Some Results on the Non-Central Multivariate Beta Distribution and Moments of Traces of Two Matrices

by

C. G. Khatri and K.C.S. Pillai

Gujarat University and University of North Carolina,

and Purdue University

Department of Statistics

Division of Mathematical Sciences

Mimeograph Series No. 35

January 1965

Some Results on the Non-Central Multivariate

Beta Distribution and Moments of Traces of Two Matrices

by

C. G. Khatri[1] and K.C.S. Pillai

Gujarat University and University of North Carolina,

and Purdue University

1. Introduction and Summary. Let $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ be two positive definite matrices of order p, $\underset{\sim}{A}_1$ having a Wishart distribution [2, 12] with f_1 degrees of freedom and $\underset{\sim}{A}_2$ (pseudo) non-central (linear) Wishart distribution [1,3,4,12,13] with f_2 degrees of freedom. Now transform

$$\underset{\sim}{A}_2 = \underset{\sim}{C}\,\underset{\sim}{Y}\,\underset{\sim}{Y}'\,\underset{\sim}{C}'$$

where $\underset{\sim}{C}$ is a lower triangular matrix such that

$$\underset{\sim}{A}_1 + \underset{\sim}{A}_2 = \underset{\sim}{C}\,\underset{\sim}{C}'$$

and the density function of $\underset{\sim}{Y}$: $p \times f_2$ is given by

$$(1.1) \qquad k_1\, e^{-\lambda^2} \sum_{j=0}^{\infty} (2\lambda\, y_{11})^j\, \Gamma\left[\tfrac{1}{2}(f_1 + f_2 + j)\right] |\underline{I}_p - \underline{YY}'|^{\frac{1}{2}(f_1 - p - 1)}/j!$$

where $\underline{I}_p$ is an indentity matrix of order p,

[1] The work of this author was supported by the Mathematics Division of the Air Force Office of Scientific Research Contract No. AF-AFOSR-84-63.

$$k_1 = \prod_{i=2}^{p} \Gamma[\tfrac{1}{2}(f_1+f_2-i+1)]/\Pi^{\frac{1}{2}pf_2} \prod_{i=1}^{p} \Gamma[(f_1-i+1)/2],$$

λ is the only non-centrality parameter in the linear case and y_{11} is the element in the top left corner of the $\underset{\sim}{Y}$ matrix.

Now $V^{(s)}$ criterion suggested by Pillai and $U^{(s)}$(a constant times Hotelling's T_0^2), [7,8,9,10] are the sums of the non-zero characteristic roots of the matrix $\underset{\sim}{Y}\underset{\sim}{Y}'$ and $(\underset{\sim}{I}_p-\underset{\sim}{Y}\underset{\sim}{Y}')^{-1}-\underset{\sim}{I}_p$ respectively. Here s is minimum (f_2,p). Also we may note that $V^{(s)}$ = trace $\underset{\sim}{Y}\underset{\sim}{Y}'$ = trace $\underset{\sim}{Y}'\underset{\sim}{Y}$ and $U^{(s)}=tr(\underset{\sim}{I}_p-\underset{\sim}{Y}\underset{\sim}{Y}')^{-1}-p=tr(\underset{\sim}{I}_{f_2}-\underset{\sim}{Y}'\underset{\sim}{Y})^{-1}-f_2$. It can be shown that the density function of $\underset{\sim}{Y}'\underset{\sim}{Y}$ for $f_2 \leq p$ can be obtained from the density function of $\underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \geq p$ if in the latter case the following changes are made: [12,5]

$$(1.2) \qquad (f_1,f_2,p) \longrightarrow (f_1-f_2+p,p,f_2).$$

Hence, for the criterion $V^{(s)}$, (and similarly for $U^{(s)}$), we shall only consider the density function of $\underset{\sim}{L} = \underset{\sim}{Y}\,\underset{\sim}{Y}'$ for $f_2 \geq p$ which is given by [6]

$$(1.3) \qquad f(\underset{\sim}{L}) = k\, e^{-\lambda^2}\, {}_1F_1\{\tfrac{1}{2}(f_1+f_2),\tfrac{1}{2}f_2,\ \lambda^2 \ell_{11}\}|\underset{\sim}{L}|^{(f_2-p-1)/2} \times |\underset{\sim}{I}_p-\underset{\sim}{L}|^{(f_1-p-1)/2}$$

where

$$k = \Pi^{-p(p-1)/4} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(f_1+f_2+1-i)]/\{\Gamma[\tfrac{1}{2}(f_1+1-i)]\ \Gamma[\tfrac{1}{2}(f_2+1-i)]\},$$

ℓ_{11} is the element in the top left corner of the matrix $\underset{\sim}{L}$ and ${}_1F_1$ denotes the confluent hypergeometric function. We shall call the distribution of $\underset{\sim}{L}: p \times p$ the non-central (linear) multivariate beta distribution with f_2 and f_1 degrees of freedom.

Pillai [11] had noted that the elements of the matrix $\underline{L}$ can be transformed into independent beta variables which he showed for $p = 2,3,4$ and 5. In this paper we give a theorem which proves the general case. In addition, when $\lambda = 0$ the first and second order moments of ℓ_{ij} are obtained and used to derive the first two moments of $V^{(s)}$ in the non-central case when $f_2 \geq p$. The moments of $V^{(s)}$ for $f_2 \leq p$ can be written down with the help of (1.2). Similar results are obtained for $U^{(p)} = t_r(\underline{I}_p - \underline{L})^{-1} - p$.

2. Independent Beta Variables. Let

$$\underline{L} = \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \underset{\sim}{\ell} & \underset{\sim}{L}_{11} \end{pmatrix} \begin{matrix} 1 \\ p-1 \end{matrix}, \quad \underset{\sim}{L}_{22} = \underset{\sim}{L}_{11} - \underset{\sim}{\ell}\underset{\sim}{\ell}'/\ell_{11},$$

$$\begin{matrix} 1 & p-1 \end{matrix}$$

and we note that

$$|\underline{L}| = \ell_{11}\, |\underline{L}_{22}|$$

and

$$|\underline{I}_p - \underline{L}| = (1-\ell_{11})\,|\underline{I}_{p-1} - \underline{L}_{22} - \underset{\sim}{\ell}\underset{\sim}{\ell}'/[\ell_{11}(1-\ell_{11})]|.$$

Then it is easy to show that

$$\ell_{11} \quad \text{and} \quad (\underline{L}_{22},\ \underline{v} = \underset{\sim}{\ell}/\sqrt{\ell_{11}(1-\ell_{11})})$$

are independently distributed and their respective distributions are

$$(2.1) \quad f_1(\ell_{11}) = [\beta(\tfrac{1}{2}f_2,\tfrac{1}{2}f_1)]^{-1}\exp(-\lambda^2)\,\ell_{11}^{\frac{1}{2}f_2-1}(1-\ell_{11})^{\frac{1}{2}f_1-1}\,{}_1F_1[\tfrac{1}{2}(f_1+f_2),\tfrac{1}{2}f_2,\lambda^2\ell_{11}]$$

and

$$(2.2) \quad f_2(\underline{L}_{22},\underline{v}) = k_2\ |\underline{L}_{22}|^{\frac{1}{2}[(f_2-1)-(p-1)-1]}\ |\underline{I}_{p-1}-\underline{L}_{22}-\underline{v}\ \underline{v}'|^{\frac{1}{2}(f_1-p-1)}$$

where

$$k_2 = k\ \beta(\tfrac{1}{2}f_2,\tfrac{1}{2}f_1).$$

For further independence, we can use two types of transformations given by

$$(2.3) \quad \underline{u} = (\underline{I}_{p-1}-\underline{L}_{22})^{-\frac{1}{2}}\ \underline{v} \quad \text{or} \quad \underline{w} = \underline{T}^{-1}\ \underline{v}$$

where $\underline{I}_{p-1}-\underline{L}_{22} = \underline{T}\ \underline{T}'$ and $\underline{T}:(p-1)\times(p-1)$ is a lower triangular matrix. It is easy to show that $\underline{u}$ (or $\underline{w}$) and $\underline{L}_{22}$ are independently distributed and their respective distributions are

$$(2.4) \qquad f_3(\underline{u}) = \Pi^{-\frac{1}{2}(p-1)}\ \frac{\Gamma(\frac{1}{2}f_1)}{\Gamma(\frac{f_1-p+1}{2})}\ (1-\underline{u}'\ \underline{u})^{\frac{1}{2}(f_1-p-1)} \qquad [\text{or } f_3(\underline{w})]$$

and

$$(2.5)\qquad f_4(\underset{\sim}{L}_{22}) = k_3\ |\underset{\sim}{L}_{22}|^{\frac{1}{2}[(f_2-1)-(p-1)-1]}\ |I-\underset{\sim}{L}_{22}|^{\frac{1}{2}[f_1-(p-1)-1]}$$

where $k_3 = \Pi^{\frac{1}{2}(p-1)} k_2$. We may note that the distribution of $\underset{\sim}{L}_{22}:(p-1)x(p-1)$ is central multivariate beta distribution with (f_2-1) and f_1 degrees of freedom, and the similar reduction from $\underset{\sim}{L}_{22}$ can be carried successively. We may also note that the transformation

$$(2.6)\qquad x_i = u_i^2/(1-u_1^2-\dots-u_{i-1}^2),\quad i = 1,2,\dots, p-1$$

in (2.4) gives us the independent beta-variates and their density functions are given by

$$(2.7)\qquad g_i(x_i) = \{\beta[\tfrac{1}{2}, \tfrac{1}{2}(f_1-i)]\}^{-1}\ x_i^{\frac{1}{2}-1}\ (1-x_i)^{\frac{1}{2}(f_1-i)-1}.$$

From the foregone, we have the following theorem:

Theorem I: If the distribution of $\underline{L} = \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \underset{\sim}{\ell} & \underset{\sim}{L}_{11} \end{pmatrix}$ is given by (1.3), then

$$\ell_{11},\ \underset{\sim}{L}_{22} = \underset{\sim}{L}_{11} - \underline{\ell}\,\underline{\ell}'/\ell_{11} \quad\text{and}\quad \underline{u} = (\underline{I}_{p-1}-\underline{L}_{22})^{-\frac{1}{2}}\ \underline{\ell}/\sqrt{\ell_{11}(1-\ell_{11})}$$

[or $\underline{w} = \underline{T}^{-1}\,\underline{\ell}/\sqrt{\ell_{11}(1-\ell_{11})}$ where $\underline{T}\,\underline{T}' = \underline{I}_{p-1}-\underline{L}_{22}$ and $\underline{T}$ is a lower triangular matrix] are independently distributed and their respective distributions are defined in (2.1), (2.5) and (2.4).

It can be verified for p=3 that from the variates ℓ_{11}, $\underline{w}$ and $\underline{L}_{22}$, we can obtain the independent β-variates exactly the same as given by Pillai [11], but the use of ℓ_{11}, $\underline{u}$ and $\underline{L}_{22}$ will give independent β-variates different from those of Pillai [11] in spite of the identical β-distributions.

3. The first and second order moments of ℓ_{ij} when $\lambda=0$. Let the density function of $\underline{L}$ be given by

$$k\,|\underline{L}|^{\frac{1}{2}(f_2-p-1)}|\underline{I}_p-\underline{L}|^{\frac{1}{2}(f_1-p-1)}, \tag{3.1}$$

where k is the same as in (1.3). It is easy to see that

$$\begin{aligned} E(\ell_{ij}) &= E(\ell_{11}) \quad \text{when } i=j \\ &= E(\ell_{12}) \quad \text{when } i\neq j \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} E(\ell_{ij}\,\ell_{i'j'}) &= E(\ell_{11}^2) && \text{when } i=j=i'=j' \\ &= E(\ell_{11}\ell_{12}) && \text{when } i=j=i',\ i\neq j' \\ &= E(\ell_{11}\ell_{22}) && \text{when } i=j, i'\neq i, i'=j' \\ &= E(\ell_{12}^2) && \text{when } i=i',\ j=j',\ i\neq i' \\ &= E(\ell_{11}\ell_{23}) && \text{when } i=j, i'\neq j'\neq i \\ &= E(\ell_{12}\ell_{13}) && \text{when } i=i', j\neq j'\neq i' \\ &= E(\ell_{12}\ell_{34}) && \text{when } i\neq j\neq i'\neq j'\,. \end{aligned} \tag{3.3}$$

It is easy to see that if $\nu = f_1 + f_2$,

$$E(\ell_{11}) = f_2/\nu, \; E(\ell_{12}) = 0,$$

and

$$E(\ell_{11}^2) = : f_2(f_2+2)/\nu(\nu+2).$$

For $E(\ell_{12}^2)$, we integrate over other variates except ℓ_{11}, ℓ_{12} and ℓ_{22}. Then as in theorem 1, $u_1 = \ell_{12}/\sqrt{(1-\ell_{11})(1-z)\ell_{11}}$, ℓ_{11} and $(\ell_{22} - \ell_{12}^2/\ell_{11}) = z$ are independently distributed. Hence

$$E(\ell_{12}^2) = E[(1-\ell_{11})\ell_{11}] \, E(1-z) \, E(u_1^2 = x_1)$$

$$= f_1 f_2/\{\nu(\nu-1)(\nu+2)\},$$

$$E(\ell_{11}\ell_{12}) = E[\ell_{11}\sqrt{\ell_{11}(1-z)(1-\ell_{11})}] \quad E(u_1) = 0,$$

and

$$E(\ell_{11}\ell_{22}) = E(\ell_{11} z) + E\{\ell_{11}(1-\ell_{11})(1-z)x_1\}$$

$$= \{f_2(f_2-1) + f_1 f_2/(\nu+2)\}/\nu(\nu-1).$$

Similarly for obtaining $E(\ell_{11}\ell_{23})$ and $E(\ell_{12}\ell_{13})$, we consider (3.1) with p=3 only. Using the successive reduction of theorem 1, it can be shown that

$$E(\ell_{11}\ell_{23}) = E(\ell_{12}\ell_{13}) = 0.$$

The same type of reduction gives us after some algebra

$$E(\ell_{12}\ell_{34}) = 0 .$$

Hence, we have the following theorem:

Theorem II: Let the distribution of $\underset{\sim}{L}$: p x p be given by (3.1). Then

$$(3.4)\qquad E(\ell_{ij}) = \begin{cases} f_2/\nu & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(3.5)\qquad E(\ell_{ij}\ell_{i'j'}) = \begin{cases} f_2(f_2+2)/\{\nu(\nu+2)\} & \text{if } i=j=i'=j' \\ f_1f_2/\{\nu(\nu-1)(\nu+2)\} & \text{if } i=i', j=j', i\neq j \text{ and } i=j', i'=j, i\neq j \\ f_2\{(f_2-1)+f_1/(\nu+2)\}/\{\nu(\nu-1)\} & \text{if } i=j, i'=j', i\neq i' \\ 0 & \text{otherwise.} \end{cases}$$

4. First two moments of $V^{(s)}$ criterion. We note that

$$(4.1)\qquad V^{(s)} = t_r L = \ell_{11} + t_r \underline{L}_{22} + (1-\ell_{11})\underline{u}'(\underline{I}_{p-1}-\underline{L}_{22})\underline{u},$$

where ℓ_{11}, $\underline{u}$ and $\underline{L}_{22}$ are independently distributed and their respective distributions are given by (2.1), (2.4) and (2.5). With the help of Theorem II, we find that

$$\text{(4.2)} \qquad E(\underline{I}_{p-1} - \underline{L}_{22}) = \underline{I}_{p-1}\{f_1/(\nu-1)\},$$

$$\text{(4.3)} \qquad E[(t_r\, \underline{L}_{22})(\underline{I}_{p-1} - \underline{L}_{22})] = \delta_1\, \underline{I}_{p-1}$$

and

$$\text{(4.4)} \qquad E(t_r\, \underline{L}_{22})^2 = \frac{(p-1)(f_2-1)}{\nu-1}\left\{\frac{f_2+1}{\nu+1} + \frac{(p-2)(f_2-2)}{\nu-2} + \frac{f_1(p-2)}{(\nu+1)(\nu-2)}\right\},$$

where

$$\text{(4.5)} \qquad \delta_1 = \frac{(f_2-1)}{(\nu-1)}\left\{(p-1) - \frac{f_2+1}{\nu+1}\frac{(f_2-2)(p-2)}{\nu-2} - \frac{f_1(p-2)}{(\nu+1)(\nu-2)}\right\}.$$

Moreover,

$$\text{(4.6)} \qquad E[\underline{u}'(\underline{I}_{p-1}-\underline{L}_{22})\underline{u}] = \{f_1/(\nu-1)\}\, E(\underline{u}'\, \underline{u}) = (p-1)/(\nu-1),$$

$$\text{(4.7)} \qquad E[(t_r\, \underline{L}_{22})\, \underline{u}'(\underline{I}_{p-1}-\underline{L}_{22})\underline{u}] = \delta_1\, E(\underline{u}'\, \underline{u}) = \delta_1(p-1)/f_1,$$

and

$$\text{(4.8)} \qquad E\{\underline{u}'(\underline{I}_{p-1}-\underline{L}_{22})\underline{u}\}^2 = E\{\underline{u}'\, \underline{S}\, \underline{u}\}^2 \quad \text{if} \quad \underline{S} = \underline{I}_{p-1}-\underline{L}_{22}$$

$$= E(s_{11}^2)\sum_{i=1}^{p-1} E(v_i^4) + \{E(s_{11}s_{22}) + 2E(s_{12}^2)\}\sum_{i\neq j=1}^{p-1} E(v_i v_j)$$

$$= \frac{3(p-1)}{(\nu-1)(\nu+1)} + \frac{(p-1)\ (p-2)}{(\nu-1)(\nu-2)(f_1+2)}\left\{(f_1-1) + \frac{3(f_2-1)}{\nu+1}\right\}.$$

Hence, we get

$$(4.9) \qquad E(V^{(s)}) = 1 + \frac{(p-1)(f_2-1)}{\nu-1} + f_1\{\frac{p-1}{\nu-1} - 1\}\, a_1$$

and

$$(4.10) \qquad E(V^{(s)})^2 = 1 + \frac{(p-1)(f_2-1)}{\nu-1}\{2 + \frac{f_2+1}{\nu+1} + \frac{(p-2)(f_2-2)}{\nu+2} + \frac{f_1(p-2)}{(\nu+1)(\nu-2)}\}$$

$$- 2[f_1\{1- \frac{p-1}{\nu-1} + \frac{(p-1)(f_2-1)}{\nu-1}\}$$

$$+ \frac{(p-1)(f_2-1)}{\nu-1}\{1-p+ \frac{f_2+1}{\nu+1} + \frac{(f_2-2)(p-2)}{\nu-2} + \frac{f_1(p-2)}{(\nu+1)(\nu-2)}\}]\, a_1$$

$$+ f_1(f_1+ 2)[1- \frac{2(p-1)}{\nu-1} + \frac{3(p-1)}{(\nu-1)(\nu+1)} +$$

$$+ \frac{(p-1)(p-2)}{(\nu-1)(\nu-2)(f_1+2)}\{f_1-1+ \frac{3(f_2-1)}{\nu+1}\}]\, a_2,$$

where

$$(4.11) \qquad a_1 = \left\{\sum_{i=0}^{\infty} (\lambda^2)^i/[i!(\nu+2i)]\right\} \exp(-\lambda^2),$$

and

$$(4.12) \qquad a_2 = \left\{\sum_{i=0}^{\infty} (\lambda^2)^i/[i!(\nu+2i)(\nu+2i+2)]\right\} \exp(-\lambda^2).$$

The expressions for the moments of $V^{(s)}$ given by (4.9) and (4.10) reduce to the results for $s=2$ given by Pillai [11] when $p=2$. However, Pillai has provided the first four moments of $V^{(2)}$ in that paper [11]. For obtaining the moments of $V^{(s)}$ when $f_2 \leq p$ replace in the expression of the moments in (4.9) and (4.10) f_1 by f_1-f_2+p, f_2 by p and p by f_2 as in (1.2).

5. The First three moments of $U^{(p)}$. We prove first the following theorem for obtaining the moments of $U^{(p)}$ [7,8,9,10].

Theorem III. Let $\underset{\sim}{M}: p \times p = (m_{ij})$ be distributed as

$$(5.1) \quad \pi^{-\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \left\{\Gamma\left(\frac{f_1+f_2-i+1}{2}\right)/\left[\Gamma\left(\frac{f_1-i+1}{2}\right)\Gamma\left(\frac{f_2-i+1}{2}\right)\right]\right\} |\underset{\sim}{M}|^{\frac{1}{2}(f_2-p-1)} |\underset{\sim}{I}_p + \underset{\sim}{M}|^{-\frac{1}{2}(f_1+f_2)} d\underset{\sim}{M}.$$

Then for $f_1 > (p+1)$,

$$(5.2) \qquad E(m_{ij}) = f_2/(f_1-p-1) \qquad \text{if } i=j$$

$$= 0 \qquad \text{otherwise}$$

and for $f_1 > (p+3)$,

$$
(5.3) \quad E(m_{ij}m_{i'j'}) = \begin{cases} f_2(f_2+2)/\{(f_1-p-1)(f_1-p-3)\} & \text{if } i=j=i'=j' \\ f_2(f_2+f_1-p-1)/\{(f_1-p)(f_1-p-1)(f_1-p-3)\} & \\ & \text{if } i=i',j=j',i\neq j \\ f_2\{(f_1-p)(f_1-p-1)\}^{-1}[(f_2-1)+(f_2+f_1-p-1)(f_1-p-3)^{-1}] & \\ & \text{if } i=j,i'=j',i\neq i' \\ 0 & \text{otherwise.} \end{cases}
$$

Proof: $\underset{\sim}{M}$ is symmetric and positive definite and for evaluating $E(m_{ij})$ and $E(m_{ij}m_{i'j'})$ it is easy to see from (3.2) and (3.3) the various cases which should be considered separately.

Moreover, we may note that

$$m_{11},\ \underset{\sim}{W}:(p-1)\times 1 = \{m_{11}(1+m_{11})\}^{-\frac{1}{2}}\ T_1^{-1}\ \underset{\sim}{m} \quad \text{and} \quad \underset{\sim}{M}_{22.1} = \underset{\sim}{M}_{11} - \underset{\sim}{m}\ \underset{\sim}{m}'/m_{11}$$

are independently distributed and their respective density functions are

$$(5.4) \qquad \{B[\tfrac{1}{2} f_2,\ \tfrac{1}{2}(f_1-p+1)]\}^{-1}\ m_{11}^{\frac{1}{2} f_2 - 1}\ (1+m_{11})^{-\frac{1}{2}(f_1+f_2-p+1)},$$

$$(5.5) \qquad \pi^{-\frac{1}{2}(p-1)} \{\Gamma(\frac{f_1+f_2-p+1}{2})\}^{-1} \{\Gamma(\frac{f_1+f_2}{2})\}\ (1+\underset{\sim}{W}'\ \underset{\sim}{W})^{-\frac{1}{2}(f_1+f_2)},$$

and

$$(5.6) \qquad \pi^{-\frac{1}{4}(p-1)(p-2)} \prod_{i=1}^{p-1} \{\Gamma(\frac{f_1+f_2-i}{2})/[\Gamma(\frac{f_1-i+1}{2})\ \Gamma(\frac{f_2-i}{2})]\} \times$$
$$|\underset{\sim}{M}_{22.1}|^{\frac{1}{2}(f_2-p-1)}\ |\underset{\sim}{I}_{p-1}+\underset{\sim}{M}_{22.1}|^{-\frac{1}{2}(f_1+f_2-1)}$$

where

$$\underset{\sim}{M}_{22.1} = (m_{ij.1},\ i,j=2,3,\ldots,p),\ \underset{\sim}{I}_{p-1} + \underset{\sim}{M}_{22.1} = \underset{\sim}{T}_1\underset{\sim}{T}_1', \qquad \underset{\sim}{T}_1:(p-1)\times(p-1)$$

is a lower-triangular matrix and $\underset{\sim}{M}_{11}$ is obtained from $\underset{\sim}{M}$ by deleting the first row and column.

From the above results, it is easy to verify the following,

$$E(m_{11}) = f_2/(f_1-p-1); \quad E(m_{12}) = (Ew_1)\ E[m_{11}(1+m_{11})(1+m_{22.1})]^{\frac{1}{2}} = 0,$$

$$E(m_{12}^2) = E(w_1^2)\ [E\ m_{11}(1+m_{11})]\ [E(1+m_{22.1})]$$

$$= f_2(f_1+f_2-p-1)/\{(f_1-p)(f_1-p-1)(f_1-p-3)\};$$

$$E(m_{11}m_{22}) = E(m_{11}m_{22.1}) + E(m_{12}^2) = f_2(f_2-1)\ \{(f_1-p)(f_1-p-1)\}^{-1} + E(m_{12}^2);$$

$$E(m_{11}m_{12}) = E(w_1)\ [E\ m_{11}^{3/2}(1+m_{11})^{1/2}\ m_{22.1}^{1/2}] = 0;$$

$$E(m_{12}m_{13}) = E\{m_{11}(1+m_{11})w_1^2\ m_{23.1}\} + E\{m_{11}(1+m_{11})\Big(\ (1+m_{33.1}) - \frac{m_{23.1}^2}{1+m_{22.1}}\Big)^{\frac{1}{2}} w_1 w_2\}$$

$$= 0,$$

where w_1 and w_2 are the first two elements in $\underset{\sim}{W}$. Again

$$E(m_{12}m_{34}) = 0.$$

This proves the theorem III.

<u>Lemma I</u>: If $\underset{\sim}{L}$: pxp is a symmetric and positive definite matrix and $U^{(p)} = t_r(\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} - p$, then

$$(5.7) \qquad 1+U^{(p)} = \{(1-\ell_{11})(1-\underset{\sim}{u}'\ \underset{\sim}{u})\}^{-1} + (1-\underset{\sim}{u}\ \underset{\sim}{u}')^{-1}(\underset{\sim}{u}'\underset{\sim}{M}\ \underset{\sim}{u}) + t_r\ \underset{\sim}{M}$$

where

$$\underset{\sim}{L} = \begin{pmatrix} \ell_{11} & \underset{\sim}{\ell}' \\ \underset{\sim}{\ell} & \underset{\sim}{L}_{11} \end{pmatrix}, \quad \underset{\sim}{\ell} : (p-1)\times 1 = \{\ell_{11}(1-\ell_{11})\}^{\frac{1}{2}} (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{\frac{1}{2}} \underset{\sim}{u},$$

$$\underset{\sim}{L}_{22} : (p-1)\times(p-1) = \underset{\sim}{L}_{11} - \underset{\sim}{\ell}\underset{\sim}{\ell}'/\ell_{11} \quad \text{and} \quad \underset{\sim}{M} : (p-1)\times(p-1) = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-1} - \underset{\sim}{I}_{p-1} .$$

Proof: We may note that

$$(\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} = \begin{pmatrix} (1-\ell_{11})^{-\frac{1}{2}} & \underset{\sim}{0} \\ \underset{\sim}{0} & (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{\ell_{11}}\, \underset{\sim}{u}' \\ -\sqrt{\ell_{11}}\, \underset{\sim}{u} & \underset{\sim}{I}_{p-1} - (1-\ell_{11})\underset{\sim}{u}\underset{\sim}{u}' \end{pmatrix}^{-1} \times \begin{pmatrix} (1-\ell_{11})^{-\frac{1}{2}} & \underset{\sim}{0} \\ \underset{\sim}{0} & (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-\frac{1}{2}} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -\sqrt{\ell_{11}}\, \underset{\sim}{u}' \\ -\sqrt{\ell_{11}}\, \underset{\sim}{u} & \underset{\sim}{I}_{p-1} - (1-\ell_{11})\underset{\sim}{u}\,\underset{\sim}{u}' \end{pmatrix}^{-1} = \begin{pmatrix} 1+\ell_{11}\, \underset{\sim}{u}'\, \underset{\sim}{u}/(1-\underset{\sim}{u}'\, \underset{\sim}{u}) & \sqrt{\ell_{11}}\, \underset{\sim}{u}'/(1-\underset{\sim}{u}'\, \underset{\sim}{u}) \\ \sqrt{\ell_{11}}\, \underset{\sim}{u}/(1-\underset{\sim}{u}'\, \underset{\sim}{u}) & \underset{\sim}{I}_{p-1} + \underset{\sim}{u}\underset{\sim}{u}'/(1-\underset{\sim}{u}'\underset{\sim}{u}) \end{pmatrix}.$$

Hence

$$t_r(\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} = 1-(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1} + \{(1-\ell_{11})(1-\underset{\sim}{u}'\underset{\sim}{u})\}^{-1} + t_r(\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-1}$$

$$+ \underset{\sim}{u}'(\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-1}\underset{\sim}{u}/(1-\underset{\sim}{u}'\underset{\sim}{u}).$$

From this, the lemma is obvious.

Theorem IV: If the distribution of $\underset{\sim}{L}$ is non-central (linear) multivariate beta distribution (1.3) and $U^{(p)} = t_r(\underset{\sim}{I}_p - \underset{\sim}{L})^{-1} - p$, then for $f_1 > (p+1)$,

$$(5.8) \qquad E(U^{(p)}) = (pf_2 + 2\lambda^2)/(f_1 - p - 1)$$

and for $f_1 > (p+3)$,

$$(5.9) \quad Var(U^{(p)}) = 2[4\lambda^4(f_1-p) + (4\lambda^2 + pf_2)(f_1-1)(f_1+f_2-p-1)]/\{(f_1-p)(f_1-p-1)^2 (f_1-p-3)\}.$$

Proof: By theorem I, we may note that ℓ_{11}, $\underset{\sim}{u}$ and $\underset{\sim}{M} = (\underset{\sim}{I}_{p-1} - \underset{\sim}{L}_{22})^{-1} - \underset{\sim}{I}_{p-1}$ are independently distributed and their respective density functions are given by (2.1), (2.4) and

$$\pi^{-\frac{1}{4}(p-1)(p-2)} \prod_{i=1}^{p-1} \{\Gamma(\frac{f_1+f_2-i}{2})/[\Gamma(\frac{f_1-i+1}{2})\Gamma(\frac{f_2-i}{2})]\} \; |\underset{\sim}{M}|^{\frac{1}{2}(f_2-p-1)} \times |\underset{\sim}{I}_{p-1} + \underset{\sim}{M}|^{-\frac{1}{2}(f_1+f_2-1)}.$$

Let $\ell_{11,0}$ be the variate whose distribution is the same as that of ℓ_{11} when $\lambda^2 = 0$. Then

$$E(1-\ell_{11})^{-1} = E(1-\ell_{11,0})^{-1} + 2\lambda^2/(f_1-2),$$

$$E(1-\ell_{11})^{-2} = E(1-\ell_{11,0})^{-2} + 4\lambda^2\{(f_1+f_2-2)+\lambda^2\}/\{(f_1-2)(f_1-4)\}.$$

If $U_0^{(p)}$ be the $U^{(p)}$ statistic when ℓ_{11} is replaced by $\ell_{11,0}$, then

$$E(U^{(p)}) = E(U_0^{(p)}) + [2\lambda^2/(f_1-2)]\ E(1-\underset{\sim}{u}'\ \underset{\sim}{u})^{-1} \tag{5.10}$$

and

$$E[1+U^{(p)}]^2 = E[1+U_0^{(p)}]^2 + \{4\lambda^2/(f_1-2)\}\ E\{(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}[t_r\underset{\sim}{M}+(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1} \times (\underset{\sim}{u}'\underset{\sim}{M}\ \underset{\sim}{u})]\} + [4\lambda^2(f_1+f_2-2+\lambda^2)/\{(f_1-2)(f_1-4)\}]E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-2}. \tag{5.11}$$

That is,

$$Var(U^{(p)}) = Var(U_0^{(p)}) + \alpha, \tag{5.11a}$$

where

$$\alpha = \{4\lambda^2/(f_1-2)\}\ E\{(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}[t_r\underset{\sim}{M}+(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}(\underset{\sim}{u}'\ \underset{\sim}{M}\ \underset{\sim}{u})]\} + [4\lambda^2(f_1+f_2-2+\lambda^2)/\{(f_1-2)(f_1-4)\}]\ E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-2} - [4\lambda^4/(f_1-2)^2] \times [E(1-\underset{\sim}{u}'\underset{\sim}{u})^{-1}]^2 - 2[2\lambda^2/(f_1-2)]\ E(1+U_0^{(p)})\ E(1-\underset{\sim}{u}'\ \underset{\sim}{u})^{-1}.$$

We note that

$$E(U_0^{(p)}) = pf_2/(f_1-p-1), \quad E(\underline{M}) = (f_2-1) \; \underline{I}_{p-1}/(f_1-p),$$

$$E(t_r\underline{M}) = (p-1)(f_2-1)/(f_1-p), \quad E(1-\underline{u}'\underline{u})^{-1} = (f_1-2)/(f_1-p-1)$$

and

$$E(1-\underline{u}'\underline{u})^{-2} = (f_1-4)(f_1-2)/\{(f_1-p-1)(f_1-p-3)\}.$$

Putting these values in α, we get

$$\alpha = \frac{8\lambda^4}{(f_1-p-1)^2(f_1-p-3)} + \frac{8\lambda^2(f_1-1)(f_1+f_2-p-1)}{(f_1-p)(f_1-p-1)^2(f_1-p-3)} \quad . \tag{5.12}$$

From theorem III, it is easy to find $Var(U_0^{(p)})$. However the first four (central) moments of $U_0^{(p)}$ are available in [7, 9, 10] and substituting the value of $Var(U_0^{(p)})$ in (5.11a), we get theorem IV.

The expressions for moments of $U^{(p)}$ given above check with those obtained by Pillai [11] for $p=2$.

REFERENCES

[1] Anderson, T.W. (1946). The non-central Wishart distribution and certain problems of multivariate statistics. Ann. Math. Statist. 17, 409-431.

[2] Anderson, T.W. (1958). An Introduction to Multivariate Statistical Analysis, Wiley, New York.

[3] Anderson, T.W. and Girschick, M.A. (1944). Some extensions of the Wishart distribution. Ann. Math. Statist., 15, 345-357.

[4] Khatri, C.G. (1964). Distribution of the 'generalized' multiple correlation matrix in the dual case. Ann. Math. Statist. 35, 1801-1806.

[5] Khatri, C.G. (1965). Classical statistical analysis based on a certain multivariate Gaussian distribution. Ann. Math. Statist. 36 (to appear)

[6] Kshirsagar, A.M. (1961). The non-central multivariate beta distribution. Ann. Math. Statist., 32, 104-111.

[7] Pillai, K.C.S. (1954). On some distribution problems in multivariate analysis. Mimeograph Series No. 88, Institute of Statistics, University of North Carolina, Chapel Hill.

[8] Pillai, K.C.S. (1955). Some new test criteria in multivariate analysis. Ann. Math. Statist. 26, 117-121.

[9] Pillai, K.C.S. (1956). Some results useful in multivariate analysis. Ann. Math. Statist. 27, 1106-1114.

[10] Pillai, K.C.S. (1960). Statistical Tables for Tests of Multivariate Hypothesis. The Statistical Center, Manila.

[11] Pillai, K.C.S. (1964). On the non-central multivariate beta distribution and the moments of traces of some matrices. Mimeograph Series No. 11, Department of Statistics, Purdue University. (to be presented at the Symposium on Multivariate Analysis, Dayton, Ohio, June 1965).

[12] Roy, S.N. (1957). Some Aspects of Multivariate Analysis, Wiley, New York.

[13] Roy, S.N. and Gnanadesiken, R. (1959). Some contributions to ANOVA in one or two dimensions: II. Ann. Math. Statist., 30, 318-340.