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The Busy Period of a Queue with Batch Service

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Summary

This paper is devoted to the distribution of the busy period for a queue in which the customers are served m at the time if there are m or more customers present and all at once if there are less than m present.

The method used is that of the imbedded semi-Markov process. The result is expressed in terms of the roots of a transcendental equation. Explicit expressions in real time may be obtained using a Lagrange expansion.

1. The imbedded semi-Markov Process

We assume that customers arrive at a service booth with a single server according to a Poisson process with rate λ. They are served-not necessarily in order of arrival-in batches of size m if there are m or more customers present. If less than m customers are present at the end of a service period, they are served together. We assume that the distribution function of a batch of size k is $H_k(x)$, $k = 1,\ldots,m$ and we also assume that the successive service times are conditionally independent, given the successive batch sizes.

Some aspects of this queue were discussed previously by Bloemena [1], Le Gall [2] and Runnenburg [4].

Let J_0 denote the queue-length at $t = 0+$ and let J_n, $n \geq 1$ be the queue size immediately after the n-th departure from the queue. Let X_1 be the time until the first departure and let X_n, $n > 1$ denote the time between the (n-1)st and the n-th departure. It follows from the properties of the Poisson process that the sequence of pairs (J_n, X_n), $n \geq 0$ with $X_0 = 0$ a.s is a general semi-Markov sequence, as defined by Pyke [3].

The transition probability matrices $\tilde{Q}(x)$ and $Q(x)$ for this sequence are defined as follows:

(1) $$\tilde{Q}_{ij}(x) = P\left\{X_1 \leq x,\ J_1 = j \mid J_0 = i\right\}, \qquad i,j = 0,1,\ldots$$

$$Q_{ij}(x) = P\left\{X_n \leq x,\ J_n = j \mid J_{n-1} = i\right\}, \qquad n > 1,\ i,j=0,1,\ldots$$

They are given by:

(2) $$Q_{0j}(x) = \int_0^x \left[1 - e^{-\lambda(x-y)}\right] e^{-\lambda y} \frac{(\lambda y)^j}{j!}\, d\, H_1(y), \qquad i = 0$$

$$Q_{ij}(x) = \int_0^x e^{-\lambda y} \frac{(\lambda y)^j}{j!}\, d\, H_i(y), \qquad 0 < i < m$$

$$Q_{ij}(x) = 0 \qquad i \geq m,\ j < i-m$$

$$Q_{ij}(x) = \int_0^x e^{-\lambda y} \frac{(\lambda y)^{j-i+m}}{(j-i+m)!}\, d\, H_m(y), \qquad i \geq m,\ j \geq i-m.$$

Let X_0 be the residual service time of the batch which is in service at time $t = 0$ and let I_0 be the batch size if different from zero. We assume that:

(3) $$P\left\{X_0 \leq x,\ I_0 = k\right\} = W_k(x), \qquad k = 1,\ldots,m.$$

Using this notation we obtain for the probabilities $Q_{ij}(x)$:

$$(4) \qquad Q_{ij}(x) = \Sigma \int_0^x e^{-\lambda y} \frac{(\lambda y)^{j-i+k}}{(j-i+k)!} \, d\, W_k(y) \qquad i \neq 0$$

$$Q_{0j}(x) = Q_{0j}(x)$$

The summation extends over all k for which $1 \leq k \leq m$ and $i - j \leq k \leq i$ and an empty sum is equal to zero.

We also obtain a general semi-Markov sequence if we only consider the lengths of the actual service times between departures. The probabilities $R_{ij}(x)$ and $R_{ij}(x)$ of this semi-Markov sequence are identical to those given above, except for the factor $1 - e^{-\lambda(x-y)}$ which is dropped.

2. The Busy Period

Provided the distributions $H_k(x)$ are not degenerate at zero, the corresponding Markov chain $\left\{ J_n, \, n \geq 0 \right\}$ is irreducible with matrix $P = \left\{ p_{ij} \right\}$ with $p_{ij} = Q_{ij}(\infty)$.

The successive busy periods are given by the times between successive recurrences to the state 0 in the semi-Markov process with matrix $R(x)$. It is clear that they form a renewal process, since the semi-Markov process is regular and irreducible.

In order to find the distribution of the busy period we may apply the usual argument for finding the first passage distribution. This argument was previously applied by Takacs [5] to the busy period of the $M|G|1$ queue.

Let $G(k,n,x)$ be the probability that a busy period consists of at least n services, which last a length of time less than or equal to x and such that at the end of the n-th service period k customers are waiting.

In semi-Markov terminology this is the probability that starting in state 0, the process with matrix $R(x)$ performs n or more transitions in a length of time x without returning to 0 and is in state k after the n-th transition.

We have:

$$G(k,1,x) = \int_0^x e^{-\lambda y} \frac{(\lambda y)^k}{k!} \, d\, H_1(y), \tag{5}$$

and for $n > 1$:

$$G(k,n,x) = \sum_{r=1}^{m-1} \int_0^x G(r,n-1,x-y)\, e^{-\lambda y} \frac{(\lambda y)^k}{k!} \, d\, H_r(y)$$

$$+ \sum_{\nu=m}^{m+k} \int_0^x G(\nu,n-1,x-y)\, e^{-\lambda y} \frac{(\lambda y)^{k+m-\nu}}{(k+m-\nu)!} \, d\, H_m(y)$$

If we denote the Laplace-Stieltjes transform of $G(k,n,x)$ by $\Gamma(k,n,s)$ and the generating function

$$\sum_{\nu=0}^{\infty} z^{\nu}\, \Gamma(\nu+m,n,s)$$

by $C_m(z,n,s)$, we obtain for $n > 1$:

$$z^m C_m(z,n,s) + \sum_{k=0}^{m-1} z^k\, \Gamma(k,n,s) = \tag{6}$$

$$\sum_{r=1}^{m-1} \Gamma(r,n-1,s)\, h_r(s+\lambda-\lambda z) + h_m(s+\lambda-\lambda z)\, C_m(z,n-1,s)$$

in which $h_r(s)$ is the L.S-transform of $H_r(x)$.

Finally we introduce the generating functions:

(8) $$D_m(z,w,s) = \sum_{n=1}^{\infty} C_m(z,n,s)\, w^n,$$

and

(9) $$E_r(w,s) = \sum_{n=1}^{\infty} \Gamma(r,n,s)\, w^n, \qquad r = 0,1,\dots,m-1.$$

whence:

(10) $$z^m D_m(z,w,s) + \sum_{r=0}^{m-1} z^r E_r(w,s) =$$

$$wh_1(s+\lambda-\lambda z) + wh_m(s+\lambda-\lambda z)\, D_m(z,w,s) +$$

$$w \sum_{r=1}^{m-1} h_r(s+\lambda-\lambda z)\, E_r(w,s)$$

Finally:

(11) $$D_m(z,w,s) =$$

$$\frac{wh_1(s+\lambda-\lambda z) - E_0(w,s) - \sum_{r=1}^{m-1} \left[z^r - wh_r(s+\lambda-\lambda z)\right] E_r(w,s)}{z^m - wh_m(s+\lambda-\lambda z)}$$

The L.S-transform of the distribution of the busy period is given by $E_0(1,s)$.

We know by lemma 1, p. 82 of Takacs [5] that the denominator of (11) has exactly m roots $z = \gamma_\rho(s,w)$ in the unit circle. If $w \neq 0$ and if $h_m(s)$ does not vanish in the right half plane, the roots are distinct.

We obtain the following system of linear equations for the unknown functions $E_r(w,s)$ by expressing that the roots $\gamma_\rho(s,w)$ are also roots of the numerator:

(12) $$E_0(w,s) + \sum_{r=1}^{m-1} \left[\gamma_\rho^r(s,w) - wh_r(s+\lambda-\lambda\gamma_\rho)\right] E_r(w,s) =$$

$$wh_1\left[s+\lambda-\lambda\,\gamma_\rho(s,w)\right], \qquad \rho = 1,\ldots,m.$$

In particular

(13) $$E_0(w,s) =$$

$$\frac{\left\| wh_1(s+\lambda-\lambda\gamma_\rho) \quad \gamma_\rho - wh_1 \quad \gamma_\rho^2 - wh_2 \quad \cdots \quad \gamma_\rho^{m-1} - wh_{m-1} \right\|}{\left\| 1 \quad \gamma_\rho - wh_1 \quad \gamma_\rho^2 - wh_2 \quad \cdots \quad \gamma_\rho^{m-1} - wh_{m-1} \right\|}$$

in determinant notation.

We now examine when $E_0(1,0) = 1$. The root $\gamma_1(s,w)$ of the equation:

$$z = \left[wh_m(s+\lambda-\lambda z)\right]^{\frac{1}{m}}$$

where the right hand side is the principal value of the m-th root, is real for $s \geq 0$, $w = 1$. We have:

$$\gamma_1(0+, 1) < 1 \quad \longleftrightarrow \quad \alpha_m/m > 1/\lambda$$

$$\gamma_1(0+, 1) = 1 \quad \longleftrightarrow \quad \alpha_m/m \leq 1/\lambda$$

in which α_m is the first moment of $H_m(x)$.

In the latter case $E_0(1,0) = 1$ as both determinants reduce to the minors of their first entries and these are equal. Hence the queue is in equilibrium if $\alpha_m/m \leq \lambda^{-1}$. This result is intuitive, because close to equilibrium the queue will behave like the queue in which only batches of size m exactly are allowed. The equilibrium conditions for both these queues are the same.

Some special cases

1. For $m = 1$ we find the busy period of the $M|G|1$ queue.

2. Let $h_r(s) = h^r(s)$ for $r = 1,\ldots,m$. This is the case in which the service time of a batch is made up of the sum of r individual service times. If we set $w = 1$ we obtain from formula (12):

$$E_0(1,s) = h\left[s+\lambda-\lambda\gamma(s,1)\right] = \gamma(s,1) \tag{14}$$

where $\gamma(s,1)$ is the unique root in the unit circle of the equation

$$z = h(s+\lambda-\lambda z)$$

This is obviously the same as for the $M|G|1$ queue, since the manner of service is irrelevant to the busy period as long as the individual service times are not affected.

3. Let $h_r(s) = h(s)$. This is the case, discussed by Bloemena, LeGall and Runnenburg. The service time does not depend on the number of customers in the batch. In this case some simplifications are obtained:

(15) $E_0(w,s) =$

$$\frac{\left\| \gamma_\rho^m(s,w) \quad \gamma_\rho(s,w) \quad \gamma_\rho^2(s,w) \quad \cdots \quad \gamma_\rho^{m-1}(s,w) \right\|}{\left\| 1 \quad \gamma_\rho - \gamma_\rho^m \quad \gamma_\rho^2 - \gamma_\rho^m \quad \cdots \quad \gamma_\rho^{m-1} - \gamma_\rho^m \right\|}$$

This expression can further be simplified in terms of symmetric functions of the roots $\gamma_\rho(s,w)$; $\rho = 1,\ldots,m$. e.g.

For $m = 2$:

(16) $$E_0(w,s) = \gamma_1(s,w)\, \gamma_2(s,w) \left[\gamma_1(s,w) + \gamma_2(s,w) - 1\right]^{-1}$$

For $m = 3$:

(17) $$E_0(w,s) = \gamma_1\gamma_2\gamma_3\left[1 - \gamma_1 - \gamma_2 - \gamma_3 + \gamma_1\gamma_2 + \gamma_2\gamma_3 + \gamma_3\gamma_1\right]^{-1}$$

Bibliography

|1|. Bloemena (1960)
On Queueing Processes with a Certain type of Bulk Service.
Bull. Inst. Int. Stat. 37, 219-227.

|2|. Le Gall (1962)
Les Systèmes avec on sans Attente et les Processus Stochastiques.
Dunod, Paris.

|3|. Pyke (1961)
Markov Renewal Processes: Definitions and Preliminary Properties.
Ann. Math. Stat. 32, 1231-1242.

|4|. Runnenburg (1964)
On the use of the Method of Collective Marks in Queueing Theory.
Report S 327, Statistical Dept. Math. Center, Amsterdam.

|5|. L. Takács (1962)
Introduction to the theory of Queues,
Oxford Univ. Press. New York.