2=E(1+U_0^{(p)})^2+(2\lambda^2)^2\eta_2^{(0)}+2(2\lambda^2)[(\nu-2)\eta_2^{(0)}+\eta_1^{(1)}]$$

(3.10) $$E(1+U^{(p)})^3=E(1+U_0^{(p)})^3+(2\lambda^2)^3\eta_3^{(0)}+3(2\lambda^2)^2[(\nu-2)\eta_3^{(0)}+\eta_2^{(1)}]$$

$$+3(2\lambda^2)[\nu-2)(\nu-4)\eta_3^{(0)}+2(\nu-2)\eta_2^{(1)}+\eta_1^{(2)}],$$

and

(3.11) $$E(1+U^{(p)})^4=E(1+U_0^{(p)})^4+(2\lambda^2)^4\eta_4^{(0)}+4(2\lambda^2)^3[(\nu-2)\eta_4^{(0)}+\eta_3^{(1)}]$$

$$+6(2\lambda^2)^2[(\nu-2)(\nu-4)\eta_4^{(0)}+2(\nu-2)\eta_3^{(1)}+\eta_2^{(2)}]$$

$$+4(2\lambda^2)[(\nu-2)(\nu-4)(\nu-6)\eta_4^{(0)}+3(\nu-2)(\nu-4)\eta_3^{(1)}+3(\nu-2)\eta_2^{(2)}+\eta_1^{(3)}].$$

Now

(3.12) $$E(\mathrm{tr}_i\underset{\sim}{M}) = \binom{p-1}{i}\prod_{j=1}^{i}[(f_2-j)/(f_1-p+j-1)],\ (i=1,2,\ldots p-1;\mathrm{tr}\underset{\sim}{M}=\mathrm{tr}_1\underset{\sim}{M})$$

(3.13) $$E(\mathrm{tr}\underset{\sim}{M})^2 = E(\mathrm{tr}\underset{\sim}{M})\{\frac{2(f_1+f_2-p-1)(f_1-1)}{(f_1-p)(f_1-p-2)(f_1-p+1)} + E(\mathrm{tr}\underset{\sim}{M})\},$$

(3.14) $$E(\mathrm{tr}\underset{\sim}{M})^3 = E(\mathrm{tr}\underset{\sim}{M})\{\frac{2^3(f_1+f_2-p-1)(f_1+2f_2-p-2)(f_1+p-2)(f_1-1)}{(f_1-p)^2(f_1-p-4)(f_1-p-2)(f_1-p+1)(f_1-p+2)}$$

$$+3E(\mathrm{tr}\underset{\sim}{M})^2-2[E(\mathrm{tr}\underset{\sim}{M})]^2\},$$

and

(3.15) $$E(\mathrm{tr}M\mathrm{tr}_2M) = E(\mathrm{tr}_2M)[\{(f_1-p)(p-1)+4\}(f_2-p-1)$$

$$+2(p+1)(p+2)]/[(f_1-p)^2-4] \quad .$$

In obtaining the expected values (2.31) - (2.33), (2.35) - (2.37) and (3.12) - (3.15), use has been made of lemmas 1 and 2 in [14] and the values of the special type of Vandermonde determinants given in [15].

Expressions for the first two moments of $U^{(p)}$ have been presented in the previous paper by the authors [6]. Now using the results (3.12) - (3.15) in (3.10) and (3.11) we get

(3.16) $$E(1+U^{(p)})^3 = E(1+U_0^{(p)})^3+A_1(2\lambda^2)^3+3A_2(2\lambda^2)^2+3A_3(2\lambda^2)$$

where

(3.17) $$A_1 = \eta_3^{(0)} = [(f_1-p-1)(f_1-p-3)(f_1-p-5)]^{-1} \ ,$$

(3.18) $$A_2 = (\nu-2)\eta_3^{(0)} + \eta_2^{(1)} \quad ,$$

where

(3.19) $$\eta_2^{(1)} = (p-1)(f_2-1)(f_1-p-4)\ A_1/(f_1-p) \quad ,$$

(3.20) $$A_3 = (\nu-2)(\nu-4)\eta_3^{(0)}+2(\nu-2)\eta_2^{(1)}+\eta_1^{(2)} \quad ,$$

where

$$(3.21)\qquad \eta_1^{(2)} = \frac{(p-1)(f_2-1)}{(f_1-p-3)(f_1-p+1)(f_1-p)}\left\{(p-2)(f_2-1)+\frac{(f_2+1)(f_1-1)}{(f_1-p-2)} + \frac{(p+1)(f_2+1)(f_1-p+1)}{(f_1-p-1)(f_1-p-2)(f_1-p-5)}\right\}.$$

Similarly

$$(3.22)\qquad E(1+U^{(p)})^4 = E(1+U_0^{(p)})^4+B_1(2\lambda^2)^4+4B_2(2\lambda^2)^3+6B_3(2\lambda^2)^2+4B_4(2\lambda^2)\ ,$$

where

$$(3.23)\qquad B_1 = \eta_4^{(0)} = A_1/(f_1-p-7)$$

$$(3.24)\qquad B_2 = (\nu-2)\ \eta_4^{(0)} + \eta_3^{(1)}$$

where

$$\eta_3^{(1)} = (p-1)(f_2-1)(f_1-p-6)\ B_1/(f_1-p)$$

$$(3.25)\qquad B_3 = (\nu-2)(\nu-4)\eta_4^{(0)}+2(\nu-2)\eta_3^{(1)}+\eta_2^{(2)}$$

where

$$(3.26)\qquad \eta_2^{(2)} = \left\{\frac{(f_1-p-4)(f_1-p-6)(p-1)(f_2-1)}{(f_1-p)^2}\left[\frac{2(f_1-1)(f_1+f_2-p-1)}{(f_1-p+1)(f_1-p-2)} + (p-1)(f_2-1)\right] - 2(p-1)(p-2)(f_2-1)(f_2-2)/\{(f_1-p)(f_1-p+1)\}\right\} B_1$$

$$(3.27)\quad B_4=(\nu-2)(\nu-4)(\nu-6)\eta_4^{(0)}+3(\nu-2)(\nu-4)\eta_3^{(1)}+3(\nu-2)\eta_2^{(2)}+\eta_1^{(3)}$$

where

$$(3.28)\quad \eta_1^{(3)}=\Big\{\frac{(f_1-p-2)(f_1-p-4)(f_1-p-6)(p-1)(f_2-1)}{(f_1-p)^3}$$

$$\Big[\frac{2^3(f_1-1)(f_1+f_2-p-1)(f_1+2f_2-p-2)(f_1+p-2)}{(f_1-p-2)(f_1-p-4)(f_1-p+1)(f_1-p+2)}$$

$$+\frac{6(f_2-1)(f_1+f_2-p-1)(p-1)(f_1-1)}{(f_1-p-2)(f_1-p+1)}+(p-1)^2(f_2-1)^2\Big]$$

$$-\frac{6(f_1-p-4)(p-1)(p-2)(f_2-1)(f_2-2)}{(f_1-p-2)(f_1-p)(f_1-p+1)(f_1-p+2)}\big[\{(f_1-p)(p-1)+4\}(f_2-p-1)+2(p+1)(p+2)\big]$$

$$+4(p-1)(p-2)(p-3)(f_2-3)(f_2-2)(f_2-1)/\{(f_1-p)(f_1-p+1)(f_1-p+2)\}\Big\}\cdot B_1 \quad .$$

4. <u>Approximations to the Distribution of $U^{(p)}$.</u> Pillai [13] has given an approximation to the distribution of $U^{(2)}$ for $f_1 > f_2$ and which is good even for very small values of f_2. The following approximation to the distribution of $U^{(p)}$ for $f_1 > (p-1)f_2$, based on its moments discussed in the preceding section and [6], generalizes Pillai's results for $U^{(2)}$ [13].

$$(4.1) \qquad g(U^{(p)}) = (U^{(p)})^{p_1-1}/(1+U^{(p)}/k)^{p_1+q_1+1}\, k^{p_1}\beta(p_1,q_1+1),\ 0 < U^{(p)} < \infty ,$$

where

$$p_1 = 2q_1/\{q_1(h-1)-2h\} ,$$

$$q_1 = 2\{c^2(f_1-p-3)h-(c+d)^2(f_1-p-1)\}/\{c^2(f_1-p-3)(h+1)-2(c+d)^2(f_1-p-1)\} ,$$

$$k = c\{q_1(h-1)-2h\}/2(f_1-p-1) ,$$

$$h = (c+1.99d)^3(f_1-p-1)/\{(c+d)^2(f_1-p-5)c\} ,$$

$$c = pf_2+2\lambda^2 \quad \text{and} \quad d = (f_1+(1-p)f_2-1)/(f_1-p)$$

A comparison of the moments from (4.1) with the respective exact ones may be made from Table 1. However, the moments for the case $p = 2$ are not presented in Table 1 since comparisons of the approximate and exact moments for $U^{(2)}$ have already been made [13]. It may further be pointed out that the case $p = 1$ is that of the non-central F [8]. Hence the accuracy of the approximation may be compared in this case with the approximation to the distribution of non-central F obtained by Patnaik and the exact distribution using Table 7 of [8]. However, it should be

Table 1

Moments (central) of $U^{(p)}$ from the exact and approximate [Eq.(4.1)] distributions for different values of $f_1 > (p-1)f_2$, λ^2 and p

Moments	$f_1=10$ $f_2=3$ $\lambda^2=0.5$ $p=1$			$f_1=10$ $f_2=15$ $\lambda^2=2$ $p=1$		
	Exact	Approximate	Ratio(A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	0.5000	0.5000	1.0000	2.3750	2.3750	1.0000
μ_2	0.2917	0.2708	0.9286	2.8385	2.6927	0.9486
μ_3	0.6458	0.5881	0.9107	16.8841	15.5835	0.9230
μ_4	5.0312	4.7122	0.9366	373.7200	336.9354	0.9016
$\sqrt{\mu_2}$	0.5401	0.5204	0.9636	1.6848	1.6409	0.9740
β_1	16.8105	17.4119	1.0358	12.4644	12.4383	0.9979
β_2	59.1429	64.2422	1.0862	46.3827	46.4695	1.0019

Moments	$f_1=10$ $f_2=15$ $\lambda^2=32$ $p=1$			$f_1=10$ $f_2=50$ $\lambda^2=2$ $p=1$		
	Exact	Approximate	Ratio(A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	9.8750	9.8750	1.0000	6.7500	6.7500	1.0000
μ_2	38.4635	35.8177	0.9312	17.6042	17.4583	0.9917
μ_3	827.1185	742.8304	0.8980	256.5521	253.0986	0.9865
μ_4	66672.3710	57685.2030	0.8652	14005.6130	13732.1110	0.9805
$\sqrt{\mu_2}$	6.2019	5.9848	0.9650	4.1957	4.1783	0.9958
β_1	12.0223	12.0084	0.9988	12.0644	12.0385	0.9979
β_2	45.0658	44.9644	0.9977	45.1930	45.0538	0.9969

Moments	$f_1=10$ $f_2=3$ $\lambda^2=32$ $p=1$			$f_1=10$ $f_2=100$ $\lambda^2=2$ $p=1$		
	Exact	Approximate	Ratio(A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	8.3750	8.3750	1.0000	13.0000	13.0000	1.0000
μ_2	28.8385	26.1927	0.9083	60.8333	60.6875	0.9976
μ_3	536.8841	464.7210	0.8656	1644.8333	1637.6403	0.9956
μ_4	37469.2820	30867.1300	0.8238	166726.9900	165521.0700	0.9928
$\sqrt{\mu_2}$	5.3702	5.1179	0.9530	7.7996	7.7902	0.9988
β_1	12.0182	12.0183	1.0000	12.0176	11.9988	0.9984
β_2	45.0535	44.9920	0.9986	45.0529	44.9422	0.9975

Table 1 (Cont'd)

Moments	$f_1=12$ $f_2=5$ $\lambda^2=0.5$ $p=3$			$f_1=12$ $f_2=5$ $\lambda^2=32$ $p=3$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	2.0000	2.0000	1.0000	9.8750	9.8750	1.0000
μ_2	1.4120	1.4077	0.9969	33.1672	32.8712	0.9911
μ_3	4.7556	5.7841	1.2163	649.5157	652.7574	1.0050
μ_4	63.1979	89.0452	1.4090	48253.3920	48599.6620	1.0072
$\sqrt{\mu_2}$	1.1883	1.1865	0.9985	5.7591	5.7333	0.9955
β_1	8.0328	11.9941	1.4932	11.5625	11.9966	1.0375
β_2	31.6964	44.9379	1.4178	43.8641	44.9782	1.0254

Moments	$f_1 = 15$ $f_2 = 4$ $\lambda^2= 4.5$ $p=4$			$f_1 = 15$ $f_2 = 4$ $\lambda^2= 32$ $p=4$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	2.5000	2.5000	1.0000	8.0000	8.0000	1.0000
μ_2	1.7170	1.6765	0.9764	16.6545	16.3640	0.9826
μ_3	5.1341	5.7869	1.1271	177.0424	176.4678	0.9968
μ_4	51.6730	61.7517	1.1950	5924.6806	5886.3824	0.9935
$\sqrt{\mu_2}$	1.3104	1.2948	0.9881	4.0810	4.0453	0.9912
β_1	5.2069	7.1062	1.3648	6.7851	7.1065	1.0474
β_2	17.5267	21.9693	1.2534	21.3599	21.9820	1.0291

Moments	$f_1=25$ $f_2=5$ $\lambda^2=12.5$ $p=5$			$f_1=25$ $f_2=5$ $\lambda^2=32$ $p=5$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	2.6316	2.6316	1.0000	4.6842	4.6842	1.0000
μ_2	0.9076	0.8768	0.9660	2.7709	2.6917	0.9714
μ_3	1.1645	1.2747	1.0947	6.7802	6.8592	1.0116
μ_4	5.8355	6.2180	1.0656	59.1995	58.6460	0.9907
$\sqrt{\mu_2}$	0.9527	0.9364	0.9829	1.6646	1.6406	0.9856
β_1	1.8137	2.4107	1.3292	2.1609	2.4123	1.1164
β_2	7.0840	8.0887	1.1418	7.7106	8.0942	1.0498

pointed out that the approximation to the distribution of $U^{(p)}$ in (4.1) has been suggested in this paper using the first three moments and with considerations of accuracy for $p > 1$. This may be noted from Table 3 where, in general, the respective exact and approximate moments seem to be closer as p increases (from 1 to 4). Table 2 gives some idea of the accuracy of the approximation when $p = 1$.

Table 2

Values of $\int_0^{U_0} g(U^{(1)})\, dU^{(1)}$ from approximate and exact distributions

f_1	f_2	λ^2	U_0	Probability		
				Approximate		Exact
				Eqn. (4.1)	Patnaik	
10	3	2	1.1124	.765	.752	.745
10	3	8	1.1124	.154	.203	.206
10	3	8	1.9656	.503	.520	.517
10	5	3	1.663	.738	.731	.731
10	5	3	2.818	.920	.913	.914
20	3	2	0.4647	.708	.706	.700
20	5	3	0.67775	.671	.665	.664
20	5	12	1.02575	.196	.244	.245

It may be observed that the approximation suggested for $U^{(1)}$ is more accurate at the upper tail end than the lower. Now, in view of the fact that the moments are closer for larger values of p in Table 3, it is reasonable to assume that the approximation to the distribution of $U^{(p)}$

Table 3

Moments (central) of $U^{(p)}$ from the exact and approximate [Eq.(4.1)] distributions for different values of $f_1 > (p-1)\ f_2$, λ^2 and p.

Moments	$f_1=10$ $f_2=3$ $\lambda^2=2$ $p=1$			$f_1=10$ $f_2=3$ $\lambda^2=2$ $p=2$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	0.8750	0.8750	1.0000	1.4286	1.4286	1.0000
μ_2	0.7135	0.5677	0.7956	1.4163	1.2610	0.8903
μ_3	2.2747	1.6241	0.7140	6.8812	6.1366	0.8918
μ_4	26.1770	17.3107	0.6613	179.7042	160.3333	0.8922
$\sqrt{\mu_2}$	0.8447	0.7535	0.8920	1.1901	1.1229	0.9436
β_1	14.2431	14.4157	1.0121	16.6666	18.7820	1.1269
β_2	51.4140	53.7110	1.0447	89.5842	100.8362	1.1256

Moments	$f_1=50$ $f_2=10$ $\lambda^2=4.5$ $p=1$			$f_1=50$ $f_2=10$ $\lambda^2=4.5$ $p=2$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	0.3958	0.3958	1.0000	0.6170	0.6170	1.0000
μ_2	0.0322	0.0245	0.7607	0.0461	0.0395	0.8568
μ_3	0.0^2628	0.0^2410	0.6539	0.0^2878	0.0^2752	0.8570
μ_4	0.0^2529	0.0^2306	0.5787	0.0^2947	0.0^2738	0.7802
$\sqrt{\mu_2}$	0.1794	0.1564	0.8722	0.2147	0.1988	0.9256
β_1	1.1822	1.1484	0.9714	0.7857	0.9174	1.1676
β_2	5.1121	5.1126	1.0000	4.4507	4.7297	1.0627

Moments	$f_1=50$ $f_2=10$ $\lambda^2=4.5$ $p=3$			$f_1=50$ $f_2=10$ $\lambda^2=4.5$ $p=4$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	0.8478	0.8478	1.0000	1.0889	1.0889	1.0000
μ_2	0.0619	0.0566	0.9144	0.0799	0.0762	0.9530
μ_3	0.0119	0.0123	1.0348	0.0158	0.0190	1.1999
μ_4	0.0159	0.0148	0.9313	0.0253	0.0266	1.0491
$\sqrt{\mu_2}$	0.2489	0.2380	0.9563	0.2827	0.2760	0.9762
β_1	0.5969	0.8358	1.4003	0.4904	0.8156	1.6633
β_2	4.1326	4.6027	1.1138	3.9660	4.5811	1.1551

provides at least a two decimal accuracy at the upper tail end. However, it should be remembered that the approximation is valid only for $f_1 > (p-1)\, f_2$ which for $p = 1$ reduces to $f_1 > 0$. In the latter case, as is evident from Table 1, in many cases, the moments are closer when $f_2 > f_1$.

Again a comparison of the probabilities in Table 2 arouses the natural curiosity to attempt a generalization of Patnaik's approximation [8]. The following is such a generalization equating the first two respective moments of the exact and approximate distributions:

$$(4.2) \qquad g_1(U^{(p)}) = (U^{(p)})^{\frac{1}{2}\nu_1 - 1} / (1+U^{(p)}/k_1)^{\frac{1}{2}(\nu_1+\nu_2)} k_1^{\frac{1}{2}\nu_1} \beta(\tfrac{1}{2}\nu_1, \tfrac{1}{2}\nu_2) \quad ,$$

$$0 < U^{(p)} < \infty$$

where

$$k_1 = (pf_2 + 2\lambda^2)/\nu_1 \quad ,$$

$$\nu_1 = (pf_2+2\lambda^2)^2 (f_1-p)/(4\lambda^2+pf_2)\{f_1+f_2(1-p)-1\} \quad ,$$

and $\nu_2 = f_1 - p + 1$.

The two approximations (4.1) and (4.2) may be compared for the same set of values of f_1, f_2, λ^2 and p as in Tables 1 and 3 using the ratios of approximate to exact moments given in Table 4 in the two cases. It is obvious from Table 4 that while for $p=1$ the ratios from Eq. (4.2) are closer to unity than the respective ratios from Eq. (4.1), as p increases the latter become closer to unity than the former (except of course for μ_2).

Table 4

Ratios of moments (Central) of $U^{(p)}$ from the exact and approximate distributions [(4.1) and (4.2)] for sets of values of f_1, f_2, λ^2 and p as in Tables 1 and 3

Moments	$f_1=10$ $f_2=3$ $\lambda^2=0.5$ p=1 Ratio (A/E)		$f_1=10$ $f_2=15$ $\lambda^2=2$ p=1 Ratio (A/E)		$f_1=10$ $f_2=15$ $\lambda^2=32$ p=1 Ratio (A/E)	
	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)
μ_1'	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
μ_2	0.9286	1.0000	0.9486	1.0000	0.9312	1.0000
μ_3	0.9107	1.0161	0.9230	1.0021	0.8980	1.0026
μ_4	0.9366	1.0326	0.9016	1.0036	0.8652	1.0042
$\sqrt{\mu_2}$	0.9636	1.0000	0.9740	1.0000	0.9650	1.0000
β_1	1.0358	1.0325	0.9979	1.0042	0.9988	1.0052
β_2	1.0862	1.0326	1.0019	1.0036	0.9977	1.0042

Moments	$f_1=10$ $f_2=50$ $\lambda^2=2$ p=1 Ratio (A/E)		$f_1=10$ $f_2=3$ $\lambda^2=32$ p=1 Ratio (A/E)		$f_1=10$ $f_2=100$ $\lambda^2=2$ p=1 Ratio (A/E)	
	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)
μ_1'	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
μ_2	0.9917	1.0000	0.9083	1.0000	0.9976	1.0000
μ_3	0.9865	1.0000	0.8656	1.0047	0.9956	1.0000
μ_4	0.9805	1.0001	0.8238	1.0076	0.9928	1.0000
$\sqrt{\mu_2}$	0.9958	1.0000	0.9530	1.0000	0.9988	1.0000
β_1	0.9979	1.0001	1.0000	1.0095	0.9984	1.0000
β_2	0.9969	1.0001	0.9986	1.0076	0.9975	1.0000

Moments	$f_1=12$ $f_2=5$ $\lambda^2=0.5$ p=3 Ratio (A/E)		$f_1=12$ $f_2=5$ $\lambda^2=32$ p=3 Ratio (A/E)		$f_1=15$ $f_2=4$ $\lambda^2=4.5$ p=4 Ratio (A/E)	
	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)
μ_1'	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
μ_2	0.9969	1.0000	0.9911	1.0000	0.9764	1.0000
μ_3	1.2163	1.2227	1.0050	1.0188	1.1271	1.1699
μ_4	1.4090	1.4207	1.0072	1.0260	1.1950	1.2573
$\sqrt{\mu_2}$	0.9985	1.0000	0.9955	1.0000	0.9881	1.0000
β_1	1.4932	1.4951	1.0375	1.0380	1.3648	1.3687
β_2	1.4178	1.4207	1.0254	1.0260	1.2534	1.2573

Table 4 (Cont'd.)

Moments	$f_1=15$ $\lambda^2=32$	$f_2=4$ $p=4$	$f_1=25$ $\lambda^2=12.5$	$f_2=5$ $p=5$	$f_1=25$ $\lambda^2=32$	$f_2=5$ $p=5$
	Ratio (A/E)		Ratio (A/E)		Ratio (A/E)	
	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)
μ_1'	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
μ_2	0.9826	1.0000	0.9660	1.0000	0.9714	1.0000
μ_3	0.9968	1.0239	1.0947	1.1563	1.0116	1.0584
μ_4	0.9935	1.0303	1.0656	1.1462	0.9907	1.0522
$\sqrt{\mu_2}$	0.9912	1.0000	0.9829	1.0000	0.9856	1.0000
β_1	1.0474	1.0485	1.3292	1.3370	1.1164	1.1203
β_2	1.0291	1.0303	1.1418	1.1462	1.0498	1.0522

Moments	$f_1=10$ $\lambda^2=2$	$f_2=3$ $p=1$	$f_1=10$ $\lambda^2=2$	$f_2=3$ $p=2$	$f_1=50$ $\lambda^2=4.5$	$f_2=10$ $p=1$
	Ratio (A/E)		Ratio (A/E)		Ratio (A/E)	
	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)
μ_1'	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
μ_2	0.7956	1.0000	0.8903	1.0000	0.7607	1.0000
μ_3	0.7140	1.0419	0.8918	1.0722	0.6539	1.0559
μ_4	0.6613	1.0772	0.8922	1.1300	0.5787	1.0479
$\sqrt{\mu_2}$	0.8920	1.0000	0.9436	1.0000	0.8722	1.0000
β_1	1.0121	1.0855	1.1269	1.1497	0.9714	1.1150
β_2	1.0447	1.0772	1.1256	1.1300	1.0000	1.0479

Moments	$f_1=50$ $\lambda^2=4.5$	$f_2=10$ $p=2$	$f_1=50$ $\lambda^2=4.5$	$f_2=10$ $p=3$	$f_1=50$ $\lambda^2=4.5$	$f_2=10$ $p=4$
	Ratio (A/E)		Ratio (A/E)		Ratio (A/E)	
	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)	Eq.(4.1)	Eq.(4.2)
μ_1'	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
μ_2	0.8568	1.0000	0.9144	1.0000	0.9530	1.0000
μ_3	0.8570	1.1233	1.0348	1.2161	1.1999	1.3143
μ_4	0.7802	1.0903	0.9313	1.1349	1.0491	1.1708
$\sqrt{\mu_2}$	0.9256	1.0000	0.9563	1.0000	0.9762	1.0000
β_1	1.1676	1.2619	1.4003	1.4788	1.6633	1.7274
β_2	1.0627	1.0902	1.1138	1.1349	1.1551	1.1708

However β_1 and β_2 are generally better in the case of Eq. (4.1) even for $p=1$. From the foregone analysis it may be concluded that approximation (4.2) is useful for smaller values of p while approximation (4.1) might be more accurate for comparatively larger values of p. Further it should be pointed out that the condition $f_1 > (p-1)f_2$ applies for both approximations.

The authors wish to thank Mrs. Louise Mao Lui, Statistical Laboratory, Purdue University, for the excellent programming of the material for the computations in this paper carried out on the IBM 7094 Computer, Purdue University's Computer Science's Center.

REFERENCES

[1] Anderson, T. W. (1946). The non-central Wishart distribution and certain problems of multivariate statistics. Ann. Math. Statist. 17, 409-431.

[2] Anderson, T. W. (1958). An Introduction to Multivariate Statistical Analysis, Wiley, New York.

[3] Anderson, T. W. and Girshick, M. A. (1944). Some extensions of the Wishart distribution. Ann. Math. Statist., 15, 345-357.

[4] Khatri, C. G. (1964). Distribution of the 'generalized' multiple correlation matrix in the dual case. Ann. Math. Statist. 35, 1801-1806.

[5] Khatri, C. G. (1965). Classical statistical analysis based on a certain multivariate complex Gaussian distribution. Ann. Math. Stat. 36, 98-114.

[6] Khatri, C. G. and Pillai, K. C. S. (1965). Some results on the non-central multivariate beta distribution and moments of traces of two matrices. Ann. Math. Statist., 36, October Issue.

[7] Kshirsagar, A. M. (1961). The non-central multivariate beta distribution. Ann. Math. Statist., 32, 104-111.

[8] Patnaik, P. B. (1949). The non-central χ^2- and F-distributions and their applications. Biometrika 34, 202-232.

[9] Pillai, K. C. S. (1954). On some distribution problems in multivariate analysis. Mimeograph Series No. 88, Institute of Statistics, University of North Carolina, Chapel Hill.

[10] Pillai, K. C. S. (1955). Some new test criteria in multivariate analysis. Ann. Math. Statist. 26, 117-121.

[11] Pillai, K. C. S. (1956). Some results useful in multivariate analysis. Ann. Math. Statist. 27, 1106-1114.

[12] Pillai, K. C. S. (1960). Statistical Tables for Tests of Multivariate Hypothesis. The Statistical Center, Manila.

[13] Pillai, K. C. S. (1964). On the non-central multivariate beta distribution and the moments of traces of some matrices. Mimeograph Series No. 11, Department of Statistics, Purdue University. (presented at the Symposium on Multivariate Analysis, Dayton, Ohio, June 1965).

[14] Pillai, K. C. S. (1964) On the moments of elementary symmetric functions of the roots of two matrices. Ann. Math. Statist., 35, 1704-1712.

[15] Pillai, K. C. S. and Mijaras, T. A. (1959). On the moments of the trace of a matrix and approximations to its distribution. Ann. Math. Statist., 30, 1135-1140.

[16] Pillai, K. C. S. and Samson, P. (1959) On Hotelling's generalization of T^2. Biometrika, 46, 160-168.

[17] Roy, S. N. (1957). Some Aspects of Multivariate Analysis, Wiley, New York.

[18] Roy, S. N. and Gnanadesikan, R. (1959). Some contributions to ANOVA in one or two dimensions. II. Ann. Math. Statist., 30, 318-340.