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Semi - Markov Analysis of a Bulk Queue

by

Marcel F. Neuts

Purdue University

Department of Statistics

Division of Mathematical Sciences

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Abstract

Customers arrive at a single counter according to a Poisson process of parameter λ. Initially and immediately after each departure the size of the next batch of customers to be served is drawn. The successive batch sizes B_1 , B_2 ,... are independent identically distributed random variables, which take on values between one and K. If after the n - th departure there are less than B_{n+1} customers present, then all customers are served at the same time. If there are B_{n+1} or more customers present, then B_{n+1} of them enter service and the other wait. The service time of the batches are independent random variables, which may depend on the batch size. The order of service is immaterial in this paper.

We point out that the queue lengths after departure points together with the times between departures form a semi-Markov process. The distribution of the busy period, the queue length after departures and the queue length at time t may be studied using properties of the imbedded semi-Markov process.

A. Description of the Model

We assume that customers arrive at a counter according to a Poisson process of parameter λ. Initially and after each departure the sizes of the successive batches to be served are drawn. Let B_1, B_2,... denote the successive batch sizes. We assume that they are independent identically distributed random variables with:

(1) $P\{B_n = k\} = b_k$, for k = 1, 2,..., k

$P\{B_n > K\} = 0$

If after the n - th departure, there are less than B_{n+1}, customers present, then all customers are served together. If there are B_{n+1} or more customers present, than B_{n+1} customers enter service and the remaining ones have to wait.

We assume that the successive service times are independent random variables and that a batch of size ν has service time distribution $H_\nu(\cdot)$, $\nu = 1,\ldots,K$.

Let ξ_0 denote the queue length at t = 0 + and let ξ_n, $n \geq 1$ denote the queue length immediately after the n - th departure from the queue.

Let $X_0 = 0$ a.s. and let X_n, n = 1, 2,... denote the time between the (n-1)th departure and the n - th departure. The random variables $\{\xi_n, n \geq 0\}$ form a Markov chain on the non - negative integers and, because of the Poisson arrivals, the random variables X_n are conditionally independent, given the ξ_n. The bivariate sequence $\{(\xi_n, X_n), n = 0, 1, \ldots\}$ is therefore a semi-Markov sequence as defined by Pyke [4].

In general t = 0 is not a point of departure, so the initial pair (ξ_1, X_1) will have a probability distribution which is different from those of later transitions. Without loss of generality and in order to simplify the derivations, we will assume that t = 0 is a point of departure, so that $\{(\xi_n, X_n)\}$ is an ordinary semi-Markov process.

We denote the transition probability distributions by Qij(X),i.e.

(2) $P\left\{\xi_n = j, X_n \geq x \mid \xi_{n-1} = i\right\} = Q_{ij}(x), \quad i,j = 1,\ldots$

and we obtain:

$$(3)\quad Q_{oj}(x) = \int_o^x \left[1 - \bar{e}^{\lambda(x-y)} \right] e^{-\lambda y} \frac{(\lambda y)^j}{j!} \, dH_1(y), \quad j = 0, 1, \ldots$$

$$Q_{1j}(x) = \int_o^x \bar{e}^{\lambda y} \frac{(\lambda y)^j}{j!} \, dH_1(y), \qquad j = 0, 1, \ldots$$

$$Q_{ij}(x) = \left(\sum_{\nu=i}^{k} b_\nu \right) \int_o^x \bar{e}^{\lambda y} \frac{(\lambda y)^j}{j!} \, dH_i(y) +$$

$$\sum_{\nu=1}^{i-1} b_\nu \int_o^x \bar{e}^{\lambda y} \frac{(\lambda y)^{j-i+\nu}}{(j-i+\nu)!} \, dH_\nu(y),$$

for $i \leq K$, $j = 0, 1, \ldots$

$$Qij(x) = \sum_{\nu=1}^{K} b_\nu \int_o^x \bar{e}^{\lambda y} \frac{(\lambda y)^{j-i+\nu}}{(j-i+\nu)!} \, dH_\nu(y),$$

for $i \geq K$, $j = 0, 1, \ldots$

We make the convention that in the above formulae on integral is zero, of a negative factorial appears in the integrand. The first k+1 rows of $Q(x)$ are non - zero, while the rows below the (K+1) th form an infinite upper triangular matrix.

If, instead of considering the time-intervals X_n between departures, we consider the time-intervals Y_n during which the server is busy between successive departures, then the bivariate sequence $\{\xi_n, Y_n\}$ is again a semi-Markov sequence. Its matrix $R(x) = \{R_{ij}(x)\}$ is the same as the one given above, except that the factor $1 - \bar{e}^{\lambda(x-y)}$ does not appear in the first integral.

B. The Distribution of the busy Period

The busy period is equal to the time it takes for the semi-Markov process with matrix $R(x)$ to reach state 0 for the first time, starting in state one. The successive busy periods form a renewal process. The initial busy period will in general have a different distribution from the following ones. We will calculate the distribution of the busy period only for those after the first. The initial busy period may be studied in a completely analogous fashion, but we will omit the details.

Let $G(k,n,x)$ denote the probability that the busy period (after the first) consists of n service periods at least, that the first n service periods last for x or less and that at the end of the n-th service k customers are waiting. We then have the following recurrence relations:

$$(4) \quad G(k,1,x) = \int_0^x e^{-\lambda y} \frac{(\lambda y)^k}{k!} \, dH_1(y),$$

and:

$$G(k,n,x) = \sum_{\nu=1}^{k-1} \int_0^x G(\nu,n-1,x-y) \Big[(b_\nu + \dots + b_k) e^{-\lambda y} \frac{(\lambda y)^k}{k!} \, dH_\nu(y) + \sum_{\alpha=1}^{\nu-1} b_\alpha e^{-\lambda y} \frac{(\lambda y)^{k-\nu+\alpha}}{(k-\nu+\alpha)!} \, dH_\alpha(y) \Big] +$$

$$\sum_{\sigma=0}^{k} \int_0^x G(K+\sigma, n-1, x-y) \, e^{-\lambda y} \sum_{\alpha=1}^{k} b_\alpha \frac{(\lambda y)^{k+\alpha-K-\sigma}}{(k+\alpha-K-\sigma)!} \, dH_\alpha(y),$$

with the same convention as above.

Let $\Gamma(k,n,s)$ and $h_\alpha(s)$, $\alpha=1,\ldots,K$ denote the Laplace-Stieltjes transforms of $G(k,n,x)$ and $H_\alpha(x)$ respectively, then we obtain:

$$(5)\quad \Gamma(k,1,s) = \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^k}{k!} \, dH_1(y)$$

and

$$\Gamma(k,n,s) =$$

$$\sum_{\nu=1}^{K-1} \Gamma(\nu,n-1,s) \Big[(b_\nu+\ldots+b_K) \int_0^\infty e^{-(\lambda+s)} \frac{(\lambda y)^k}{k!} \, dH_\nu(y) +$$

$$\sum_{\alpha=1}^{\nu-1} b_\alpha \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^{k-\nu+\alpha}}{(k-\nu+\alpha)!} \, dH_\alpha(y) \Big] +$$

$$\sum_{\sigma=0}^{k} \Gamma(K+\sigma,n-1,s) \sum_{\alpha=1}^{K} b_\alpha \int_0^\infty e^{-(\lambda+s)y} \frac{(\lambda y)^{k+\alpha-K-\sigma}}{(k+\alpha-K-\sigma)!} \, dH_\alpha(y),$$

We now introduce the generating functions:

$$(6)\quad \sum_{k=0}^\infty z^k \, \Gamma(K+k,n,s) = C_K(z,n,s)$$

$$\sum_{n=1}^\infty w^n \, C_K(z,n,s) = D_K(z,w,s)$$

$$\sum_{n=1}^\infty w^n \, \Gamma(r,n,s) = E_r(w,s), \quad \text{for } r = 0,\ldots,K-1.$$

and obtain successively:

$$(7) \quad \sum_{k=0}^{\infty} z^k \ \Gamma(k,1,s) = h_1(s+\lambda-\lambda z)$$

and for $n > 1$:

$$\sum_{k=0}^{K-1} z^k \ \Gamma(k,n,s) + z^K C_K(z,n,s) =$$

$$\sum_{\nu=1}^{K-1} \Gamma(\nu,n-1,s) \left[(b_\nu+\ldots+b_K) \ h_\nu \ (s+\lambda-\lambda z) + \sum_{\alpha=1}^{\nu-1} b_\alpha \ z^{\nu-\alpha} \ h_\alpha(s+\lambda-\lambda z)\right] +$$

$$+ \sum_{\alpha=1}^{K} z^{K-\alpha} \ b_\alpha \ h_\alpha \ (s+\lambda-\lambda z). \quad C_K \ (z,n-1,s)$$

and

$$(8) \quad D_K \ (z,w,s) =$$

$$\left[z^K - w \sum_{\alpha=1}^{K} b_\alpha z^{K-\alpha} \ h_\alpha(s+\lambda-\lambda z)\right]^{-1}.$$

$$\left\{ w h_1(s+\lambda-\lambda z) - E_o(w,s) - \sum_{k=1}^{K-1} E_k(w,s) \ \left[z^k - w(b_\nu+\ldots+b_K) \ h_\nu(s+\lambda-\lambda z)\right.\right.$$

$$\left.\left. - w\sum_{\alpha=1}^{\nu-1} b_\alpha z^{\nu-\alpha} h_\alpha \ (s+\lambda-\lambda z)\right] \right\}$$

For Re $s > o$, $|w| \leq 1$ or Re $s \geq o$, $|w| < 1$ we have

$$(9) \quad \left| w \sum_{\alpha=1}^{K} b_\alpha z^{K-\alpha} \ h_\alpha \ (s+\lambda-\lambda z) \right| < |z|^K = 1$$

on the unit - circle $|z| = 1$. It follows by Rouché's theorem, that the denominator in (8) has exactly K roots inside the unit - circle $|z| < 1$. The function $D_K(z,w,s)$ is analytic in z,w,s in the region $|z| \leq 1$, Re $s \geq 0$, $|w| \leq 1$ so the K roots $\gamma_1(w,s),\ldots\gamma_K(w,s)$ must also be zeros of the numerator.

This leads to the following system of K equations in K unknowns for the functions $E_o(w,s),\ldots, E_{K-1}(w,s)$:

$$(10) \quad wh_1\left[s+\lambda-\lambda\gamma_\rho(w,s)\right] =$$

$$E_o(w,s) + \sum_{k=1}^{K-1} E_k(w,s)\Big[\gamma_\rho^h(w,s) - w(b_\nu+\ldots+b_K)\,h_\nu\,(s+\lambda-\lambda\gamma_\rho(w,s))$$

$$-w\sum_{\alpha=1}^{\nu-1} b_\alpha\gamma_\rho^{\nu-\alpha}(w,s)\,h_\alpha\,(s+\lambda-\lambda\gamma_\rho(w,s))\Big]$$

for $\rho = 1,\ldots,K$

The Laplace - Stieltjes transform of the distribution of the busy period is given by $E_o(1,s)$ and it is easy to obtain this as the ratio of two determinant from formula (10) i.e.

$$(11) \quad E_o(w,s) = \frac{\left\| \; wh_1(s+\lambda-\lambda\gamma_\rho) \quad A_\rho^1(w,s) \; --- \; A_\rho^{K-1}(w,s) \; \right\|}{\left\| \; 1 \quad A_\rho^1(w,s) \; --- \; A_\rho^{K-1}(w,s) \; \right\|}$$

where

$$A_{\rho}^{k}(w,s) = \gamma_{\rho}^{k}(w,s) - w(b_{\nu}+\ldots+b_{K})\, h_{\nu}(s+\lambda-\lambda\gamma_{\rho})$$

$$- w\sum_{\alpha=1}^{\nu-1} b_{\alpha}\, \gamma_{\rho}^{\nu-\alpha}(w,s)\, h_{\alpha}(s+\lambda-\lambda\gamma_{\rho})$$

If we set w=1 and let $s \longrightarrow 0+$ then $E_{o}(1, 0+)$ gives us the probability that the busy period is of finite duration. We note that as $s \longrightarrow 0+$ and w=1 one of the roots, say $\gamma_1(w,s)$ will tend to the real root of the equation.

$$(12) \qquad X^{K} = \sum_{\alpha=1}^{K} b_{\alpha}\, X^{K-\alpha}\, h_{\alpha}(\lambda-\lambda X)$$

in the unit - circle.

This equation clearly has two roots in $[0,1]$. One root is X=1 and the other root is $X = \gamma_1(1,0+)$. This follows by continuity and by a consideration of the graphs.

By the usual argument, we find:

$$\gamma_1(1,0+) = 1 \text{ if and only if } \sum_{\alpha=1}^{K} b_{\alpha}(\lambda m_{\alpha} - \alpha) \leq 0$$

and

$$\gamma_1(1,0+) < 1 \text{ if and only if } \sum_{\alpha=1}^{K} b_{\alpha}(\lambda m_{\alpha} - \alpha) > 0$$

where m_{α} is the first moment of the distribution $H_{\alpha}(\cdot)$

It is clear from formula (11) that $\gamma_1(1,0+) = 1$ implies $E_{o}(1,0+) = 1$.

[illegible]

[illegible].

Under mild additional conditions it also follows that $E_o(1,0+) < 1$ if the contrary inequality holds.

An important particular case

If $b_1 = \ldots = b_{K-1} = 0$ and $b_K = 1$ then we obtain a queueing model which has been studied previously by Bloemena [1], Le Gall [2], Runnenburg [6]. The distribution of the busy period, given by formula (11), reduces to that previously found by Neuts [3]

In order to avoid messy calculations we will now limit ourselves to this special case in our derivation of the queue length distributions. The same arguments apply in the general case.

C. The Queue length in Discrete Time

In the particular case at hand, the transition probability distributions become:

$$(13) \quad Q_{oj}(x) = \int_o^x \left[1-e^{-\lambda(x-y)}\right] e^{-\lambda y} \frac{(\lambda y)^j}{j!} \, dH_1(y), \qquad j = 0, 1, \ldots$$

$$Q_{ij}(x) = 0, \qquad i \geq K, \; j < i-K$$

$$Q_{ij}(x) = \int_o^x e^{-\lambda y} \frac{(\lambda y)^j}{j!} \, dH_i(y), \qquad 0 < i < K, \; j = 0, 1, \ldots$$

$$Q_{ij}(x) = \int_o^x e^{-\lambda y} \frac{(\lambda y)^{j+K-i}}{(j+K-i)!} \, dH_K(y) \qquad i \geq K, \; j \geq i-K$$

We first calculate the iterates of the matrix $Q(x)$, given by

$$Q^{(m+n)}(x) = \sum_{\nu=0}^{\infty} Q_{i\nu}^{(m)} * Q_{\nu j}^{(n)} (x)$$

$$Q_{ij}^{(o)}(x) = \delta_{ij} I_o(x)$$

where $I_o(x)$ is the distribution, degenerate at zero. $Q_{ij}^{(n)}(x)$ is the joint distribution of the n - th departure time, together with ξ_n, conditional upon the fact that there are i customers in line at t = 0+

We have the following recurrence relation:

$$(14) \quad Q_{ij}^{(n+1)}(x) = \int_o^x Q_{io}^{(n)}(x-u) \int_o^u \lambda e^{-\lambda u} \frac{(\lambda y)^j}{j!} dH_1(y)\, du +$$

$$\sum_{\nu=1}^{K-1} \int_o^x Q_{i\nu}^{(n)}(x-u)\; e^{-\lambda u} \frac{(\lambda u)^j}{j!} dH_\nu(u) +$$

$$\sum_{\nu=K}^{K+j} \int_o^x Q_{i\nu}^{(n)}(x-u)\; e^{-\lambda u} \frac{(\lambda u)^{j-\nu+K}}{(j-\nu+K)!} dH_m(u),$$

for all i and j.

Taking Laplace - Stisltjes transforms we obtain:

$$(15) \quad q_{ij}^{(n+1)}(s) = \frac{\lambda}{\lambda+s} q_{is}^{(n)}(s) \int_o^\infty e^{-(s+\lambda)y} \frac{(\lambda y)^j}{j!} dH_1(y) +$$

$$\sum_{\nu=1}^{K-1} q_{i\nu}^{(n)}(s) \int_o^\infty e^{-(s+\lambda)y} \frac{(\lambda y)^j}{j!} dH_\nu(y) +$$

$$\sum_{\nu=K}^{K+j} q_{i\nu}^{(n)}(s) \int_o^\infty e^{-(s+\lambda)y} \frac{(\lambda y)^{j+K-\nu}}{(j+K-\nu)!} dH_m(y)$$

Let us set

$$(16) \quad \sum_{\nu=0}^{\infty} q_{i,\nu+K}^{(n)}(s)\, z^\nu = U_i^{(n)}(z,s)$$

then we obtain:

$$(17)\quad \sum_{j=0}^{K-1} q_{ij}^{(n+1)}(s)\, z^j + z^K\, U_i^{(n+1)}(z,s) =$$

$$\frac{\lambda}{\lambda+s}\, q_{io}^{(n)}(s)\, h_1(s+\lambda-\lambda z) + \sum_{\nu=1}^{K-1} q_{i\nu}^{(n)}(s)\, h_\nu(s+\lambda-\lambda z)$$

$$+ U_i^{(n)}(z,s)\, h_K(s+\lambda-\lambda z)$$

Now set:

$$(18)\quad \sum_{n=0}^{\infty} q_{ij}^{(n)}(s)\, w^n = W_{ij}(w,s), \qquad \text{for } j = 0, 1, \ldots, K-1$$

and $$\sum_{n=0}^{\infty} U_i^{(n)}(z,s)\, w^n = V_i(z,s,w)$$

Multiplying by w^{n+1} in (17) and summing up, we obtain

$$(19)\quad V_i(z,s,w) =$$

$$\left[z^K - w\, h_K(s+\lambda-\lambda z)\right]^{-1} \left\{z^i - W_{io}(w,s)\left[1 - w\frac{\lambda}{\lambda+s}\, h_1(s+\lambda-\lambda z)\right]\right.$$

$$\left. - \sum_{\nu=1}^{K-1} W_{i\nu}(w,s)\, \left[z^\nu - wh_\nu(s+\lambda-\lambda z)\right]\right\}$$

The unknown functions $W_{io}(w,s), \ldots, W_{i,K-1}(w,s)$ may be determined by the standard argument. The demoninator has **K** roots inside the unit-circle $z = 1$ for Re $s > 0$, $w \leq 1$ or Re $s \geq 0$, $w < 1$. Let us denote them by $\gamma_\rho(w,s)$, $\rho = 1, \ldots, K$ then the functions $W_{i\nu}(w,s)$, $\nu = 0, 1, \ldots, K-1$ are the solutions of the K equations:

$$(20).\quad W_{io}(w,s)\left[1 - w\,\frac{\lambda}{\lambda+s}\;h_1\;(s+\lambda-\lambda\gamma_\rho)\right]$$

$$+\sum_{\nu=1}^{K-1} W_{i\nu}\;(w,s)\left[\gamma_\rho^{\nu}(w,s) - wh_\nu\;(s+\lambda-\lambda\gamma_\rho(w,s))\right] = \gamma_\rho^{i}\;(w,s)$$

We also calculate the actual generating function of the $q_{ij}^{(n)}(s)$ and obtain:

$$(21)\quad \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} z^j\,w^n\;q_{ij}^{(n)}\;(s) =$$

$$\left[z^K - wh_K(s+\lambda-\lambda z)\right]^{-1}\left\{z^{K+i} + \left[z^K\,\frac{\lambda}{\lambda+s}\;h_1(s+\lambda-\lambda z) - h_K(s+\lambda-\lambda z)\right] w\,W_{io}(w,s)\right.$$

$$\left. + w\sum_{\nu=1}^{K-1} W_{i\nu}(w,s)\;\left[z^{K-\nu}h_\nu(s+\lambda-\lambda z) - h_K(s+\lambda-\lambda z)\right] z^\nu\right\}$$

For s=0+ we find the generating function of the transition probabilities

$$P_{ij}^{(n)} = P\left\{\xi_n = j \;\middle|\; \xi_o = i\right\}$$

for the imbedded Markov chain:

$$(22)\quad \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} z^j w^n\,P_{ij}^{(n)} =$$

$$\left[z^K - wh_K(\lambda-\lambda z)\right]^{-1}\left\{z^{K+i} - wW_{io}(w,o)\;\left[z^K h_1(\lambda-\lambda z) - h_K(\lambda-\lambda z)\right]\right.$$

$$\left. + w\sum_{\nu=1}^{K-1} W_{i\nu}(w,o)\;z^\nu\left[z^{K-\nu}\,h_\nu\;(\lambda-\lambda z) - h_K\;(\lambda-\lambda z)\right]\right\}$$

The imbedded Markov chain is irreducible and aperiodic. In order to find the stationary transition probabilities, we multiply equation (22) by (1-w) and let w tend to one. We obtain:

$$(23)\quad \sum_{j=0}^{\infty} z^j \pi_j =$$

$$\frac{\pi_o \left[h_K(\lambda-\lambda z) - z^K h_1(\lambda-\lambda z)\right] + \sum_{\nu=1}^{K-1} \pi_\nu z^\nu \left[h_K(\lambda-\lambda z) - z\, h_\nu(\lambda-\lambda z)\right]}{z^K - h_K(\lambda-\lambda z)}$$

where

$$\pi_\nu = \lim_{w\to 1} (1-w)\, W_{i\nu}(w,o), \qquad \text{for } \nu = 0, 1, \ldots, K-1$$

now, in determinant notation

$$(24)\quad W_{i\nu}(w,o) =$$

$$\frac{\left\| 1-wh_1(\lambda-\lambda\gamma_\rho) \;---\; \overset{\nu}{\downarrow}\gamma_\rho^{i}(w,o) \;---\; \gamma_\rho^{K-1}(w,o) - wh_{K-1}(\lambda-\lambda\gamma_\rho) \right\|}{\left\| 1-wh_1(\lambda-\lambda\gamma_\rho) \;---\; \gamma_\rho^{\nu}(w,o) - wh_\nu(\lambda-\lambda\gamma_\rho) \;---\; \gamma_\rho^{K-1}(w,o) - wh_{K-1}(\lambda-\lambda\gamma_\rho) \right\|}$$

If $\gamma_1(1,0+) < 1$ then $\pi_\nu = 0$ for $\nu = 0, 1, \ldots, K-1$ and therefore all π_j are zero.

If $\gamma_1(1,0+ = 1$ we leave the numerator in (24) as it is and divide' the first row in the determinant by 1-w.

Taking limits the numerator tends to $(-1)^\nu$ times the minor of the element $\gamma_1^i(w,0)$ evaluated at $w=1$. This minor does not depend on i.

The first row of the determinant in the denominator tends to a row of constants $A_o, A_1, \ldots, A_{K-1}$ which are given by:

$$(25) \qquad A_o = \left[1+\lambda\alpha_1\gamma_1'(1, 0+)\right]^{-1}$$

$$A_\nu = \left[1 - (\nu-\lambda\alpha_\nu)\ \gamma_1'\ (1, 0+)\right]^{-1}$$

From the equation:

$$(26) \qquad \gamma_1^K\ (w,o = wh_K\left[\lambda-\lambda\gamma_1(w, 0)\right]$$

we obtain after differentiation that:

$$(27) \qquad \gamma_1'\ (1, 0+) = \frac{1}{K-\lambda\alpha_K}$$

If $K = \alpha_K\lambda$ then again all $\pi_j = 0$ and the chain is null - recurrent
If $K > \alpha_K\lambda$ then we obtain:

$$(28) \qquad A_o = \frac{K - \lambda\alpha_K}{K - \lambda\alpha_K + \lambda\alpha_1},$$

$$A_\nu = \frac{K - \lambda\alpha_K}{K - \lambda\alpha_K + \lambda\alpha_\nu - \nu}, \qquad \text{for} \quad \nu = 1,\ldots, K-1.$$

The minors of these elements in the determinant go to well - defined limits as $w \longrightarrow 1$.

If we let z tend to one in (23) and assume $K > \alpha_K \lambda$ then we obtain:

$$(29) \quad \Pi_o \left[\lambda \frac{\alpha_K - \alpha_1}{K} - 1\right] + \sum_{\nu=1}^{K-1} \Pi_\nu \left[\lambda \frac{\alpha_K - \alpha_\nu}{K} - \frac{K-\nu}{K}\right]$$

as the limit.

If we substitute the expressions found for $\pi_o, \ldots, \pi_{K-1}$ then we see that the resulting sum is unity, which verifies again that in this case the π_j form a bona - fide probability distribution.

D. <u>The Queue length in Continuous Time</u>

Let $M_{ij}(t)$ denote the expected numbers of visits to state j in the interval $(o,t]$, starting in state i, in the semi-Markov process with matrix $Q(x)$.

It is known that

$$(30) \quad M_{ij}(t) = \sum_{\nu=0}^{\infty} Q_{ij}^{(\nu)}(t)$$

Let $\mu_{ij}(s)$ denote the Laplace - Stisltjes transform of $M_{ij}(t)$, then it follows from formula (30) that

$$(31) \quad \sum_{j=0}^{\infty} \mu_{ij}(s)\, z^j = \sum_{j=0}^{K-1} \mu_{ij}(s)\, z^j + z^K \sum_{\nu=0}^{\infty} \mu_{i,\nu+K}(s)\, z^\nu =$$

$$\sum_{j=0}^{K-1} W_{ij}(1,s)\, z^j + z^K\, V_i(z,s,1)$$

The functions $W_{ij}(1,s)$, $j = 0, 1, \ldots, K-1$ and $V_i\ (z,s,1)$ may be calculated using formulae (19) and (20)

If we know the renewal functions $M_{ij}(t)$, $i, j, = 0, 1, \ldots$ then the distribution of the queue length at times t is obtained easily

Let

$$(32) \quad P_{ij}(t) = P\left\{\xi(t) = j \;\middle|\; \xi_o = i\right\}$$

It then follows that:

$$(33) \quad P_{io}(t) = \int_o^t e^{-\lambda(t-\tau)} \; dM_{io}(\tau),$$

and for $j > 0$,

$$(34) \quad P_{ij}\ (t) =$$

$$\int_o^t dM_{io}(t) \int_o^{t-\tau} \lambda e^{-\lambda u} \left[1-H_1(t-\tau-u)\right] e^{-\lambda(t-\tau-u)} \frac{\left[\lambda(t-\tau-u)\right]^{j-1}}{(j-1)!} du$$

$$+ \sum_{\nu=1}^{j} \int_o^t \left[1-\sum_{\alpha=}^{\infty} Q_{\nu\alpha}(t-\tau)\right] e^{-\lambda(t-\tau)} \left[\lambda(t-\tau)\right]^{j-\nu} dM_{i\nu}(\tau),$$

in which

$$1 - \sum_{\alpha=0}^{\infty} Q_{\nu\alpha}\ (t-\tau) = 1 - H_\nu(t-\tau), \quad \text{for} \quad \nu = 1, \ldots, K-1$$

$$= 1 - H_K(t-\tau), \quad \text{for } \nu \geq K.$$

Taking Laplace - transforms of the $P_{ij}(t)$ and denoting them by $p_{ij}(s)$ we obtain:

$$(35)\quad p_{io}(s) = (s+\lambda)^{-1}\,\mu_{io}(s)$$

$$p_{ij}(s) = \frac{\lambda}{\lambda+s}\,\mu_{io}(s)\int_0^\infty e^{-(s+\lambda)\zeta}\left[1 - H_1(\zeta)\right]\frac{(\lambda\zeta)^{j-1}}{(j-1)!}\,d\zeta$$

$$+\sum_{\nu=1}^{j}\mu_{i\nu}(s)\int_0^\infty e^{-(s+\lambda)\zeta}\left[1 - H_\nu(\zeta)\right]\frac{(\lambda\zeta)^{j-\nu}}{(j-\nu)!}\,d\zeta$$

for $j = 1,\dots, K-1$

$$p_{ij}(s) = \frac{\lambda}{\lambda+s}\,\mu_{io}(s)\int_0^\infty e^{-(s+\lambda)\zeta}\left[1 - H_1(\zeta)\right]\frac{(\lambda\zeta)^{j-1}}{(j-1)!}\,d\zeta$$

$$+\sum_{\nu=1}^{K-1}\mu_{i\nu}(s)\int_0^\infty e^{-(s+\lambda)\zeta}\left[1 - H_\nu(\zeta)\right]\frac{(\lambda\zeta)^{j-\nu}}{(j-\nu)!}\,d\zeta$$

$$+\sum_{\nu=K}^{j}\mu_{i\nu}(s)\int_0^\infty e^{-(s+\lambda)\zeta}\left[1 - H_K(\zeta)\right]\frac{(\lambda\zeta)^{j-\nu}}{(j-\nu)!}\,d\zeta\,,$$

for $j \geq K$.

The generating function of the $p_{ij}(s)$ is given by:

$$(36)\quad \sum_{j=0}^{\infty} z^j\,p_{ij}(s) =$$

$$\frac{\lambda}{s+\lambda} \; \mu_{io}(s) \left[\frac{1}{\lambda} + \frac{z}{s+\lambda+\lambda z} \left[1 - h_1(s+\lambda-\lambda z) \right] \right] +$$

$$\frac{1}{s+\lambda-\lambda z} \sum_{\nu=1}^{K-1} \mu_{i\nu}(s) \, z^{\nu} \left[1 - h_{\nu}(s+\lambda-\lambda z) \right] +$$

$$\frac{z^K}{s+\lambda-\lambda z} \left[1 - h_K(s+\lambda-\lambda z) \right] \sum_{\rho=0}^{\infty} \mu_{i,K+\rho}(s) \, z^{\rho} =$$

$$\frac{\lambda}{s+\lambda} \, W_{is}(1,s) \left[\frac{1}{\lambda} + \frac{z}{s+\lambda-\lambda z} \left[1 - h_1(s+\lambda-\lambda z) \right] \right] +$$

$$\frac{1}{s+\lambda-\lambda z} \sum_{\nu=1}^{K-1} W_{i\nu}(1,s) \, z^{\nu} \left[1 - h_{\nu}(s+\lambda-\lambda z) \right] +$$

$$\frac{z^K}{s+\lambda-\lambda z} \left[1 - h_K(s+\lambda-\lambda z) \right] V_i(z,s,1)$$

These functions $W_{i\nu}(1,s)$, $\nu = 0, 1, \ldots, K-1$ and $V_i(z,s,1)$ were determined above.

We proceed by studying the asymptatic behavior of the probabilities $P_{ij}(t)$.

Let μ_o^* , $\mu_1^*, \ldots, \mu_2^*, \ldots$ denote the mean recurrence times of the states $0, 1, \ldots$ in the semi-Markov process with matrix $Q(x)$. Applying the Key renewal theorem to the integrals in formulae (33) and (34) we obtain that the limits P_j^* of the probabilities $P_{ij}(t)$ exist and are given by:

$$(37) \qquad P_0^* = \frac{1}{\lambda\mu_0^*}$$

$$P_j^* = \frac{1}{\mu_0^*} \int_0^\infty \left[1 - H_1(v)\right] e^{-\lambda v} \frac{(\lambda v)^{j-1}}{(j-1)!} dv$$

$$+ \sum_{\nu=1}^{j} \frac{1}{\mu_\nu^*} \int_0^\infty \left[1 - H_\nu(v)\right] e^{-\lambda v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} dv,$$

for $j = 1, \ldots K-1$

$$P_j^* = \frac{1}{\mu_0^*} \int_0^\infty \left[1 - H_1(v)\right] e^{-\lambda v} \frac{(\lambda v)^{j-1}}{(j-1)!} dv +$$

$$\sum_{\nu=1}^{K-1} \frac{1}{\mu_\nu^*} \int_0^\infty \left[1 - H_\nu(v)\right] e^{-\lambda v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} dv +$$

$$\sum_{\nu=K}^{j} \frac{1}{\mu_\nu^*} \int_0^\infty \left[1 - H_K(v)\right] e^{-\lambda v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} dv,$$

for $j \geq K$.

in which $\frac{1}{\mu_\nu^*} = 0$ if $\mu_\nu^* = \infty$

It follows that all $P_j^* = 0$ if $K \leq \alpha_K \lambda$ since the semi-Markov process is then either transient or null - recurrent.

If $K > \alpha_K \lambda$ then all μ_j^*, $j = 0, 1, \ldots$ are finite and the P_j^* are well defined.

The formulae (37) relate the probabilities P_j^* in an interesting way to the stationary probabilities of the semi-Markov process with matrix $Q(x)$. Let η_j, $j = 0, 1, \ldots$ denote the first moments of the distribution function $\sum_{\alpha=0}^{\infty} Q_{j\alpha}(x)$ then it is known that the stationary probabilities for the semi-Markov process are given by: Pyke [5].

$$(38) \qquad \pi_j^* = \eta_j \mu_j^{*-1}$$

The π_j^* are different from the π_j found above.

As an example the last of the formulae (37) may be written as:

$$(39) \quad P_j^* = \pi_0^* \int_0^\infty \frac{1 - H_1(v)}{\alpha_1 + \lambda^{-1}} \; e^{-\lambda v} \frac{(\lambda v)^{j-1}}{(j-1)!} \, dv +$$

$$\sum_{\nu=1}^{K-1} \pi_\nu^* \int_0^\infty \frac{1 - H_\nu(v)}{\alpha_\nu} \; e^{-\lambda v} \frac{(\lambda v)^{j-\nu}}{(j-\nu)!} \, dv +$$

$$\sum_{\nu=K}^{j} \pi_\nu^* \int_0^\infty \frac{1 - H_K(v)}{\alpha_K} \; \frac{e^{-\lambda v} (\lambda v)^{j-\nu}}{(j-\nu)!} \, dv$$

for $j \geq K$.

If the initial conditions of the semi-Markov process are chosen so that it is stationary - Pyke [5] - then formula (39) expresses the total probability that $\xi(t) = j$ by saying that at time t the semi-Markov process is in some state $\nu = 0, 1, \ldots, j$ and that during the busy part of the backwards recurrence time the required number of customers have arrived.

There appears to be no single way to relate the P_j^* to the stationary probabition π_j for the imbadded Markov chain.

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